A STUDY ON VAGUE SET

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

C. AMUTHA

Reg. No: 19SPMT01

Under the guidance of

Ms. P. SUGANYA M.Sc., M.Phil., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

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CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON VAGUE SET"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **C. AMUTHA (Reg. No: 19SPMT01)**

P. Sr---

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON VAGUE SET"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. P. Suganya M.Sc., M.Phil., SET.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

C. Amutha

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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CONTENT

1. PRELIMINARIES

Definition: 1.1

A **set** is a collection of well-defined objects. The objects in a set are called elements or members of that set.

Definition: 1.2

Let A and B be any two non-empty sets. A **function** f from A to B is a relation $f \subseteq A \times B$ such that the following hold:

(i) Domain of f is A.

(ii) For each $x \in A$, there is only one $y \in B$ such that $(x, y) \in f$.

Definition: 1.3

A universe of **discourse or universal set** is the set which, with reference to a particular context, contains all possible elements having the same characteristics and from which sets can be formed. It is denoted by U or W.

Definition: 1.4

An element x is said to be a number of a set A if x belongs to the set A. The **membership** is indicated by $\in$ and is pronounced as 'belongs to'. Thus, $x \in A$ means x belongs to A and $x \notin A$ means x does not belong to A.

Definition: 1.5

A **fuzzy set** $A = \{\langle u, \mu_A(u)\rangle | u \in U\}$ in a universe of discourse U is characterized by a membership function, μ_A, as follows: $\mu_A: U \rightarrow [0,1]$.

Definition: 1.6

A **vague set** in the universe of discourse X is a pair (t_A, f_A), where $t_A: X \to [0,1], f_A: X \to [0,1]$ are mappings such that $t_A(x) + f_A(x) \leq 1, \forall x \in X$. The functions t_A and f_A are called true membership function and false membership function respectively.

Definition: 1.7

The interval $[t_A(x), 1 - f_A(x)]$ is called the **vague value** of x in A and is denoted by $V_A(x)$ that is $V_A(x) = [t_A(x), 1 - f_A(x)]$.

Definition: 1.8

A vague set $A = (t_A, f_A)$ of a set X with $t_A(x) = 0$ and $f_A(x) = 1, \forall x \in X$ is called **zero vague set** of X.

Definition: 1.9

A vague set $A = (t_A, f_A)$ of a set X with $t_A(x) = 1$ and $f_A(x) = 0, \forall\, x \in X$ is called **unit vague set** of X.

Definition: 1.10

A vague set $A = (t_A, f_A)$ of a set X with $t_A(x) = \alpha$ and $f_A(x) = 1 - \alpha, \forall x \in X$ is called **α-vague set** of X, where$\alpha \in [0,1]$.

Definition: 1.11

Let $A = \{\langle x, t_A(x), f_A(x)\rangle | x \in U\}$ and $B = \{\langle x, t_B(x), f_B(x)\rangle | x \in U\}$ be two vague sets of the universe of discourse U, then

(i) **equality:** $A = B$ iff $\forall x \in U, t_A(x) = t_B(x)$ and $f_A(x) = f_B(x)$.

(ii) **inclusion:** $A \subseteq B$ iff $\forall x \in U, t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$.

(iii) **intersection:** $C = A \wedge B$ iff $\forall x \in U$,

$t_C(x) = t_A(x) \wedge t_B(x) = inf\big(t_A(x), t_B(x)\big)$ and

$f_c(x) = f_A(x) \vee f_B(x) = sup\big(f_A(x), f_B(x)\big)$.

(iv) **union:** $D = A \vee B$ and $\forall x \in U$,

$t_D(x) = t_A(x) \vee t_B(x) = sup\big(t_A(x), t_B(x)\big)$ and

$f_D(x) = f_A(x) \wedge f_B(x) = inf\big(f_A(x), f_B(x)\big)$.

(v) **complement:** $A^c = (A)^c$ iff $\forall x \in U, t_A^c(x) = f_A(x)$ and$f_A^c(x) = t_A(x)$.

Definition: 1.12

The normalized Euclidean distance between two vague sets $A = (t_A, f_A)$ and $B = (t_B, f_B)$ in $X = \{x_1, x_2, \dots, x_n\}$ is

$$\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\big(t_A(x_i) - t_B(x_i)\big)^2 + \big(f_A(x_i) - f_B(x_i)\big)^2\right]}$$

Definition: 1.13

Let ρ be an equivalence relation defined on a set S. Let $x \in S$. The **equivalence class**$[\boldsymbol{x}]$ determined by the element x is defined by

$$[x] = \{y \in S | x\ \rho\ y\}.$$

Since $x\ \rho\ x, x \epsilon [x]$ so that any equivalence class is non empty.

Definition: 1.14

A **De-Morgan algebra** is an algebra $\big(D, (\cup, \cap, ', 0,1)\big)$ where $\cup$ and $\cap$ are binary operations,' is a unary operation and 0,1 are nullary operations satisfying

(i) $(D,\cup,\cap,0,1)$ is a bounded lattice.

(ii) $(a \cup b)' = a' \cap b'$; $(a \cap b)' = a' \cup b'$

(iii) $(a')' = a$

Definition: 1.15

A relation ρ is defined on a set S is said to be antisymmetric if $a \, \rho \, b$ and $b \, \rho \, a \Longrightarrow a = b$. A relation defined on a set S which is reflexive, anti-symmetric and transitive is called a partial ordering on S. A set S with a partial ordering ρ defined on it is called a partially **ordered set** or a **poset** and is denoted by (S, ρ).

Definition: 1.16

A poset $(L, \leq)$ is called a **lattice** if $Sup\{a, b\}$ (also denote by $a \vee b$) and $Inf\{a, b\}$(also denote by $a \wedge b$) exists for every pair of elements $a, b \in L$.

Definition: 1.17

A Lattice $(L, \leq)$ is said to be **complete**, if each of its subsets has 1. u. b and g. 1. b in it.

Definition: 1.18

A lattice L is called a **Distributive Lattice** if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\forall a, b, c \in L$.

Definition: 1.19

$A = (A, \wedge, \vee, ', 0_L, 1_L)$ is called a **Boolean Lattice** if A is distributive lattice with 0_L and 1_L and A is complemented, that is for every $a \in A$ there is a unique element $a' \in A$ such that $a \vee a' = 1_L$ and $a \wedge a' = 0_L$.

Definition: 1.20

Let X be a non-empty set. Then A is called an **Intuitionistic Fuzzy Set (IFS)** of X, if it is an object having the form $A = \{\langle x, \mu_A, \gamma_A \rangle | x \in X\}$ where the function $\mu_A: X \to [0,1]$ and $\gamma_A: X \to [0,1]$ denote the degree of membership $\mu_A(x)$ and degree of non-membership $\gamma_A(x)$ of each element $x \in X$ to the set A and satisfies the condition that $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$.

Definition: 1.21

Let X be a universe of discourse. A **Pythagorean Fuzzy Set (PFS)** P in X is given by $P = \left\{\langle x, \mu_p(x), \nu_p(x) \rangle \middle| x \in X\right\}$ where $\mu_p: X \to [0,1]$ denotes the degree of membership and $\nu_p: X \to [0,1]$ denotes the degree of non-membership of the element $x \in X$ to the set P, respectively with the condition that $0 \leq \left(\mu_p(x)\right)^2 + \left(\nu_p(x)\right)^2 \leq 1$

2. BOOLEAN VAGUE SETS

2.1. Relations and Operations of Boolean Vague Sets:

Definition: 2.1.1

Let X be a non-empty set. A **Boolean vague set** A in X is a pair $A = (t_A, f_A)$ where $t_A: X \to L,\ f_A\ : X \to L$ are mappings such that $t_A \leq f'_A$. The function $t_A\ : X \to L$ define the degree of membership and the function $f_A : X \to L$ define the degree of non-membership of the element $x \in X$, to A respectively, the functions t_A and f_A should satisfy the condition $t_A \leq f'_A$ i.e., $t_A(x)\ \leq\ (f_A(x))'$ where $(f_A(x))'$ is the Boolean complement of $f_A(x)$ in the Boolean lattice L.

Definition: 2.1.2

The Boolean vague set $(\bar{0}, \bar{1})$ where $\bar{0}$ is the constant function $\bar{0}(x) = 0$ for all $x \in X$ is called the **Zero Boolean vague set** and it is denoted by 0.

Definition: 2.1.3

The Boolean Vague Set $(\bar{0}, \bar{1})$ where $\bar{1}$ is the constant function $\bar{1}(x) = 1$ for all $x \in X$ is called the **Unit Boolean Vague Set** and it is denoted by 1.

Note: 2.1.4

A Boolean Vague Set A is contained in another Boolean Vague Set $B, A\ \subset B$, if and only if, $t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$ for all $x \in X$. Here $\subset$ is a partial ordering on the set of Boolean Vague Sets defined on domain X.

Corollary: 2.1.5

$(\mathbb{V}, \subset)$ is a poset.

Proof:

A ⊂ A for all $A \in \mathbb{V}$, shows ⊂ is reflexive. Let $A \subset B$ and $B \subset A$ implies that $A = B$. This ⊂ is anti-symmetric. Let $A \subset B, B \subset C$ then $A \subset C$, shows that ⊂ is transitive and hence is a partial ordering.

Definition: 2.1.6

If A is the Boolean Vague Set (t_A, f_A), then (t_A, f_A) is also a Boolean Vague Set and is defined as the **complement** of A and is denoted by A'.

Definition: 2.1.7

The **Intersection** of two Boolean vague sets A and B is a Boolean vague set, written as $A \cap B$ and is defined by $A \cap B = (t_A \wedge t_B,\ f_A \vee f_B)$.

Definition: 2.1.8

The **Union** of two Boolean vague sets A and B is a Boolean vague set, written as $A \cup B$ and is defined by $A \cup B = (t_A \vee t_B,\ f_A \wedge f_B)$.

Definition: 2.1.9

The **addition** of two Boolean vague set A and B is a Boolean vague set, written as $A + B$ and is defined by $A + B = \big((t_A \vee t_B) \wedge (t_A \wedge t_B)', f_A \wedge f_B\big)$

Definition: 2.1.10

The **multiplication** of two Boolean vague sets A and B is a Boolean vague set, written as $A.B$ and is defined by $A.B = \big(t_A \wedge t_B, (f_A \vee f_B) \wedge (f_A \wedge f_B)\big)'$.

The operations of Union, intersection and complementation defined on Boolean vague sets, is easy to extend many basic identities. In fact, the class of all Boolean vague sets defined on a domain X with the Union, intersection and complementation forms a complete De Morgan algebra.

Theorem: 2.1.11

$(\mathbb{V}, \cup, \cap, ', 0,1)$ is a complete De Morgan Algebra.

Proof:

Let $\mathbb{V} = \{(t_\alpha, f_\alpha) | \alpha \in \Delta\}$ be the class of Boolean vague set of X.

Let $s = \{t_\alpha | (t_\alpha, f_\alpha) \in \mathbb{V}\}$ and $T = \{f_\alpha | (t_\alpha, f_\alpha) \in \mathbb{V}\}$

Let $t_0 = \wedge_{\alpha\in\Delta} t_\alpha (\inf S)$ and $f_0 = \vee_{\alpha\in\Delta} f_\alpha (\sup T)$.

We show that (t_0, f_0) is a Boolean vague set and is equal to $\mathbb{V}$.

To prove that (t_0, f_0) is a Boolean vague set, we have to prove $t_0 \leq f_0'$.

Since $t_0 = \wedge_{\alpha\in\Delta} t_\alpha$ and $f_0 = \vee_{\alpha\in\Delta} f_\alpha$ and $t_\alpha \leq f'_\alpha$ for all $\alpha \in \Delta$, which gives that

$\wedge_{\alpha\in\Delta} t_\alpha \leq \vee_{\alpha\in\Delta} f_\alpha$.

$$\Rightarrow \wedge_{\alpha\in\Delta} t_\alpha \leq (\vee_{\alpha\in\Delta} f_\alpha)'$$

$$\Rightarrow t_0 \leq f_0{}'$$

Hence (t_0, f_0) is a Boolean vague set.

By the definition, $t_0 \leq t_\alpha$ and $f_0 \leq f_\alpha$ for all $\alpha \in \Delta$.

Hence $(t_0, f_0) \leq (t_\alpha, f_\alpha)$ for all $\alpha \in \Delta$.

$\Rightarrow (t_0, f_0)$ is lower bound of $\mathbb{V}$

Let (t_k, f_k) be another lower bound of $\mathbb{V}$

Then $(t_k, f_k) \subset (t_\alpha, f_\alpha)$ for all $\alpha \in \Delta$.

i.e., $t_k \leq t_\alpha$ and $f_k \geq f_\alpha$ for all $\alpha \in \Delta$.

This gives that t_k is lower bound of S and f_k is lower bound of T.

But $t_0 = \wedge_{\alpha\in\Delta} t_\alpha$ and $f_0 = \vee_{\alpha\in\Delta} f_\alpha$ and hence $t_k \leq t_0$ and $f_0 \geq f_k$

$$\Rightarrow (t_k, f_k) \subset (t_0, f_0)$$

Hence (t_0, f_0) is infimum of $\mathbb{V}$.

Thus $\mathbb{V}$ is complete.

Let A, B, C be any three Boolean vague sets of $\mathbb{V}$. Then we make the following observation.

Idempotency:

$$A \cap A = (t_A \wedge t_A,\ f_A \vee f_A)$$

$$= (t_A, f_A) = A$$

Therefore, $A \cap A = A$.

$$A \cup A = (t_A \vee t_A,\ f_A \wedge f_A)$$

$$= (t_A, f_A) = A$$

Therefore, $A \cup A = A$.

Commutativity:

$$A \cap B = (t_A \wedge t_B,\ f_A \vee f_B)$$

$$= (t_B \wedge t_A,\ f_B \vee f_A)$$

$$= B \cap A$$

Therefore, $A \cup B = B \cap A$

$$A \cup B = (t_A \vee t_B,\ f_A \wedge f_B)$$

$$= (t_B \vee t_A, f_A \wedge f_B)$$

$$= B \cup A$$

Therefore, $A \cup B = B \cup A$.

Associativity:

$$A \cap (B \cap C) = \big(t_A \wedge (t_B \wedge t_c)\,, f_A \vee (f_B \vee f_C)\big)$$

$$= \big((t_A \wedge t_B) \wedge t_C\,, (f_A \vee f_B) \vee f_C\big)$$

$$= (t_A \wedge t_B\,, f_A \vee f_B) \cap (t_C, f_C)$$

$$= (A \cap B) \cap C$$

Therefore, $A \cap (B \cap C) = (A \cap B) \cap C$

$$A \cup (B \cup C) = \big(t_A \vee (t_B \vee t_c)\,, f_A \wedge (f_B \wedge f_C)\big)$$

$$= \left((t_A \vee t_B) \vee t_C\,, (f_A \wedge f_B) \wedge f_C\right)$$

$$= (t_A \vee t_B\,, f_A \wedge f_B) \cap (t_C, f_C)$$

$$= (A \cup B) \cup C$$

Therefore, $A \cup (B \cup C) = (A \cup B) \cup C$

Absorption:

$$A \cap (A \cup B) = \left(t_A \wedge (t_A \vee t_B)\,, f_A \vee (f_A \wedge f_B)\right)$$

$$= \left((t_A \wedge t_A) \vee (t_A \wedge t_B)\,, (f_A \vee f_A) \wedge (f_A \vee f_B)\right)$$

$$= (t_A, f_A) = A$$

$$A \cup (A \cap B) = \left(t_A \vee (t_A \wedge t_B)\,, f_A \wedge (f_A \vee f_B)\right)$$

$$= \left((t_A \vee t_A) \wedge (t_A \vee t_B)\,, (f_A \wedge f_A) \vee (f_A \wedge f_B)\right)$$

$$= (t_A, f_A) = A$$

Therefore, $A \cap \left(A \cup B = A = A \cup (A \cap B)\right)$

Distributivity:

$$A \cup (B \cap C) = \left(t_A \vee (t_B \wedge t_C)\,, f_A \wedge (f_B \vee f_C)\right)$$

$$= \left((t_A \vee t_B) \wedge (t_A \vee t_C)\,, (f_A \wedge f_B) \vee (f_A \wedge f_C)\right)$$

$$= (t_A \vee t_B\,, f_A \wedge f_B) \cap (t_A \vee t_C\,, f_A \wedge f_C)$$

$$= (A \cup B) \cap (A \cup C)$$

Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$A \cap (B \cup C) = \left(t_A \wedge (t_B \vee t_C)\,, f_A \vee (f_B \wedge f_C)\right)$$

$$= \left((t_A \wedge t_B) \vee (t_A \wedge t_C)\,, (f_A \vee f_B) \vee (f_A \vee f_C)\right)$$

$$= (t_A \wedge t_B\,, f_A \vee f_B) \cap (t_A \wedge t_C\,, f_A \vee f_C)$$

$$= (A \cap B) \cup (A \cap C)$$

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Therefore $(\mathbb{V}, \cup, \cap, ', 0, 1)$ is a bounded distributive lattice, further.

De Morgan Laws:

$(A \cup B)' = (t_A \vee t_B \ , f_A \wedge f_B)'$

$= (f_A \wedge f_B \ , t_A \vee t_B)$

$= (f_A, t_A) \cap (f_B, t_B)$

$= A' \cap B'$

Therefore, $(A \cup B)' = A' \cap B'$

$(A \cap B)' = (t_A \wedge t_B \ , f_A \vee f_B)'$

$= (f_A \vee f_B \ , t_A \wedge t_B)$

$= (f_A, t_A) \cup (f_B, t_B)$

$= A' \cup B'$

Therefore, $A \cap B' = A' \cup B'$

Involution:

$(A')' = (f_A, t_A)'$

$= (t_A, f_A)$

$= A$

Therefore, $(A')' = A$

Therefore, $(\mathbb{V}, \cup, \cap, ', 0,1)$ is a complete De Morgan Algebra.

We shall define over the set of all Boolean Vague Sets two operators in some model logics that is for any Booleans Vague Set A,

$$\Delta A = (t_A, t'_A), \ \nabla A = \left(f'_A, f_A\right)$$

Clearly ΔA and ∇A are Boolean Vague Sets.

Theorem: 2.1.12

For every Boolean vague set A

(i) $\Delta A = (\nabla \mathrm{A}')'$

(ii) $\nabla A = (\Delta \mathrm{A}')'$

Proof:

(i) $A = (t_A, f_A)$

$\Delta A = (t_A, t'_A)$

We have $A' = (f_A, t_A)$

$\nabla A' = (t'_A, t_A)$

Therefore $(\nabla A')' = (t_A, t'_A) = \Delta A$

(ii) $\nabla A = \left(f'_A, f_A\right)$

We have $A' = (f_A, t_A)$

$\Delta A' = \left(f_A, f'_A\right)$

Therefore $(\Delta A')' = \left(f'_A, f_A\right) = \nabla A$

Theorem: 2.1.13

For every two Boolean vague sets A and B, we have

(i) $\Delta(A \cup B) = \Delta A \cup \Delta B$

(ii) $\nabla(A \cup B) = \nabla A \cup \nabla B$

(iii)$\Delta(A \cap B) = \Delta A \cap \Delta B$

Proof:

(i) $\Delta(A \cup B) = \Delta(t_A \vee t_B, f_A \wedge f_B)$

$= (t_A \vee t_B, (t_A \vee t_B)')$

$= (t_A \vee t_B, t'_A \wedge t'_B)$

$= (t_A, t'_A) \cup (t_B, t'_B)$

$= \Delta A \cup \Delta B$

Therefore $\Delta(A \cup B) = \Delta A \cup \Delta B$

(ii) $\nabla(A \cup B) = \nabla(t_A \vee t_B, f_A \wedge f_B)$

$= (\,(f_A \wedge f_B)', f_A \wedge f_B)$

$$= \left(f'_A \vee f'_B, f_A \wedge f_B\right)$$

$$= \left(f'_A, f_A\right) \cup \left(f'_B, f_B\right)$$

$$= \nabla A \cup \nabla B$$

Therefore $\nabla(A \cup B) = \nabla A \cup \nabla B$

(iii) $\Delta(A \cap B) = \Delta(t_A \wedge t_B, f_A \vee f_B)$

$$= ((t_A \wedge t_B, (t_A \wedge t_B)')$$

$$= (t_A \wedge t_B, t'_A \vee t'_B)$$

$$= (t_A, t'_A) \cap (t_B, t'_B)$$

$$= \Delta A \cap \Delta B$$

Therefore $\Delta(A \cap B) = \Delta A \cap \Delta B$.

For any given Boolean vague set we can determine two Boolean vague set with its the four numbers. Let $A = (t_A, f_A)$ be a Boolean vague set.

Write $P_A = \vee_{x \in X} t_A(x)$; $Q_A = \wedge_{x \in X} f_A(x)$

$p_A = \wedge_{x \in X} t_A(x)$; $q_A = \vee_{x \in X} f_A(x)$

Then the sets $\mu(A) = (P_A, Q_A)$ and $\vartheta(A) = (\, p_A \,, q_A \,)$ Clearly, the set $\mu(A)$ and $\vartheta(A)$ are Boolean Vague Set.

For, let $A = (t_A(x)\,, f_A(x)\,)$ be a Boolean Vague Set

Then $t_A(x) \leq f'_A(x)$ for all $x \in X$

This gives that $\vee_{x \in X} t_A(x) \ \leq\ \wedge_{x \in X} (f_A(x))'$

i.e., $P_A \leq Q'_A$

Therefore $\mu(A)$ is a Boolean vague set. Analogouly $\vartheta(A)$ is also Boolean vague set.

Theorem: 2.1.14

Let $(\mathbb{V}, \subseteq)$ is a Lattice, then the operator $C: \mathbb{V} \to \mathbb{V}$ defined by $C(A) = \mu(A)$, satisfies the following properties.

(i) A$\subset C(A)$

(ii) $C\big(C(A)\big) = C(A)$

(iii) If $A \subset B$, then $C(A) \subset C(B)$ for all $A, B \in \mathbb{V}$

Proof:

(i) Let $A \in \mathbb{V}$.

Then $A = (t_A(x), f_A(x))$ with $t_A(x) \leq f_A'(x)$ for all $x \in X$.

By definition of $\mu(A)$, $C(A) = (\vee_{x \in X} t_A(x), \wedge_{x \in X} f_A(x))$.

Clearly, $t_A(x) \leq \vee_{x \in X} t_A(x)$ and ,$f_A(x) \geq \wedge_{x \in X} f_A(x)$.

Therefore $A \subset C(A)$.

(ii) $C\big(C(A)\big) = C(\vee_{x \in X} t_A(x), \wedge_{x \in X} f_A(x))$

$= \mu(\vee_{x \in X} t_A(x), \wedge_{x \in X} f_A(x))$

$= \Big(\vee_{x \in X}\big(\vee_{x \in X} t_A(x)\big), \wedge_{x \in X}\big(\wedge_{x \in X} f_A(x)\big)\Big)$

$= \big(\vee_{x \in X} t_A(x), \wedge_{x \in X} f_A(x)\big)$

$= C(A)$

Therefore $C\big(C(A)\big) = C(A)$.

(iii) Let $A \subset B$

That is $t_A(x) \leq t_B(x)$ and $f_A(x) \leq f_B(x)$ for all $x \in X$.

This gives that $\vee_{x \in X} t_A(x) \leq \vee_{x \in X} t_B(x)$ and

$\wedge_{x \in X} f_A(x) \geq \wedge_{x \in X} f_B(x)$.

By definition of containment, $C(A) \subset C(B)$.

Theorem: 2.1.15

For any two Boolean vague sets A and B

(i) $\mu(A \cup B) = \mu(A) \uplus \mu(B)$

(ii) $\Delta A \subset \mu(A)$

(iii) $\vartheta(A) \subset \nabla A$

(iv) $\left(\vartheta(A')\right)' = \mu(A)$

Proof:

(i) $A = (t_A, f_A)$, and $B = (t_B, f_B)$

$A \cup B = \left(t_A(x) \vee t_B(x),\ f_A(x) \wedge f_B(x)\right)$

$\mu(A \cup B) = \mu\left(t_A(x) \vee t_B(x),\ f_A(x) \wedge f_B(x)\right)$

$= (\vee_{x\in X}\left(t_A(x) \vee t_B(x)\right), \wedge_{x\in X}\left(f_A(x)\right) \wedge f_B(x)))$

$= (\left(\vee_{x\in X} t_A(x)\right) \vee \left(\vee_{x\in X} t_B(x)\right), \left(\wedge_{x\in X} f_A(x)\right) \wedge \left(\wedge_{x\in X} f_B(x)\right))$

$= \left(\vee_{x\in X} t_A(x), \wedge_{x\in X} f_A(x)\right) \cup \left(\vee_{x\in X} t_B(x), \wedge_{x\in X} f_B(x)\right)$

$= \mu(A) \uplus \mu(B)$

Therefore, $\mu(A \cup B) = \mu(A) \uplus \mu(B)$.

(ii) $\Delta A = \left(t_A(x), t'_A(x)\right)$ and

$\mu(A) = \left(\vee_{x\in X} t_A(x), \wedge_{x\in X} f_A(x)\right)$.

Clearly, $t_A(x) \leq \vee_{x\in X} t_A(x)$ and ,$f_A(x) \geq \wedge_{x\in X} f_A(x)$.

Hence by containment, $\Delta A \subset \mu(A)$.

(iii) $\vartheta(A) = (\wedge_{x\in X} t_A(x), \vee_{x\in X} f_A(x))$ and

$\nabla A = (f'_A(x), f_A(x))$.

Clearly, $\wedge_{x\in X} t_A(x) \leq f'_A(x)$ and $\vee_{x\in X} f_A(x) \geq f_A(x)$.

Hence by containment, $\vartheta(A) \subset \nabla A$.

(iv) $A' = \left(f_A(x), t_A(x)\right)$

$\vartheta(A') = (\wedge_{x\in X} f_A(x), \vee_{x\in X} t_A(x))$

$\left(\vartheta(A')\right)' = \left(\vee_{x\in X} t_A(x), \wedge_{x\in X} f_A(x)\right) = \mu(A)$

Therefore, $\left(\vartheta(A')\right)' = \mu(A)$

2.2 Cartesian product and its properties:

Let X_1 and X_2 be two nonempty sets and $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be two Boolean Vague Sets of X_1 and X_2 respectively. The Cartesian product of A and B denoted by $A \times B$, defined as $A \times B = (t_A \wedge \mathrm{t_B}, \mathrm{f_A} \wedge \mathrm{f_B})$. Since A is Boolean vague set of X_1 and B is Boolean vague set of X_2, this gives that $t_A \leq f'_A$ on X_1 and $t_B \leq f'_B$ on X_2 which implies $t_A \wedge \mathrm{t_B} \leq t_A \leq f'_A \leq f'_A \vee f'_B = (\mathrm{f_A} \wedge \mathrm{f_B})'$. This shows that $A \times B$ is a Boolean vague set.

Theorem: 2.2.1

Let X_1, X_2, X_3 be three non-empty sets and A, B be Boolean vague sets of X_1, C be a Boolean vague set of X_2, D be a Boolean vague set of X_3 then

(i) $A \times C = C \times A$

(ii) $(A \times C) \times D = A \times (C \times D)$

(iii)$(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof:

(i) $A \times C = (t_A \wedge t_C, f_A \wedge f_C)$

$= (t_C \wedge t_A, f_C \wedge f_A) = C \times A$

Therefore, $A \times C = C \times A$.

(ii) $(A \times C) \times D = (t_A \wedge t_C, f_A \wedge f_C) \times (t_D, f_D)$

$= ((t_A \wedge t_C) \wedge t_D, (f_A \wedge f_C) \wedge f_D)$

$= (t_A \wedge (t_C \wedge t_D), f_A \wedge (f_C \wedge f_D))$

$= (t_A, f_A) \times (t_C \wedge t_D, f_C \wedge f_D)$

$= A \times (C \times D)$

Therefore, $(A \times C) \times D = A \times (C \times D)$

(iii) $(A \cup B) \times C = (t_A \vee t_B, f_A \wedge f_B) \times (t_C, f_C)$

$$= \left((t_A \vee t_B) \wedge t_C, (f_A \wedge f_B) \wedge f_C\right)$$

$$= \left((t_A \wedge t_C) \vee (t_B \wedge t_C), (f_A \wedge f_C) \wedge (f_B \wedge f_C)\right)$$

$$= (t_A \wedge t_C, f_A \wedge f_C) \cup (t_B \wedge t_C, f_B \wedge f_C)$$

$$= (A \times C) \cup (B \times C)$$

Therefore, $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Theorem: 2.2.2

For every two non-empty sets X_1, X_2 with Boolean vague sets A, B over them

Then (i) $\Delta(A \times B) \subset \Delta A \times \Delta B$

(ii) $\nabla(A \times B) \supset \nabla \mathrm{A} \times \nabla \mathrm{B}$

Proof:

(i) $A \times B = (t_A \wedge t_B, f_A \wedge f_B)$

$$\Delta(A \times B) = \left((t_A \wedge \mathrm{t_B}), (t_A \wedge \mathrm{t_B})'\right)$$

$$= (t_A \wedge \mathrm{t_B}, \mathrm{t'_A} \vee \mathrm{t'_B})$$

$$\subset (t_A \wedge \mathrm{t_B}, \mathrm{t'_A} \wedge \mathrm{t'_B})$$

$$= (t_A, t'_A) \times (t_B t'_B)$$

$$= \Delta A \times \Delta B$$

Therefore, $\Delta(A \times B) \subset \Delta A \times \Delta B$.

(ii) $\nabla(A \times B) = \left((f_A \wedge f_B)', (f_A \wedge f_B)\right)$

$$= \left(f'_A \vee f'_B, f_A \wedge f_B\right)$$

$$\supset \left(f'_A \wedge f'_B, f_A \wedge f_B\right)$$

$$= \left(f'_A, f_A\right) \times \left(f'_B, f_B\right)$$

$$= \nabla A \times \nabla B$$

Therefore, $\nabla(A \times B) \supset \nabla \mathrm{A} \times \nabla \mathrm{B}$

2.3 Rough Approximations of a Boolean Vague Set

Let X denote a set of objects called universe and let $R \subset X \times X$ be an equivalence relation on X. The pair $P = (X, R)$ is called Pawlak approximation space. For $u, v \in X$ and $(u, v) \in R, u$ and v belong to the same equivalence class and we say that they are indistinguishable in A. Therefore, the relation R is called an indiscernibility relation. Let $[x]_R$ denote an equivalence class of R containing element x. The lower and upper approximations for a subset $Y \subseteq X$ in A denoted $P_*(Y)$ and $P^*(Y)$ respectively and defined as follows,

$$P_*(Y) = \{x \in X | [x]_R \subset Y\},$$

$$P^*(Y) = \{x \in X | [x]_R \cap Y \neq \emptyset\}$$

If an object x belongs to the lower approximation space of Y in P then "x surely belongs to Y in P", x ∈ P* (Y) means that "x possibly belongs to Y in P".

Definition: 2.3.1

Let $A = (t_A, f_A)$ be a Boolean vague Set of X. The **lower approximation $\boldsymbol{A_L}$** and **upper approximation $\boldsymbol{A^U}$** of A in the Pawlak approximation space $< X, R >$ are defined as $A_L = (t_{AL}, f_{AL})$ and $A^U = (t_{A^U}, f_{A^U})$ respectively, where for all $x \in X$,

$$t_{AL}(x) = \wedge\{t_A(y) | y \in [x]_R, x \in X\},$$

$$f_{AL}(x) = \vee\{f_A(y) | y \in [x]_R, x \in X\},$$

$$t_{A^U}(x) = \vee\{t_A(y) | y \in [x]_R, x \in X\},$$

$$f_{A^U}(x) = \wedge\{f_A(y) | y \in [x]_R, x \in X\},$$

Here $[x]_R$ is the equivalence class of the element x.

Corollary: 2.3.2

If A is a Boolean vague set then A_L and A^U are Boolean vague sets.

Proof:

Let $A = (t_A, f_A)$ be a Boolean vague set

$\Rightarrow t_A(x) \leq (f_A(x))'$ for all $x \in X$

$\Rightarrow t_A(y) \leq (f_A)'(y)$ for all $y \in [x]_R \subset X$

$\Rightarrow \wedge t_A(y) \leq \wedge f'_A(y)$ for all $y \in [x]_R \subset X$

$\Rightarrow \wedge t_A(y) \leq (\vee f_A(y))'$ for all $y \in [x]_R \subset X$

Hence $\wedge\{t_A(y)|y \in [x]_R\} \leq (\vee\{f_A(y)|y \in [x]_R\})'$.

This gives $t_{AL}(x) \leq \left(f_{AL}(x)\right)'$ for every $x \in X$.

Hence $A_L = (t_{AL}, f_{AL})$ is a Boolean vague set.

Let $A = (t_A, f_A)$ be a Boolean vague set

$\Rightarrow t_A(x) \leq (f_A(x))'$ for all $x \in X$

$\Rightarrow t_A(y) \leq (f_A)'(y)$ for all $y \in [x]_R \subset X$

$\Rightarrow \vee t_A(y) \leq \vee f'_A(y)$ for all $y \in [x]_R \subset X$

$\Rightarrow \vee t_A(y) \leq (\wedge f_A(y))'$ for all $y \in [x]_R \subset X$

Hence $\vee\{t_A(y)|y \in [x]_R\} \leq (\wedge\{f_A(y)|y \in [x]_R\})'$.

This gives $t_{A^U}(x) \leq \left(f_{A^U}(x)\right)'$ for every $x \in X$.

Hence, A^U is a Boolean vague set.

Theorem: 2.3.3

If A is a Boolean vague set then $A_L \subseteq A \subseteq A^U$.

Proof:

Let $A = (t_A, f_A)$ be a Boolean vague set.

From this definitions of A_L and A^U, we have,

$t_{AL}(x) = \wedge\{t_A(y) | y \in [x]_R\}$, $f_{AL}(x) = \vee\{f_A(y) | y \in [x]_R\}$ and

$t_{A^U}(x) = \vee\{t_A(y) | y \in [x]_R\}$, $f_{A^U}(x) = \wedge\{f_A(y) | y \in [x]_R\}$

$\Rightarrow t_{AL}(x) \leq t_A(x) \leq t_{A^U}(x)$ for all $x \in X$ and

$f_{AL}(x) \leq f_A(x) \leq f_{A^U}(x)$ for all $x \in X$.

Clearly, $(t_{AL}, f_{AL}) \subseteq (t_A, f_A) \subseteq (t_{A^U} \leq f_{A^U})$.

Hence $A_L \subseteq A \subseteq A^U$.

Theorem: 2.3.4

If A and B be a Boolean vague sets, then the following holds:

(i) $(A \cup B)^{\cup} = A^{\cup} \cup B^{\cup}$

(ii) $(A \cap B)_L = A_L \cap B_L$

Proof:

Let $A = (t_A, f_A)$ be a Boolean vague set.

(i) $(A \cup B)^{\cup} = \left(t_{(A\cup B)^{\cup}}, f_{(A\cup B)^U}\right)$ and

$A^{\cup} \cup B^{\cup} = (t_{A^{\cup}} \vee t_{B^{\cup}}, f_{A^{\cup}} \wedge f_{B^{\cup}})$

Now, for all $x \in X$,

$$\begin{aligned} t_{(A\cup B)^{\cup}}(x) &= \vee\{t_{(A\cup B)}(y) | y \in [x]_R\} \\ &= \vee\{t_A(y) \vee t_B(y) | y \in [x]_R\} \\ &= (\vee\{t_A(y) | y \in [x]_R\}) \vee (\vee\{t_B(y) | y \in [x]_R\}) \\ &= t_{A^{\cup}}(x) \vee t_{B^{\cup}}(x) \\ &= (t_{A^{\cup}} \vee t_{B^{\cup}})(x) \end{aligned}$$

$$\begin{aligned} f_{(A\cup B)^{\cup}}(x) &= \wedge\{f_{(A\cup B)}(y) | y \in [x]_R\} \\ &= \wedge\{f_A(y) \wedge f_B(y) | y \in [x]_R\} \\ &= (\wedge\{f_A(y) | y \in [x]_R\}) \wedge (\wedge\{f_B(y) | y \in [x]_R\}) \\ &= f_{A^{\cup}}(x) \wedge f_{B^{\cup}}(x) \end{aligned}$$

$= (f_{A^{\cup}} \wedge f_{B^{\cup}})(x)$

Hence $(A \cup B)^{\cup} = A^{\cup} \cup B^{\cup}$

(ii) $(A \cap B)_L = \left(t_{(A\cap B)_L}, f_{(A\cap B)_L}\right)$

$(A_L \cap B_L) = t_{A_L} \wedge t_{B_L}, f_{A_L} \wedge f_{B_L}$

For all $x \in X$,

$$t_{(A\cap B)_L}(x) = \wedge\{t_{(A\cap B)}(y) | y \in [x]_R\}$$

$$= \wedge\{t_A(y) \wedge t_B(y) | y \in [x]_R\}$$

$$= (\wedge\{t_A(y) | y \in [x]_R\}) \wedge (\wedge\{t_B(y) | y \in [x]_R\})$$

$$= t_{A_L}(x) \wedge t_{B_L}(x)$$

$$= \left(t_{A_L} \wedge t_{B_L}\right)(x)$$

$$f_{(A\cap B)_L}(x) = \vee\{f_{(A\cap B)}(y) | y \in [x]_R\}$$

$$= \vee\{f_A(y) \vee f_B(y) | y \in [x]_R\}$$

$$= (\vee\{f_A(y) | y \in [x]_R\}) \vee (\vee\{f_B(y) | y \in [x]_R\})$$

$$= f_{A_L}(x) \vee f_{B_L}(x)$$

$$= \left(f_{A_L} \vee f_{B_L}\right)(x)$$

Hence $(A_L \cap B_L) = A_L \cap B_L$

Theorem: 2.3.5

If A be Boolean vague set, then the following holds

(i) $(A \cap B)^{\cup} = A^{\cup} \cap B^{\cup}$

$(ii)$$(A \cup B)_L = A_L \cup B_L$

Proof:

Let $A = (t_A, f_A)$ be a Boolean vague set.

(i) $(A \cap B)^{\cup} = \left(t_{(A\cap B)^{\cup}}, f_{(A\cap B)^{U}}\right)$ and

$A^{\cup} \cup B^{\cup} = (t_{A^{\cup}} \vee t_{B^{\cup}}, f_{A^{\cup}} \wedge f_{B^{\cup}})$

Now, for all $x \in X$,

$$t_{(A\cap B)^{\cup}}(x) = \vee\{t_{(A\cap B)}(y) | y \in [x]_R\}$$

$$= \vee\{t_A(y) \wedge t_B(y) | y \in [x]_R\}$$

$$= (\vee\{t_A(y) | y \in [x]_R\}) \wedge (\vee\{t_B(y) | y \in [x]_R\})$$

$$= t_{A^{\cup}}(x) \wedge t_{B^{\cup}}(x)$$

$$= (t_{A^{\cup}} \wedge t_{B^{\cup}})(x)$$

$$f_{(A\cap B)^{\cup}}(x) = \wedge\{f_{(A\cap B)}(y) | y \in [x]_R\}$$

$$= \wedge\{f_A(y) \vee f_B(y) | y \in [x]_R\}$$

$$= (\wedge\{f_A(y) | y \in [x]_R\}) \vee (\wedge\{f_B(y) | y \in [x]_R\})$$

$$= f_{A^{\cup}}(x) \vee f_{B^{\cup}}(x)$$

$$= (f_{A^{\cup}} \vee f_{B^{\cup}})(x)$$

Hence $(A \cap B)^{\cup} = A^{\cup} \cap B^{\cup}$

(ii) $(A \cup B)_L = \left(t_{(A\cup B)_L}, f_{(A\cup B)_L}\right)$

$(A_L \cup B_L) = t_{A_L} \vee t_{B_L}, f_{A_L} \wedge f_{B_L}$

For all $x \in X$,

$$t_{(A\cup B)_L}(x) = \wedge\{t_{(A\cup B)}(y) | y \in [x]_R\}$$

$$= \wedge\{t_A(y) \vee t_B(y) | y \in [x]_R\}$$

$$= (\wedge\{t_A(y) | y \in [x]_R\}) \vee (\wedge\{t_B(y) | y \in [x]_R\})$$

$$= t_{A_L}(x) \vee t_{B_L}(x)$$

$$= \left(t_{A_L} \vee t_{B_L}\right)(x)$$

$$f_{(A\cup B)_L}(x) = \vee\{f_{(A\cup B)}(y) | y \in [x]_R\}$$

$$= \vee\{f_A(y) \wedge f_B(y) | y \in [x]_R\}$$

$$= (\vee\{f_A(y) | y \in [x]_R\}) \wedge (\vee\{f_B(y) | y \in [x]_R\})$$

$$= f_{A_L}(x) \wedge f_{B_L}(x)$$

$$= \left(f_{A_L} \wedge f_{B_L}\right)(x)$$

Hence $(A_L \cup B_L) = A_L \cup B_L$

Theorem: 2.3.6

If A and B be Boolean vague sets such that $A \subseteq B$, then $A_L \subseteq B_L$ and $A^U \subseteq B^U$

Proof:

$t_{A_L}(x) = \wedge\{t_A(y) | y \in [x]_R\}$

$\leq \wedge\{t_B(y) | y \in [x]_R\}$

$= t_{B_L}(x)$

$t_{A_L}(x) \leq t_{B_L}(x)$

$f_{A_L}(x) = \vee\{f_A(y) | y \in [x]_R\}$

$\geq \vee\{f_B(y) | y \in [x]_R\}$

$= f_{B_L}(x)$

$f_{A_L}(x) \geq f_{B_L}(x)$

Therefore $A_L \subseteq B_L$

$t_{A^{\cup}}(x) = \vee\{t_A(y) | y \in [x]_R\}$

$\leq \vee\{t_B(y) | y \in [x]_R\}$

$= t_{B^{\cup}}(x)$

$t_{A^{\cup}}(x) \leq t_{B^{\cup}}(x)$

$f_{A^{\cup}}(x) = \wedge\{f_A(y) | y \in [x]_R\}$

$\geq \wedge\{f_B(y) | y \in [x]_R\}$

$= f_{B^{\cup}}(x)$

$f_{A^{\cup}}(x) \geq f_{B^{\cup}}(x)$

Therefore $A^U \subseteq B^U$

2.4 Roughness of a Boolean Vague Set:

The roughness measure of an ordinary set in the universe of discourse developed by Pawlak. M.Banerjee, in 1996 introduced roughness measure of a fuzzy set μ, which is expressed as,

$$\rho_{\mu}^{\alpha\beta} = 1 - \frac{|\mu_{\alpha_L}|}{|\mu_{\beta^U}|}$$

Where $0< \beta \leq \alpha \leq 1$ and $\mu_{\alpha_L} = \{x \in U|\mu_L(x) \geq \alpha\}, \mu_{\beta^U} = \{x \epsilon U|\mu^U(x) \geq \beta\}$.

Definition: 2.4.1

The ***αβ* level sets** of A_L, A^U denoted by $A_{\alpha\beta_L}$ and $A_{\alpha\beta^U}$, are defined as,

$$A_{\alpha\beta_L} = \{x \in X|t_{A_L}(x) \geq \alpha, f_{A_L}(x) \leq \beta\},$$

$$A_{\alpha\beta^U} = \{x \in X|t_{A^U}(x) \geq \alpha, f_{A^U}(x) \leq \beta\}$$

Where $0< \alpha, \beta \leq 1, \alpha \vee \beta \leq 1$ (0 and 1 are minimum and maximum elements of L).

Here $\alpha \vee \beta \leq 1$, otherwise $A_{\alpha\beta_L}$and $A_{\alpha\beta^U}$ are respectively reduced to

$$A_{\alpha\beta_L} = \{x \in X|t_{AL}(x) \geq \alpha\},$$

$$A_{\alpha\beta^U} = \{x \in X|t_{A^U}(x) \geq \alpha\}.$$

Definition: 2.4.2

A **roughness measure** $\rho_A^{\alpha\beta}$ of the Boolean vague set of A of X with respect to parameters α, β in L, and the approximation $\langle X, R\rangle$ is defined as,

$$\rho_A^{\alpha\beta} = 1 - \frac{|A_{\alpha\beta_L}|}{|A_{\alpha\beta^U}|}$$

Theorem: 2.4.3

Let A and B be two vague sets of X. If $A \subseteq B$ then

(i) If $A_{\alpha\beta^U} = B_{\alpha\beta^U}$ then $\rho_B^{\alpha\beta} \leq \rho_A^{\alpha\beta}$

(ii) If $A_{\alpha\beta_L} = B_{\alpha\beta_L}$ then $\rho_B^{\alpha\beta} \geq \rho_A^{\alpha\beta}$

Proof:

From the theorem 2.3.6, $A \subseteq B$ then we have $A_L \subseteq B_L$ and $A^U \subseteq B^U$. Moreover $A_{\alpha\beta^U} \subseteq B_{\alpha\beta^U}$ and $A_{\alpha\beta_L} \subseteq B_{\alpha\beta_L}$

(i) From the definition 2.4.2,

$$\rho_A^{\alpha\beta} = 1 - \frac{|A_{\alpha\beta_L}|}{|A_{\alpha\beta^U}|} \quad , \quad \rho_B^{\alpha\beta} = 1 - \frac{|B_{\alpha\beta_L}|}{|B_{\alpha\beta^U}|}$$

Now, $\rho_A^{\alpha\beta} - \rho_B^{\alpha\beta} = 1 - \frac{|A_{\alpha\beta_L}|}{|A_{\alpha\beta^U}|} - 1 + \frac{|B_{\alpha\beta_L}|}{|B_{\alpha\beta^U}|} = \frac{|B_{\alpha\beta_L}|}{|B_{\alpha\beta^U}|} - \frac{|A_{\alpha\beta_L}|}{|A_{\alpha\beta^U}|}$

Since $A_{\alpha\beta^U} = B_{\alpha\beta^U}$, $\rho_A^{\alpha\beta} - \rho_B^{\alpha\beta} = \frac{|B_{\alpha\beta_L}| - |A_{\alpha\beta_L}|}{|A_{\alpha\beta^U}|} \geq 0$

Hence $\rho_B^{\alpha\beta} \leq \rho_A^{\alpha\beta}$

(ii) $\rho_A^{\alpha\beta} - \rho_B^{\alpha\beta} = \frac{|B_{\alpha\beta_L}|}{|B_{\alpha\beta^U}|} - \frac{|A_{\alpha\beta_L}|}{|A_{\alpha\beta^U}|}$

Since $A_{\alpha\beta_L} = B_{\alpha\beta_L}$, , $\rho_A^{\alpha\beta} - \rho_B^{\alpha\beta} = |A_{\alpha\beta_L}| \left(\frac{1}{|B_{\alpha\beta^U}|} - \frac{1}{|A_{\alpha\beta^U}|} \right) \leq 0$

Hence $\rho_B^{\alpha\beta} \geq \rho_A^{\alpha\beta}$

3. PYTHAGOREAN VAGUE SETS (PVS)

3.1 Pythagorean Vague Sets

Let X be a universe of discourse. A Pythagorean Vague Set (PVS) A in X is given by $A = \{< x, t_A(x), 1 - f_A(x) > | x \in X\}$, where $t_A(x) : X \to [0,1]$ denotes the truth value and $1 - f_A(x) : X \to [0,1]$ denotes the false value of the element $x \in X$ to the set A, respectively, with the condition that $0 \leq \left(t_A(x)\right)^2 + \left(1 - f_A(x)\right)^2 \leq 1$.

Definition: 3.1.1

Let $P =< t_p, 1 - f_p >$, $P_1 =< t_{p1}, 1 - f_{p1} >$, $P_2 =< t_{p2}, 1 - f_{p2} >$ be the Pythagorean vague elements and $\lambda > 0$, satisfies the following operations.

1. $P^\lambda \quad = \left[\left(t_p\right)^\lambda, \sqrt{1 - \left(1 - \left(1 - f_p\right)^2\right)^\lambda}\right],$
2. $\lambda P \quad = \left[\sqrt{\left(1 - \left(1 - t_p\right)^2\right)^\lambda}, \left(1 - f_p\right)^\lambda\right]$
3. $P_1 \oplus P_2 \quad = \left[\sqrt{\left(t_{p1}\right)^2 + \left(t_{p2}\right)^2 - {t_{p1}}^2 . {t_{p2}}^2}, \left(1 - f_{p1}\right)\left(1 - f_{p2}\right)\right]$
4. $P_1 \otimes P_2 \quad = \left[\sqrt{\left(1 - f_{p1}\right)^2 + \left(1 - f_{p2}\right)^2 - \left(1 - f_{p1}\right)^2 . \left(1 - f_{p2}\right)^2}, \left(t_{p1}\right)\left(t_{p2}\right)\right]$

Example: 3.1.2

Let $X = \{a, b\}$ and P, P_1 and P_2 be Pythagorean vague sets in X where $\lambda \geq 0$

$P = \{< x, [0.4,0.6], [0.3,0.5] >\}$ $\quad P_1 = \{< x, [0.5,0.7], [0.4,0.5] >\}$

$P_2 = \{< x, [0.5,0.6], [0.3,0.4] >\}$ and Let $\lambda = 2$

1. $P^{\lambda} \quad = \left[\left(t_p\right)^{\lambda}, \sqrt{1-\left(1-\left(1-f_p\right)^2\right)^{\lambda}}\right]$

$$= \left([0.4^2, 0.3^2], \left[\sqrt{1-(1-0.6^2)^2}, \sqrt{1-(1-0.5^2)^2}\right]\right)$$

$$= \left([0.16,0.09], \left[\sqrt{1-(1-0.36)^2}, \sqrt{1-(1-0.25)^2}\right]\right)$$

$$= \left([0.16,0.09], \left[\sqrt{1-(0.64)^2}, \sqrt{1-(0.75)^2}\right]\right)$$

$$= \left([0.16,0.09], \left[\sqrt{1-0.4096}, \sqrt{1-0.5625}\right]\right)$$

$$= \left([0.16,0.09], \left[\sqrt{0.5904}, \sqrt{0.4375}\right]\right)$$

$$P^{\lambda} \quad = ([0.16,0.09], [0.77,0.66]).$$

2. $\lambda P \quad = \left[\sqrt{\left(1-\left(1-t_p\right)^2\right)^{\lambda}}, \left(1-f_p\right)^{\lambda}\right]$

$$= \left(\left[\sqrt{1-(1-0.4^2)^2}, \sqrt{1-(1-0.3^2)^2}\right], [0.6^2, 0.5^2]\right)$$

$$= \left(\left[\sqrt{1-(1-0.16)^2}, \sqrt{1-(1-0.09)^2}\right], [0.36,0.25]\right)$$

$$= \left(\left[\sqrt{1-(0.84)^2}, \sqrt{1-(0.91)^2}\right], [0.36,0.25]\right)$$

$$= \left(\left[\sqrt{1-0.7056}, \sqrt{1-0.8281}\right], [0.36,0.25]\right)$$

$$= \left(\left[\sqrt{0.2944}, \sqrt{0.1719}\right], [0.36,0.25]\right)$$

$$\lambda P = ([0.54,0.41], [0.36,0.25])$$

3. Given that $P_1 = \{< x, [0.5,0.7], [0.4,0.5] >\}$ and $P_2 = \{< x, [0.5,0.6], [0.3,0.4] >\}$

$$P_1 \oplus P_2 = \left[\sqrt{\left(t_{p1}\right)^2 + \left(t_{p2}\right)^2 - {t_{p1}}^2 . {t_{p2}}^2}, \left(1-f_{p1}\right)\left(1-f_{p2}\right)\right]$$

$$= \left(\left[\sqrt{0.5^2 + 0.5^2 - 0.5^2 . 0.5^2}, (0.7)(0.6)\right], \left[\sqrt{0.4^2 + 0.3^2 - 0.4^2 . 0.3^2}, (0.5)(0.4)\right]\right)$$

$$= \left(\left[\sqrt{0.25 + 0.25 - (0.25)(0.25)}, 0.42\right], \left[\sqrt{0.16 + 0.09 - (0.16)(0.09)}, 0.20\right]\right)$$

$$= \left(\left[\sqrt{0.50 - 0.0625}, 0.42\right], \left[\sqrt{0.25 - 0.0144}, 0.20\right]\right)$$

$$= \left(\left[\sqrt{0.4375}, 0.42\right], \left[\sqrt{0.2356}, 0.20\right]\right)$$

$$= ([0.66, 0.42], [0.49, 0.20])$$

Hence

$P_1 \oplus P_2 = ([0.66,0.42],[0.49,0.20])$.

4. Given that $P_1 = \{< x, [0.5,0.7], [0.4,0.5] >\}$ and $P_2 = \{< x, [0.5,0.6], [0.3,0.4] >\}$

$$P_1 \otimes P_2 = \left[\sqrt{\left(1 - f_{p1}\right)^2 + \left(1 - f_{p2}\right)^2 - \left(1 - f_{p1}\right)^2 . \left(1 - f_{p2}\right)^2}, \left(t_{p1}\right)\left(t_{p2}\right)\right]$$

$$= \left(\left[\sqrt{0.7^2 + 0.6^2 - (0.7)^2 . (0.6)^2}, (0.5)(0.5)\right], \left[\sqrt{0.5^2 + 0.4^2 - (0.5)^2 (0.4)^2}, (0.4)(0.53)\right]\right)$$

$$= \left(\left[\sqrt{0.49 + 0.36 - (0.49)(0.36)}, (0.25)\right], \left[\sqrt{0.25 + 0.16 - (0.25)(0.16)}, 0.12\right]\right)$$

$$= \left(\left[\sqrt{0.85 - 0.1764}, 0.25\right], \left[\sqrt{0.41 - 0.04}, 0.12\right]\right)$$

$$= \left(\left[\sqrt{0.6736}, 0.25\right], \left[\sqrt{0.37}, 0.12\right]\right)$$

$$= ([0.82, 0.25], [0.61, 0.12])$$

Hence $P_1 \otimes P_2 = ([0.82,0.25],[0.61,0.12])$.

Theorem: 3.1.3

Let $P = \langle a, b\rangle$ and $P_1 = \langle a_1, b_1\rangle, P_2 = \langle a_2, b_2\rangle$ be the Pythagorean vague sets and $\lambda > 0$ then

(i) $P_1 \oplus P_2 = P_2 \oplus P_1$

(ii) $P_1 \otimes P_2 = P_2 \otimes P_1$

(iii) $\lambda(P_1 \oplus P_2) = \lambda P_1 \oplus \lambda P_2$

Proof:

$(i)\ P_1 \oplus P_2 = \left[\sqrt{(a_1)^2 + (a_2)^2 - a_1{}^2.a_2{}^2}, (b_1)(b_2)\right]$

$= \left[\sqrt{(a_2)^2 + (a_1)^2 - a_2{}^2.a_1{}^2}, (b_2)(b_1)\right]$

$= P_2 \oplus P_1$

Therefore, $P_1 \oplus P_2 = P_2 \oplus P_1$.

$(ii)\ P_1 \otimes P_2 = \left[\sqrt{(b_1)^2 + (b_2)^2 - (b_1)^2.(b_2)^2}, (a_1)(a_2)\right]$

$= \left[\sqrt{(b_2)^2 + (b_1)^2 - (b_2)^2.(b_1)^2}, (a_2)(a_1)\right]$

$= P_2 \otimes P_1$

Therefore, $P_1 \otimes P_2 = P_2 \otimes P_1$.

(iii) **LHS:**

$\lambda(P_1 \oplus P_2) = \lambda\left[\sqrt{(a_1)^2 + (a_2)^2 - a_1{}^2.a_2{}^2}, (b_1)(b_2)\right]$

$= \left[\sqrt{1 - (1 - (a_1{}^2 + a_2{}^2 - a_1{}^2.a_2{}^2))^\lambda}, (b_1 b_2)^\lambda\right]$

RHS:

$\lambda P_1 \oplus \lambda P_2 = \left[\sqrt{1 - (1 - a_1{}^2)^\lambda}, (b_1)^\lambda\right] \oplus \left[\sqrt{1 - (1 - a_2{}^2)^\lambda}, (b_2)^\lambda\right]$

$=$

$$\left[\sqrt{1-(1-{a_1}^2)^\lambda+1-(1-{a_2}^2)^\lambda-\left(1-(1-{a_1}^2)^\lambda\right).(1-(1-{a_2}^2)^\lambda)},((b_1)(b_2))^\lambda\right]$$

$$=\left[\sqrt{\begin{matrix}1-(1-{a_1}^2)^\lambda+1-(1-{a_2}^2)^\lambda-\\ \left[1-(1-{a_2}^2)^\lambda-(1-{a_1}^2)^\lambda+(1-{a_1}^2)^\lambda(1-{a_2}^2)^\lambda\right]\end{matrix}},(b_1b_2)^\lambda\right]$$

$$=\left[\sqrt{\begin{matrix}1-(1-{a_1}^2)^\lambda+1-(1-{a_2}^2)^\lambda-\\ 1+(1-{a_2}^2)^\lambda+(1-{a_1}^2)^\lambda-(1-{a_1}^2)^\lambda(1-{a_2}^2)^\lambda\end{matrix}},(b_1b_2)^\lambda\right]$$

$$=\left[\sqrt{1-(1-{a_1}^2)^\lambda(1-{a_2}^2)^\lambda},(b_1b_2)^\lambda\right]$$

$$=\left[1-[(1-{a_1}^2)(1-{a_2}^2)]^\lambda\right]$$

$$=\left[\sqrt{1-(1-a_2^2-a_1^2+a_1^2a_2^2)^\lambda,}(b_1b_2)^\lambda\right]$$

$$=\left[\sqrt{1-(1-({a_1}^2+a_2^2-a_1^2.a_2^2))^\lambda,}(b_1b_2)^\lambda\right]$$

$$=\lambda(P_1\oplus P_2)$$

Therefore, $\lambda(P_1\oplus P_2)=\lambda P_1\oplus \lambda P_2$.

Definition: 3.1.4

Let $P=\{<x,(a,b)>|x\in X\}$ and $P_1=\{<x,(a_1,b_1)>|x\in X\}$, $P_2=\{<x,(a_2,b_2)>|x\in X\}$ be the Pythagorean vague sets and $\lambda\geq 0$, then

$*\ P_1\cup P_2=\left(max(a_1,a_2),max(b_1,b_2)\right)$

$*\ P_1\cap P_2=\left(min(a_1,a_2),min(b_1,b_2)\right)$

$* P^C = (b, a)$

Theorem: 3.1.5

Let $P = \{< x, (a, b) > | x \in X\}$ and $P_1 = \{< x, (a_1, b_1) > | x \in X\}$,

$P_2 = \{< x, (a_2, b_2) > | x \in X\}$ be the Pythagorean vague sets and $\lambda \geq 0$, then

(i) $P_1 \cup P_2 = P_2 \cup P_1$

(ii) $P_1 \cap P_2 = P_2 \cap P_1$

(iii) $\lambda(P_1 \cup P_2) = \lambda P_1 \cup \lambda P_2$

Proof:

$(i)\ P_1 \cup P_2 = [max(a_1, a_2), max(b_1, b_2)]$

$= [max(a_2, a_1), max(b_2, b_1)]$

$= P_2 \cup P_1$

Hence, $P_1 \cup P_2 = P_2 \cup P_1$

$(ii)\ P_1 \cap P_2 = [min(a_1, a_2), min(b_1, b_2)]$

$= [min(a_1, a_2), min(b_1, b_2)]$

$= P_1 \cap P_2$

Hence, $P_1 \cap P_2 = P_2 \cap P_1$

(iii) **LHS:**

$\lambda(P_1 \cup P_2) = \lambda[max(a_1, a_2), max(b_1, b_2)]$

$$= \left[\sqrt{1 - \left(1 - max({a_1}^2, {a_2}^2)\right)^{\lambda}}, max\left({b_1}^{\lambda}, {b_2}^{\lambda}\right)\right]$$

RHS:

Given that $P_1 = \{< x, (a_1, b_1) > | x \in X\}$, $P_2 = \{< x, (a_2, b_2) > | x \in X\}$

$$\lambda P_1 = \left[\sqrt{1 - (1 - {a_1}^2)^\lambda}, {b_1}^\lambda\right]$$

$$\lambda P_2 = \left[\sqrt{1 - (1 - {a_1}^2)^\lambda}, {b_2}^\lambda\right]$$

$$\lambda P_1 \cup \lambda P_2 = \left[\sqrt{1 - (1 - {a_1}^2)^\lambda}, {b_1}^\lambda\right] \cup \left[\sqrt{1 - (1 - {a_1}^2)^\lambda}, {b_2}^\lambda\right]$$

$$= \left[\sqrt{1 - \left(1 - max({a_1}^2, {a_2}^2)\right)^\lambda}, max\left({b_1}^\lambda, {b_2}^\lambda\right)\right]$$

$$= \lambda(P_1 \cup P_2)$$

Hence, $\lambda(P_1 \cup P_2) = \lambda P_1 \cup \lambda P_2$

Theorem: 3.1.6

Let $P_1 = \{< x, (a_1, b_1) > | x \in X\}$, $P_2 = \{< x, (a_2, b_2) > | x \in X\}$ be the Pythagorean vague sets, then

i. ${P_1}^C \cup {P_2}^C = (P_1 \cup P_2)^C$

ii. ${P_1}^C \oplus {P_2}^C = (P_1 \otimes P_2)^C$

Proof:

(i) **LHS:**

$${P_1}^C \cup {P_2}^C = (b_1, a_1) \cup (b_2, a_2)$$

$$= (max(b_1, b_2), max(a_1, a_2))$$

RHS:

$$(P_1 \cup P_2)^C = (max(a_1, a_2), max(b_1, b_2))^C$$

$$= (max(b_1, b_2), max(a_1, a_2))$$

$$= {P_1}^C \cup {P_2}^C$$

Hence, ${P_1}^C \cup {P_2}^C = (P_1 \cup P_2)^C$

(ii) **LHS:**

$${P_1}^C \oplus {P_2}^C = (b_1, a_1) \oplus (b_2, a_2)$$

$$= \left[\sqrt{{b_1}^2 + {b_2}^2 - {b_1}^2{b_2}^2}, a_1. a_2\right]$$

RHS:

$$(P_1 \otimes P_2)^C = \left[\sqrt{{b_1}^2 + {b_2}^2 - {b_1}^2{b_2}^2}, a_1. a_2\right]$$

$$= {P_1}^C \oplus {P_2}^C$$

Hence, ${P_1}^C \oplus {P_2}^C = (P_1 \otimes P_2)^C$

3.2 Distance for Pythagorean Vague Sets

Definition: 3.2.1

Let $A = \{< x, (t_A(x), 1 - f_A(x)) > | x \in X\}$ and

$B = \{< x, (t_B(x), 1 - f_B(x)) > | x \in X\}$ be Pythagorean vague sets in X.

1. Hamming Distance

$$H_{PV}\ (A,B)=\frac{1}{2}\sum_{i=1}^{n}(|t_A(x_i)-t_B(x_i)|+|[1-f_A(x_i)]-[1-f_B(x_i)]|$$
$$+|\pi_A(x_i)-\pi_B(x_i)|)$$

2. Normalized Hamming Distance

$$H_{PV}\ (A,B)=\frac{1}{2n}\sum_{i=1}^{n}(|t_A(x_i)-t_B(x_i)|+|[1-f_A(x_i)]-[1-f_B(x_i)]|$$
$$+|\pi_A(x_i)-\pi_B(x_i)|)$$

3. Euclidean Distance

$E_{PV}\ (A,B)$

$$=\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(\left(t_A(x_i)-t_B(x_i)\right)^2+([1-f_A(x_i)]-[1-f_B(x_i)])^2+\left(\pi_A(x_i)-\pi_B(x_i)\right)^2\right)}$$

Where $\pi_A(x_i)=1-t_A(x_i)-\left(1-f_A(x_i)\right),\ \pi_B(x_i)=1-t_B(x_i)-\left(1-f_B(x_i)\right)$ be the degree of indeterminacy of x in A and B.

4. Normalized Euclidean Distance

$NE_{PV}\ (A,B)$

$$=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left(\left(t_A(x_i)-t_B(x_i)\right)^2+([1-f_A(x_i)]-[1-f_B(x_i)])^2+\left(\pi_A(x_i)-\pi_B(x_i)\right)^2\right)}$$

Distance measures satisfies the following conditions

$$0\le H_{VS}(A,B)\le 2n$$

$$0\le N_{VS}(A,B)\le 2$$

$0 \leq E_{VS}(A,B) \leq \sqrt{2n}$

$0 \leq NE_{VS}(A,B) \leq \sqrt{2}$

Example 3.2.2:

Let $X = \{1,2\}$ and let A and B be the Pythagorean vague set in X defined by $A = \{< x, (0.1,0.4), (0.3,0.6) >\}, B = \{< x, (0.4,0.6), (0.5,0.7) >\}$

1. Hamming Distance

$$H_{PV}\,(A,B) = \frac{1}{2}\sum_{i=1}^{n}(|t_A(x_i) - t_B(x_i)| + |[1 - f_A(x_i)] - [1 - f_B(x_i)]| + |\pi_A(x_i) - \pi_B(x_i)|)$$

$$= \tfrac{1}{2}[|0.1 - 0.4| + |0.3 - 0.5| + |0.4 - 0.6| + |0.6 - 0.7| + |(1 - 0.1 - 0.4) - (1 - 0.4 - 0.6)| + |(1 - 0.3 - 0.6) - (1 - 0.5 - 0.7)|]$$

$$= \tfrac{1}{2}[0.3 + 0.2 + 0.2 + 0.1 + |0.5 - 0| + |0.1 - (-0.2)|]$$

$$= \tfrac{1}{2}[0.8 + 0.5 + 0.3]$$

$$= \tfrac{1}{2}(1.6) = 0.8$$

$H_{PV}\,(A,B) = 0.8$

2. Normalized Hamming Distance

$$H_{PV}\,(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|t_A(x_i) - t_B(x_i)| + |[1 - f_A(x_i)] - [1 - f_B(x_i)]| + |\pi_A(x_i) - \pi_B(x_i)|)$$

$$= \tfrac{1}{4}(1.6) = 0.4$$

$H_{PV}\ (A, B) = 0.4$

3. Euclidean Distance

$\mathrm{E_{PV}}\ (\mathrm{A, B}) =$

$$\sqrt{\tfrac{1}{2}\sum_{i=1}^{n}\left(\left(t_A(x_i) - t_B(x_i)\right)^2 + ([1 - f_A(x_i)] - [1 - f_B(x_i)])^2 + \left(\pi_A(x_i) - \pi_B(x_i)\right)^2\right)}$$

$$= \sqrt{\frac{1}{2}[(0.3)^2 + (0.2)^2 + (0.2)^2 + (0.1)^2 + (0.5)^2]}$$

$$= \sqrt{\frac{1}{2}(0.52)} = \sqrt{0.26} = 0.17$$

$\mathrm{E_{PV}}\ (\mathrm{A, B}) = 0.17$

4. Normalized Euclidean Distance

$NE_{PV}\ (A, B)$

$$= \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left(\left(t_A(x_i) - t_B(x_i)\right)^2 + ([1 - f_A(x_i)] - [1 - f_B(x_i)])^2 + \left(\pi_A(x_i) - \pi_B(x_i)\right)^2\right)}$$

$$= \sqrt{\frac{1}{4}(0.52)} = \sqrt{0.13} = 0.36$$

$NE_{PV}\ (A, B) = 0.36$

Distance in Pythagorean vague sets should be calculated by taking truth membership and false membership function it also satisfies the following conditions

$$0 \le H_{PV}(A, B) \le 2n$$

$$0 \le 0.8 \le 4$$

$$0 \le H_{PV}(A, B) \le 2n$$

$$0 \le 0.4 \le 2$$

$$0 \le E_{PV}(A,B) \le \sqrt{2n}$$

$$0 \le 0.51 \le 2$$

$$0 \le NE_{PV}(A,B) \le \sqrt{2}$$

$$0 \le 0.36 \le 1.$$

4. CORRELATION COEFFICIENT OF VAGUE SETS

4.1 CORRELATION OF VAGUE SETS

Let $X = \{x_1, x_2, \dots, x_n\}$ be the finite universal set and $A, B \in VS(x)$ be given by A= $\{\langle x, [t_A(x), 1 - f_A(x)]\rangle | x \in X\}$, $B = \{\langle x, [t_B(x), 1 - f_B(x)]\rangle | x \in X\}$.

And the length of the vague values are given by

$$\pi_A(x) = 1 - t_A(x) - f_A(x),$$

$$\pi_B(x) = 1 - t_B(x) - f_B(x)$$

Now for each $A \in VS(x)$, the informational vague energy of A is defined as follows:

$$E_{VS}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[t_A^2(x_i) + \left(1 - f_A(x_i)\right)^2 + \pi_A^2(x_i)\right]$$

And for each $B \in VS(x)$, the informational vague energy of B is defined as follows:

$$E_{VS}(B) = \frac{1}{n}\sum_{i=1}^{n}\left[t_B^2(x_i) + \left(1 - f_B(x_i)\right)^2 + \pi_B^2(x_i)\right]$$

The correlation of A and B is given by the formula:

$$C_{VS}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left[t_A(x_i)t_B(x_i) + \left(1 - f_A(x_i)\right)\left(1 - f_B(x_i)\right) + \pi_A(x_i)\pi_B(x_i)\right]$$

Furthermore, the correlation coefficient of A and B is defined by the formula:

$$K_{VS}(A, B) = \frac{C_{VS}(A, B)}{\sqrt{E_{VS}(A)\,.\,E_{VS}(B)}}$$

Where $0 \le K_{VS}(A, B) \le 1$.

Proposition: 4.1.1

For $A, B \in VS(X)$, the following are true:

(i) $0 \leq C_{VS}(A, B) \leq 1$,

(ii) $C_{VS}(A, B) = C_{VS}(B, A)$,

(iii) $K_{VS}(A, B) = K_{VS}(B, A)$,

Theorem: 4.1.2

For $A, B \in VS(X)$, then $0 \leq K_{VS}(A, B) \leq 1$.

Proof:

Since $C_{VS}(A, B) \geq 0$, it can be proved that $K_{VS}(A, B) \leq 1$.

For any arbitrary real number ξ, the following inequality is true:

$$0 \leq \frac{1}{\mathrm{n}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[\left(t_A(x_i) - \xi t_B(x_i)\right)^2 + \left(f_A(x_i) - \xi f_B(x_i)\right)^2 + \left(\pi_{\mathrm{A}}(\mathrm{x_i}) - \xi \pi_{\mathrm{B}}(\mathrm{x_i})\right)^2 \right]$$

$$= \sum_{i=1}^{n} \Big[\left(t^2{}_A(x_i) + f^2{}_A(x_i) + \pi^2{}_A(x_i)\right) - 2\xi\big(t_A(x_i)t_B(x_i) + f_A(x_i)f_B(x_i) + \pi_A(x_i)\pi_B(x_i)\big) + \xi^2 \left(t^2{}_B(x_i) + f^2{}_B(x_i) + \pi^2{}_B(x_i)\right) \Big]$$

Hence ,

$$\left\{\sum_{i=1}^{n} \left[\left(t_A(x_i)t_B(x_i) + f_A(x_i)f_B(x_i) + \pi_A(x_i)\pi_B(x_i)\right)\right]\right\}^2 \leq \left(\sum_{i=1}^{n} \left\{\left(t^2{}_A(x_i) + f^2{}_A(x_i) + \pi^2{}_A(x_i)\right)\right\} \times \sum_{i=1}^{n} \left\{\left(t^2{}_B(x_i) + f^2{}_B(x_i) + \pi^2{}_B(x_i)\right)\right\}\right)$$

The above inequality can be written as:

$$\frac{\left\{\sum_{i=1}^{n}\left[\left(t_A(x_i)t_B(x_i)+f_A(x_i)f_B(x_i)+\pi_A(x_i)\pi_B(x_i)\right)\right]\right\}^2}{\sum_{i=1}^{n}\left\{\left(t^2{}_A(x_i)+f^2{}_A(x_i)+\pi^2{}_A(x_i)\right)\right\}\times\sum_{i=1}^{n}\left\{\left(t^2{}_B(x_i)+f^2{}_B(x_i)+\pi^2{}_B(x_i)\right)\right\}}$$

$$\leq 1$$

Therefore $\frac{[C_{VS}(A,B)]^2}{E_{VS}(A)\,.\,E_{VS}(B)} \leq 1$

Hence $K_{VS}(A,B) = \frac{C_{VS}(A,B)}{\sqrt{E_{VS}(A)\,.\,E_{VS}(B)}} \leq 1$

Theorem: 4.1.3

$K_{VS}(A,B) = 1 \Leftrightarrow A = B$

Proof:

Considering the inequality in the proof of theorem 4.1.2, the equality holds if and only if the following are true:

1. $t_A(x_i) = \xi t_B(x_i)$
2. $f_A(x_i) = \xi f_B(x_i)$
3. $\pi_A(x_i) = \xi \pi_B(x_i)$, for some positive real ξ.

As $t_A(x_i) + f_A(x_i) + \pi_A(x_i) = t_B(x_i) + f_B(x_i) + \pi_B(x_i) = 1$,

Adding 1, 2 and 3, we get

$$t_A(x_i) + f_A(x_i) + \pi_A(x_i) = \xi[t_B(x_i) + f_B(x_i) + \pi_B(x_i)]$$

As $t_A(x_i) + f_A(x_i) + \pi_A(x_i) = t_B(x_i) + f_B(x_i) + \pi_B(x_i) = 1$, which gives that

$1 = \xi(1)$

$\Rightarrow \xi = 1$

Therefore, $A = B$.

Theorem: 4.1.4

$C_{VS}(A,B) = 0 \Leftrightarrow A$ and B are non-fuzzy sets and satisfy the condition

$t_A(x_i) + t_B(x_i) = 1 \; or \; f_A(x_i) + f_B(x_i) = 1 \; or \; \pi_A(x_i) + \pi_B(x_i) = 1, \forall x_i \in X.$

Proof:

For all $x_i \in X$, we have, $\left(t_A(x_i)t_B(x_i) + f_A(x_i)f_B(x_i) + \pi_A(x_i)\pi_B(x_i)\right) \geq 0$.

If $C_{VS}(A,B) = 0$ for all $x_i \in X$, then it should be that:

$t_A(x_i).\, t_B(x_i) = 0 \,, f_A(x_i).\, f_B(x_i) = 0 \,, and \; \pi_A(x_i).\, \pi_B(x_i) = 0$

If $t_A(x_i) = 1$

Then $t_B(x_i) = 0 \; and \; f_A(x_i) = \pi_A(x_i) = 0$

If $t_B(x_i) = 1$

Then $t_A(x_i) = 0 \; and \; f_B(x_i) = \pi_B(x_i) = 0$

Hence, $t_A(x_i) + t_B(x_i) = 1$

Conversely, when A and B are non-fuzzy sets and $t_A(x_i) + t_B(x_i) = 1$,

If $t_A(x_i) = 1$

Then $t_B(x_i) = 0 \; and \; f_A(x_i) = \pi_A(x_i) = 0$

If $t_B(x_i) = 1$

Then$t_A(x_i) = 0 \; and \; f_B(x_i) = \pi_B(x_i) = 0$

Therefore, $C_{VS}(A,B) = 0$

The cases $f_A(x_i) + f_B(x_i) = 1 \; and \; \pi_A(x_i) + \pi_B(x_i) = 1$ can be proved similarly.

Theorem: 4.1.5

$C_{VS}(A, A) = 1 \Leftrightarrow A$ is a non-fuzzy set.

Proof:

If A is a non-fuzzy set, then $C_{VS}(A, A) = 1$ is obvious.

Conversely, it can be proved by the method of contradiction.

Assume A is a non-fuzzy set.

Then $0 \leq t_A(x_i) < 1, 0 \leq f_A(x_i) < 1 \; and \; 0 \leq \pi_A(x_i) < 1 \; for \; some \; x_i$

Hence $t_A^2(x_i) + f_A^2(x_i) + \pi_A^2(x_i) < 1$.

Then $C_{VS}(A, A) = E_{VS}(A)$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(t_A^2(x_i) + \left(1 - f_A(x_i)\right)^2 + \pi_A^2(x_i) \right) < 1.$$

$C_{VS}(A, A) < 1$.

This is a contradiction. Hence A is a non-fuzzy set.

4.2 Numerical Example for Correlation of Vague Sets

In a sensor database application, suppose there are a set of ten sensors in a testing region $\{S_1, S_2, S_3, \ldots, S_{10}\}$. Let there be ten corresponding measurements, $\{20,22,20,21,20,-,20,20,-,20\}$ at a certain time t. Hence $" - "$ means that the sensor data is not reachable/ accessible at time t (i.e. we have six 20, one 21, one 22 and two missing values). Now, the results can be formalized to a vague set V as follows. There are six occurrences of 20, but two values (21 and 22) are against it.

There are also two missing values (neutral). Hence the true membership t is 0.6 and the false membership f is 0.2

(that is $1 - f = 0.8$). Thus, the vague membership value is $[0.6,0.8]$ for 20. Similarly, the vague membership value is $[0.1,0.3]$ for 21 and $[0.1,0.3]$ for 22. Combining these results, one gets the VS,

$$V_t = [0.6,0.8]/20 + [0.1,0.3]/21 + [0.1,0.3]/22.$$

Equally, one can have the IFS,

$$A_t = \{\langle 20,0.6,0.2\rangle, \langle 21,0.1,0.7\rangle, \langle 22,0.1,0.7\rangle\}$$

Data sets with vague values

Vague set A	Vague set B
$[0.313,0.628]$	$[0.411,0.536]$
$[0.235,0.712]$	$[0.316,0.481]$
$[0.183,0.697]$	$[0.288,0.663]$
$[0.439,0.511]$	$[0.387,0.400]$
$[0.299,0.600]$	$[0.149,0.811]$
$[0.199,0.723]$	$[0.412,0.523]$
$[0.418,0.532]$	$[0.319,0.611]$
$[0.315,0.489]$	$[0.272,0.593]$
$[0.163,0.700]$	$[0.313,0.568]$
$[0.296,0.483]$	$[0.400,0.513]$

The correlation coefficient between VS is calculated as follows:

$$E_{VS}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[t_A^2(x_i) + \left(1 - f_A(x_i)\right)^2 + \pi_A^2(x_i)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[t_A^2(x_i) + \left(1 - f_A(x_i)\right)^2 + \left(1 - t_A(x_i) - f_A(x_i)\right)^2\right]$$

$$= \frac{1}{10}\{[(0.313)^2 + (0.628)^2 + (0.628 - 0.313)^2]$$

$$+ [(0.235)^2 + (0.712)^2 + (0.712 - 0.235)^2]$$

$$+ [(0.183)^2 + (0.697)^2 + (0.697 - 0.183)^2]$$

$$+ [(0.439)^2 + (0.511)^2 + (0.511 - 0.439)^2]$$

$$+ [(0.299)^2 + (0.600)^2 + (0.600 - 0.299)^2]$$

$$+ [(0.199)^2 + (0.723)^2 + (0.723 - 0.199)^2]$$

$$+ [(0.418)^2 + (0.532)^2 + (0.532 - 0.418)^2]$$

$$+ [(0.315)^2 + (0.489)^2 + (0.489 - 0.315)^2]$$

$$+ [(0.163)^2 + (0.700)^2 + (0.700 - 0.163)^2]$$

$$+ [(0.296)^2 + (0.483)^2 + (0.483 - 0.296)^2]\}$$

$$= \frac{1}{10}[0.591578 + 0.789698 + 0.783494 + 0.459926 +$$

$$0.540002 + 0.836906 + 0.470744 + 0.368622 + 0.804938 + 0.355874]$$

$$= \frac{6.001782}{10}$$

$$= 0.6002$$

$$E_{VS}(A) = 0.6002$$

$$E_{VS}(B) = \frac{1}{n}\sum_{i=1}^{n}\left[t_B^2(x_i) + \left(1 - f_B(x_i)\right)^2 + \pi_B^2(x_i)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[t_B^2(x_i) + \left(1 - f_B(x_i)\right)^2 + \left(1 - t_B(x_i) - f_B(x_i)\right)^2\right]$$

$$\begin{aligned}= \frac{1}{10}\{&[(0.411)^2 + (0.536)^2 + (0.536 - 0.411)^2] \\ &+ [(0.316)^2 + (0.481)^2 + (0.481 - 0.316)^2] \\ &+ [(0.288)^2 + (0.663)^2 + (0.663 - 0.288)^2] \\ &+ [(0.387)^2 + (0.400)^2 + (0.400 - 0.387)^2] \\ &+ [(0.149)^2 + (0.811)^2 + (0.811 - 0.149)^2] \\ &+ [(0.412)^2 + (0.523)^2 + (0.523 - 0.412)^2] \\ &+ [(0.319)^2 + (0.611)^2 + (0.611 - 0.319)^2] \\ &+ [(0.272)^2 + (0.593)^2 + (0.593 - 0.272)^2] \\ &+ [(0.313)^2 + (0.568)^2 + (0.568 - 0.313)^2] \\ &+ [(0.400)^2 + (0.513)^2 + (0.513 - 0.400)^2]\}\end{aligned}$$

$$\begin{aligned}= \frac{1}{10}[&0.471842 + 0.358442 + 0.663138 + 0.309938 + 1.118166 \\ &+ 0.455594 + 0.560346 + 0.528674 + 0.485618 \\ &+ 0.435938]\end{aligned}$$

$$= \frac{5.387696}{10}$$

$$= 0.5388$$

$$E_{VS}(B) = 0.5388$$

$$C_{VS}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[t_A(x_i)t_B(x_i) + \left(1 - f_A(x_i)\right)\left(1 - f_B(x_i)\right) + \pi_A(x_i)\pi_B(x_i)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\Big[t_A(x_i)t_B(x_i) + \big(1 - f_A(x_i)\big)\big(1 - f_B(x_i)\big)$$

$$+ \big(1 - t_A(x_i) - f_A(x_i)\big)\big(1 - t_B(x_i) - f_B(x_i)\big)\Big]$$

$$= \frac{1}{10}\{[(.313)(.411) + (.628)(.536) + (.628 - .313)(.536 - .411)]$$

$$+ [(.235)(.316) + (.712)(.481)$$

$$+ (.712 - .235)(.481 - .316)]$$

$$+ [(.183)(.288) + (.697)(.663)$$

$$+ (.697 - .183)(.663 - .288)]$$

$$+ [(.439)(.387) + (.511)(.400)$$

$$+ (.511 - .439)(.400 - .387)]$$

$$+ [(.299)(.149) + (.600)(.811)$$

$$+ (.600 - .299)(.811 - .149)]$$

$$+ [(.199)(.412) + (.723)(.523)$$

$$+ (.723 - .199)(.523 - .412)]$$

$$+ [(.418)(.319) + (.532)(.611)$$

$$+ (.532 - .418)(.611 - .319)]$$

$$+ [(.315)(.272) + (.489)(.593)$$

$$+ (.489 - .315)(.593 - .272)]$$

$$+ [(.163)(.313) + (.700)(.568)$$

$$+ (.700 - .163)(.568 - .313)]$$

$$+ [(.296)(.400) + (.483)(.513)$$

$$+ (.483 - .296)(.513 - .400)]\}$$

$$= \frac{1}{10}[.504626 + .495437 + .707565 + .375229 + .730413 + .518281$$

$$+ .491682 + .431511 + .585554 + .38731]$$

$$= \frac{5.211207}{10}$$

$$= 0.5211$$

$C_{VS}(A, B) = 0.5211$

$$K_{VS}(A, B) = \frac{C_{VS}(A, B)}{\sqrt{E_{VS}(A). E_{VS}(B)}}$$

$$= \frac{0.5211}{\sqrt{(0.6002)(0.5388)}}$$

$$= \frac{0.5211}{\sqrt{0.3234}}$$

$$= \frac{0.5211}{0.5687}$$

$$K_{VS}(A, B) = 0.9163$$

where $0 \leq K_{VS}(A, B) \leq 1$.

5. SOME APPLICATIONS IN VAGUE SETS

MEDICAL DIAGNOSIS:

The field of medicine is one of the most fruitful and interesting area of application of vague set theory. In this chapter, Bhargavi and Eswarlal (1) study a novel application of vague set in a medical diagnosis by applying the normalized Euclidean distance method to measure the distance between each IT workers and each health problem. In this connection, they have taken a survey from friends and relatives, which are working in software industries in India. The most common problems, which are identified in IT professionals are namely Musculoskeletal Discomfort, Computer Vision Syndrome, some of them were staying away from their family and their regular source of meal was hotel, overweight, were not satisfied with a time they spent with their family etc. In the present study, particularly about the Stress, Ulcer, Vision problem, Spinal problem and Blood pressure and finally, they obtain the solution, which determines the health problem of the IT worker.

Application of vague sets in medical diagnosis:

They have taken survey from the IT professional in which thsey are facing many problems in which we have chosen the most common problems. Among the workers they have consider only four workers, let the workers be W_1, W_2, W_3, W_4 are denoted by the set W= $\{W_1, W_2, W_3, W_4\}$ and the set of symptoms

S=$\{Head\ ache, acidity, burning\ eyes, back\ pain\ and\ depression\}$.

Let the set of health problems be

H=$\{Stress, ulcer, vision\ problem, spinal\ problem\ and\ blood\ pressure\}$.

Table 1: Represents the workers and their symptoms

	Head ache	Acidity	Burning eyes	Back pain	Depression
W_1	(0.9,0.1)	(0.7,0.2)	(0.1,0.9)	(0.7,0.2)	(0.2,0.7)
W_2	(0,0.7)	(0.4,0.5)	(0.6,0.2)	(0.2,0.7)	(0.1,0.2)
W_3	(0.7,0.1)	(0.7,0.1)	(0,0.5)	(0.1,0.7)	(0,0.6)
W_4	(0.5,0.1)	(0.4,0.3)	(0.4,0.5)	(0.8,0.2)	(0.3,0.4)

Table 2: Represents related health problems

	Stress	Ulcer	Vision Problem	Spinal Problem	Blood Pressure
Head ache	(0.3,0)	(0,0.6)	(0.2,0.2)	(0.2,0.8)	(0.2,0.9)
Acidity	(0.3,0.5)	(0.2,0.6)	(0.5,0.2)	(0.1,0.5)	(0,0.7)
Burning eyes	(0.2,0.8)	(0,0.8)	(0.1,0.7)	(0.7,0)	(0.2,0.8)
Back pain	(0.7,0.3)	(0.5,0)	(0.2,0.6)	(0.1,0.7)	(0.1,0.8)
Depression	(0.2,0.6)	(0.1,0.8)	(0.1,0.9)	(0.2,0.7)	(0.8,0.1)

Using definition 1.13 above to calculate the distance between each workers and each health problem with reference to the symptoms, we get the table below.

Table3: Workers vs Health Problem

	Stress	Ulcer	Vision Problem	Spinal Problem	Blood Pressure
W_1	0.2569	0.3987	0.3225	0.5666	0.5771
W_2	0.4111	0.4147	0.3728	0.2145	0.3633
W_3	0.3435	0.4505	0.2191	0.4528	0.5263
W_4	0.1732	0.3478	0.3256	0.4347	0.4868

From the above table3, the shortest distance gives the health problems of four IT workers,

This is represented by the chart diagram given as below:

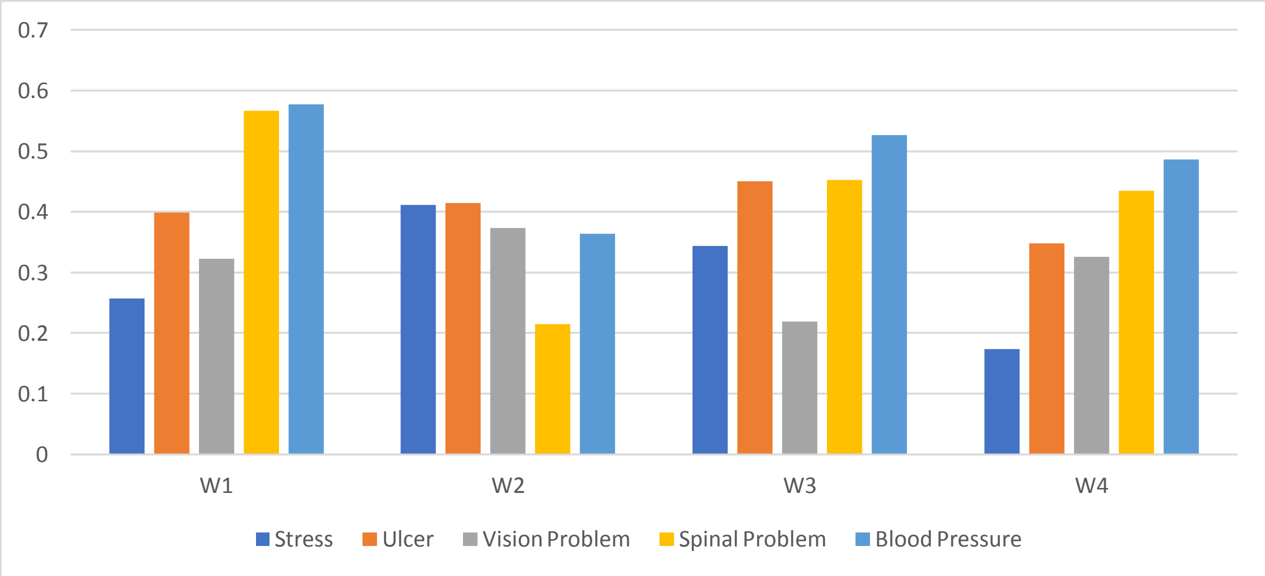

Fig1: Graph of Workers Vs Health Problems

CONCLUSION:

Overall the normalized Euclidean distance method gives the final result of four IT workers health problems that is from Table 3, we see that-

(i) The shortest value of W_1 is 0.2569 and therefore W_1 faces stress.

(ii) The shortest value of W_2 is 0.2145 and therefore W_2 faces spinal problem.

(iii) The shortest value of W_3 is 0.2191 and therefore W_3 faces vision problem.

(iv) The shortest value of W_4 is 0.1732 and therefore W_4 faces stress.

A STUDY ON FUZZY SOFT IDEALS IN NEAR RINGS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

R. ANGEL

Reg.No: 19SPMT02

Under the guidance of

Ms. M. KANAGA , M.Sc., B.Ed., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON FUZZY SOFT IDEALS IN NEAR RINGS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **R. ANGEL (Reg. No: 19SPMT02)**

M. Kanaga
Signature of the Guide

Signature of the Coordinator

Joyce
Signature of the Director
Director
Self Supporting Courses
St. Mary's College (Autonor
Thoothukudi - 628 001.

Lucia Rose
Signature of the Principal
Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON FUZZY SOFT IDEALS IN NEAR RINGS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. M. Kanaga M.Sc., B.Ed., SET.,** Assistant Professor, Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi.**

R. Angel

Signature of the Student

Station: Thoothukudi

Date: 10 · 04 · 2021

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Place: Thoothukudi

Date: 10·04·2021

CONTENT

CHAPTER I

PRELIMINARIES

Definition : 1.1

A non-empty set R together with two binary operations + and . are called ring. Which satisfies the following condition.

(i) $(R, +)$ is an abelian group.

(ii) $(R, .)$ is an associative.

(iii) Multiplication is distributive over addition.

Definition : 1.2

A near ring is a set N together with two binary operations + and .

such that

(i) $(N, +)$ is a group.

(ii) $(N, .)$ is a semi-group.

(iii) For all $x, y, z \in N$ it holds that $(x + y) .z = (x. z) + (y. z)$.

Definition : 1.3

A non-empty subset S of a near-ring R is called a subnear-ring of R if

(i) $x - y \in S$

(ii) $xy \in S$ for all $x, y \in S$.

Definition : 1.4

A Subgroup B of N is a bi-ideal of N if $BNB \subseteq B$

Definition : 1.5

Let I be an ideal of R. For each $a + I$, $b + I$ in the factor group R / I. We define $(a + I) + (b + I) = (a + b) + I$ and $(a + I)(b + I) = (ab) + I$. Then R / I is a near-ring which we call the residue class near-ring of R with respect to I.

Definition : 1.6

A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy subset of X.

Definition : 1.7

Let R be a near-ring and μ be a fuzzy subset of R. Then μ is a fuzzy ideal of R if:

(i) $\mu(x - y) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(y + x - y) \geq \mu(x)$

(iii) $\mu(xy) \geq \mu(y)$

(iv) $\mu((x + z)y - xy) \geq \mu(z)$ for all $x, y, z \in R$.

A fuzzy subset with $(i) - (iii)$ is called a fuzzy left ideal of R, whereas a fuzzy subset with (i), (ii) and (i) is called a fuzzy right ideal of R.

Definition : 1.8

A fuzzy set μ in N is a fuzzy sub near ring of N if for all $x, y \in N$,

(i) $\mu(x - y) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$

Definition : 1.9

A fuzzy set μ in N is a fuzzy bi-ideal of N if for all $x, y \in N$,

(i) $\mu(x - y) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(xyz) \geq min\{\mu(x), \mu(z)\}$

Definition : 1.10

Let X be a non-empty set. A mapping $\mu : X \rightarrow D\,[0,1]$ is called interval-valued fuzzy set, where $D[0,1]$ denote the family of all closed sub intervals of $[0,1]$ and $\mu(x) = [\mu^{-}(x), \mu^{+}(x)]$ for all $x \in X$ where μ^{-} and μ^{+} are fuzzy subsets of X such that $\mu^{-}(x) \leq \mu^{+}(x)$ for all $x \in X$.

Definition : 1.11

Let X be a non-empty set. A cubic set A in X is a structure $A = \{< x, \mu_A(x), f_A(x) > : x \in X\}$ which is briefly denoted by $A = < \mu_A, f_A >$, where $\mu_A = [\mu_A{}^{-}, \mu_A{}^{+}]$ is an interval-valued fuzzy set in X and f is fuzzy set in X.

In this case, we will use

$$A(x) = < \mu_A(x), f_A(x) >$$

$$= < [\mu^{-}(x), \mu^{+}(x)], f_A(x) > \text{ for all } x \in X.$$

Definition : 1.12

Let $A = < \mu, \gamma >$ be a cubic set of S. Define $U\,(A;\, t,\, n) = \{x \in S \mid \mu(x) \geq t \text{ and } \gamma(x) \leq n\}$ where $t \in D[0,1]$ and $n \in [0,1]$ is called the cubic level set of A.

Definition : 1.13

Let R be a near ring. Given two subsets A and B of R, we define the following products $AB = \{ab \mid a \in A, b \in B\}$ and $A * B = \{(a'+ b)a - a'a \mid a, a' \in A, b \in B\}$.

Definition : 1.14

A subgroup B of $(R, +)$ is said to be bi-ideal of R if $BRB \cap B * RB \subseteq B$.

Definition : 1.15

A subgroup B of $(R, +)$ is said to be weak bi-ideal of R if $BBB \subseteq B$.

Definition : 1.16

Let μ and λ be any two fuzzy subsets of R. Then $\mu\lambda$ is fuzzy subset of R defined by

$$\mu\lambda(x) = \begin{cases} \sup\limits_{x=yz} min\{\mu(y), \lambda(z)\} & if\ x = yz\ for\ all\ x, y, z \in R \\ 0 & otherwise \end{cases}$$

Definition : 1.17

Let R be a near-ring and μ be a fuzzy subset of R. We say μ is a fuzzy subnear –ring of R if

(i) $\mu(x - y) \geq min\ \{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq min\{\mu(x), \mu(y)\}\ for\ all\ x, y \in R.$

Definition : 1.18

Let R be a near-ring and μ be a fuzzy subset of R. Then μ is called the fuzzy ideal of R if

(i) $\mu(x - y) \geq min\ \{\mu(x), \mu(y)\}$

(ii) $\mu(x + y - y) \geq \mu(x)$

(iii) $\mu(xy) \geq \mu(y)$

(iv) $\mu(x+z)y - xy) \geq \mu(z)$ for all $x, y \in R$.

A fuzzy subset with (*i*) to (iii) is called a fuzzy left ideal of *R*, whereas a fuzzy subset with (*i*), (ii) and (iv) are called a fuzzy right ideal of *R*.

Definition : 1.19

A fuzzy subset μ of a near-ring *R* is called a fuzzy *R*-subgroup of *R* if

(i) μ is a fuzzy subgroup of (R, +)

(ii) $\mu(xy) \geq \mu(y)$

(iii) $\mu(xy) \geq \mu(x)$ for all *x*, *y* ∈ *R*.

A fuzzy subset with (*i*) and (ii) is called a fuzzy left *R*-subgroup of *R*, whereas a fuzzy subset with (i) and (iii) is called a fuzzy right *R*-subgroup of *R*.

Definition : 1.20

A fuzzy subgroup μ of *R* is called fuzzy weak bi-ideal of *R* if

$\mu(xyz) \geq min\{\mu(x), \mu(y), \mu(z)\}$

Definition : 1.21

Let X be a non-empty set. A mapping $\mu : X \to D[0,1]$ is called an interval-valued fuzzy subset of X, if for all $x \in X, \mu(x) = [\mu^-(x), \mu^+(x)]$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$. Thus $\mu(x)$ is an interval and not a number from the interval [0,1] as in the case of fuzzy set.

CHAPTER - II

CUBIC IDEALS OF NEAR RINGS

Definition : 2.1

A cubic set $A = < \mu_A,\ \lambda_A >$ of a near – ring R is called a cubic subnear – ring of R if

(1) $\mu_A(x - y) \geq min\{\mu_A(x), \mu_A(y)\}$,

(2) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$,

(3) $\lambda_A(x - y) \leq max\{\lambda_A(x), \lambda_A(y)\}$,

(4) $\lambda_A(xy) \leq max\{\lambda_A(x), \lambda_A(y)\}$, for all $x, y \in R$.

Definition : 2.2

Let $A = < \mu_A,\ \lambda_A >$ be a cubic set of R. We say A is a cubic ideal of R if it satisfies the following:

(1) $\mu_A(x - y) \geq min\{\mu_A(x), \mu_A(y)\}$,

(2) $\mu_A(y + x - y) \geq \mu_A(x)$,

(3) $\mu_A(xy) \geq \mu_A(y)$,

(4) $\mu_A((x + z)y - xy) \geq \mu_A(z)$,

(5) $\lambda_A(x - y) \leq max\{\lambda_A(x), \lambda_A(y)\}$,

(6) $\lambda_A(y + x - y) \leq \lambda_A(x)$,

(7) $\lambda_A(xy) \leq \lambda_A(y)$,

(8) $\lambda_A((x + z)y - xy) \leq \lambda_A(z)$, for all $x, y \in R$.

Example : 2.3

Let $R = \{a, b, c, d\}$ be a set with two binary operations defined as:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(R, +, \ .)$ is near-ring.

Define a cubic set $A = < \mu_A, \ \lambda_A >$ by $\mu_A(a) = [0.8, 0.9]$, $\mu_A(b) = [0.6, 0.7]$ and $\mu_A(c) = [0.5, 0.5] = \mu_A(d)$,

$\lambda_A(a) = 0.2$, $\lambda_A(b) = 0.6$ and $\lambda_A(c) = 0.8 = \lambda_A(d)$.

Then, $A = < \mu_A, \ \lambda_A >$ is a cubic ideal of R.

Theorem : 2.4

If $A = < \mu_A, \ \lambda_A >$ is a cubic ideal of R if and only if every non empty cubic level set of $A = < \mu_A, \ \lambda_A >$ is a left (resp. right) ideal of R.

Proof:

Assume that $A = < \mu_A, \ \lambda_A >$ is a cubic ideal of R.

Let $x, y \in U(A; [s, t], r)$ for all $[s, t] \in D[0,1]$ and $r \in [0,1]$.

Then $\mu_A(x) \geq [s, t]$, $\mu_A(y) \geq [s, t]$, $\lambda_A(x) \leq r$, $\lambda_A(y) \leq r$.

By the definition of cubic ideal of $\mu_A(x - y) \geq min\{\mu_A(x), \ \mu_A(y)\} \geq [s, t]$ and

$\lambda_A(x - y) \leq \max\{\lambda_A(x), \lambda_A(y)\} \leq r$.

Hence $x - y \in U(A; [s,t],\ r)$ and $y \in R$.

Let $x \in U(A; [s,t], r$ and $y \in R$.

Then $\mu_A(x) \ \geq\ [s,t]$ and $\lambda_A(x) \ \leq\ r$.

We know that $\mu_A(y + x - y) \ \geq\ \mu_A(x) \geq [s,t]$ and

$\lambda_A(y + x - y) \ \leq\ \lambda_A(x) \ \leq\ r$ implies $y + x - y \ \in\ U(A; [s,t],\ r)$.

Thus $U(A; [s,t],\ r)$ is a normal subgroup of R.

Let $y \in U(A; [s,t],\ r)$ and $x \in R$.

We know it $\mu_A(xy) \ \geq\ \mu_A(y) \ \geq\ [s,t]$ and $\lambda_A(xy) \ \leq\ \lambda_A(y) \ \leq\ r$.

This implies that $xy \in U(A; [s,t], r)$.

Therefore $U(A; [s,t], r)$ is a left ideal.

As we know $\mu_A((x + z)y - xy \ \geq\ \mu_A(z) \ \geq\ [s,t]$ and

$\lambda_A\big((x + z)y - xy\big) \ \leq\ \lambda_A(z) \ \leq\ r$.

Which implies that $((x + z)y - xy) \ \in U(A; [s,t], r)$.

Thus $U(A; [s,t], r)$ is an ideal of R.

Conversely,

Assume that $U(A; [s,t],\ r)$ is an ideal of R.

Let $r \in [0,1]$ and [s,t] $\in D[0,1]$ be such that $U(A; [s,t],\ r) \neq 0$.

1) Suppose we assume that $\mu_A(x - y) \ <\ min\ \{\mu_A(x), \mu_A(y)\}$ or

   $\lambda_A(x - y) \ >\ max\ \{\lambda_A(x), \lambda_A(y)\}$.

(i) If $\mu_A(x-y) < min\{\mu_A(x), \mu_A(y)\}$ then we can find an interval

$\mu_A(x-y) < [s_1, t_1] < min\{\mu_A(x), \mu_A(y)$ for some $[s_1, t_1] \in D[0,1]$.

Hence $x, y \in U(A; [s_1, t_1], max\{\lambda_A(x), \lambda_A(y)\})$

but $x-y \notin U(A; [s_1, t_1], max\{\lambda_A(x), \lambda_A(y)\})$ which is a contradiction.

(i) If $\lambda_A(x-y) > max\{\lambda_A(x), \lambda_A(y)\}$ then there exist $r_1 \in [0,1]$ such that $\lambda_A(x-y) > r_1 > \max\{\lambda_A(x), \lambda_A(y)\}$.

Thus $x, y \in U(A; min\{\mu_A(x), \mu_A(y)\}, r_1)$

but $x-y \notin U(A; min\{\mu_A(x), \mu_A(y)\}, r_1)$ which is a contradiction.

Therefore $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(x-y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$.

Let $x, y \in R$.

2) Suppose $\mu_A(y+x-y) < \mu_A(x)$ or $\lambda_A(y+x-y) > \lambda_A(x)$.

(i) If $\mu_A(y+x-y) < \mu_A(x)$ then we can find an interval $[s_1, t_1] \in D[0,1]$ such that $\mu_A(y+x-y) < [s_1, t_1] < \mu_A(x)$.

So, $x \in U(A; [s_1, t_1], \lambda_A(x))$

but $y+x-y \notin U\big(A; [s_1, t_1], \lambda_A(x)\big)$ which is a contradiction.

(ii) If $\lambda_A(y+x-y) > \lambda_A(x)$ then there exist r_1 such that $\lambda_A(y+x-y) > r_1 > \lambda_A(x)$ implies $x \in U(A; \mu_A(x), r_1)$

but $y+x-y \notin U(A; \mu_A(x), r_1)$ which is a contradiction.

Therefore $\mu_A(y+x-y) \geq \mu_A(x)$ and $\lambda_A(y+x-y) \leq \lambda_A(x)$.

3) Suppose assume that $\mu_A(xy) < \mu_A(y)$ or $\lambda_A(xy) > \lambda_A(y)$.

(i) If $\mu_A(xy) < \mu_A(y)$ then we can find an interval $[s_1, t_1]$ such that

$\mu_A(xy) < [s_1, t_1] < \mu_A(y)$ implies $y \in$ U(A; $[s_1, t_1]$, $\lambda_A(y)$)

but $xy \notin$ U(A; $[s_1, t_1]$, $\lambda_A(y)$) which is a contradiction.

(ii) If $\lambda_A(xy) > \lambda_A(y)$ then we can find an r_1 such that

$\lambda_A(xy) > r_1 > \lambda_A(y)$ implies $y \in$ U(A; $\mu_A(y)$, r_1) but $xy \notin (A;\ \mu_A(y),\ r_1)$ which is contradiction.

Therefore $\mu_A(xy) \geq \mu_A(y)$ and $\lambda_A(xy) \leq \lambda_A(y)$.

4) Suppose assume that $\mu_A\big((x+z)y-xy\big) < \mu_A(z)$ or

$\lambda_A\big((x+z)y-xy\big) > \lambda_A(z)$.

(i) If $\mu_A((x+z)y-xy) < \mu_A(z)$ we can find an interval $[s_1, t_1]$, such that

$\mu_A((x+z)y-xy) < [s_1, t_1] < \mu_A(z)$.

Then $\mu_A(z) \in$ U(A; $[s_1, t_1]$, $\lambda_A(z)$) but $(x+z)y-xy \notin$ U(A; $[s_1, t_1]$, $\lambda_A(z)$) which is a contradiction.

(ii) If $\lambda_A((x+z)y-xy) > \lambda_A(z)$ we can find r_1 such that

$\lambda_A((x+z)y-xy) > r_1 > \lambda_A(z)$.

Then $\lambda_A(z) \in$ U(A; $\mu_A(z)$, r_1) but $(x+z)y-xy \notin$ U(A; $\mu_A(z)$, r_1) which is a contradiction.

Therefore $\mu_A\big((x+z)y - xy\big) \geq \mu_A(z)$ and $\lambda_A((x+z)y - xy) \leq \lambda_A(z)$

Hence A $= < \mu_A, \lambda_A >$ is cubic ideal of R.

Theorem : 2.5

Intersection of any family of cubic ideals (sub near-ring) of R is also a cubic ideal (sub near-ring) of R.

Proof :

Let $A_i = < \mu_{A_i}, \lambda_{A_i} >$ be cubic ideals of R where $i \in \Omega$ any index set.

Let $x, y \in R$ then,

(i) $\cap_{i\in\Omega} \mu_{A_i}(x-y) = inf\ \{\mu_{A_i}(x-y)/\ i \in \Omega\}$

$\geq inf\{min\ \{\mu_{A_i}(x), \mu_{A_i}(y)\}/i \in \Omega\}$

$= min\ \{inf\ \{\mu_{A_i}(x)/i \in \Omega\},\ inf\{\mu_{A_i}(y)/\ i \in \Omega\}$

$= min\{\cap_{i\in\Omega} \mu_{A_i}(x), \cap_{i\in\Omega} \mu_{A_i}(y)\}$

(ii) $\cup_{i\in\Omega} \lambda_{A_i}(x-y) = sup\ \{\lambda_{A_i}(x-y)/\ i \in \Omega\}$

$\leq sup\ \{max\ \{\lambda_{A_i}(x), \lambda_{A_i}(y)\}/i \in \Omega\}$

$= max\ \{sup\ \{\lambda_{A_i}(x)/i \in \Omega\}, sup\{\lambda_{A_i}(y)/\ i \in \Omega\}$

$= min\{\ \cup_{i\in\Omega} \lambda_{A_i}(x), \cup_{i\in\Omega} \lambda_{A_i}(y)\}$

(iii) $\cap_{i\in\Omega} \mu_{A_i}(y+x-y) = inf\ \{\mu_{A_i}(y+x-y)/\ i \in \Omega\}$

$\geq inf\ \{\mu_{A_i}(x)/i \in \Omega\}$

$= \bigcap_{i \in \Omega} \mu_{A_i}(x)$

(iv) $\bigcup_{i \in \Omega} \lambda_{A_i}(y + x - y) = sup\, \{\lambda_{A_i}(y + x - y)/\, i \in \Omega\}$

$\leq sup\, \{\lambda_{A_i}(x)/i \in \Omega\}$

$= \bigcup_{i \in \Omega} \lambda_{A_i}(x)$

(v) $\bigcap_{i \in \Omega} \mu_{A_i}(xy) = inf\, \{\mu_{A_i}(xy)/\, i \in \Omega\}$

$\geq inf\, \{\, \mu_{A_i}(y)/i \in \Omega\}$

$= \bigcap_{i \in \Omega} \mu_{A_i}(y)$

(vi) $\bigcup_{i \in \Omega} \lambda_{A_i}(xy) = sup\, \{\lambda_{A_i}(xy)/\, i \in \Omega\}$

$\leq sup\, \{\lambda_{A_i}(y)/i \in \Omega\}$

$= \bigcup_{i \in \Omega} \lambda_{A_i}(y)$

(vii) $\bigcap_{i \in \Omega} \mu_{A_i}((x + z)y - xy) = inf\, \{\mu_{A_i}(x + z)y - xy)/\, i \in \Omega\}$

$\geq inf\, \{\, \mu_{A_i}(z)/i \in \Omega\}$

$= \bigcap_{i \in \Omega} \mu_{A_i}(z)$

(viii) $\bigcup_{i \in \Omega} \lambda_{A_i}((x + z)y - xy) = sup\, \{\lambda_{A_i}((x + z)y - xy)/\, i \in \Omega\}$

$\leq sup\, \{\lambda_{A_i}(z)/i \in \Omega\}$

$= \bigcup_{i \in \Omega} \lambda_{A_i}(z)$

Therefore, intersection of any cubic ideals of R is also a cubic ideal of R.

The following theorem is relation between crisp ideals and cubic ideals.

Definition : 2.6

Let A_i be cubic ideals of near-rings R_i for $i = 1,2, \dots, n$. Then the direct product of $A_i (i = 1,2, \dots n)$ is a function

$\mu_{A_1} \times \mu_{A_2} \times \dots . \times \mu_{A_n} : R_1 \times R_2 \times \dots . \times R_n \rightarrow D[0,1]$,

$\lambda_{A_1} \times \lambda_{A_2} \times \dots . \times \lambda_{A_n} : R_1 \times R_2 \times \dots . \times R_n \rightarrow D[0,1]$ defined by

$\mu_{A_1} \times \mu_{A_2} \times \dots . \times \mu_{A_n}(x_1, x_2, \dots x_n) = min \{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \dots, \mu_{A_n}(x_n)\}$ and

$\lambda_{A_1} \times \lambda_{A_2} \times \dots . \times \lambda_{A_n}(x_1, x_2, \dots x_n) = max \{\lambda_{A_1}(x_1), \lambda_{A_2}(x_2), \dots, \lambda_{A_n}(x_n)\}$.

Theorem : 2.7

The direct product of cubic ideals of near-rings is also cubic ideal of near-rings.

Proof:

Let $A_i = < \mu_{A_i}, \lambda_{A_i} >$ be cubic ideals of near-rings R_i where $i = 1,2, \dots n$.

Let $x = (x_1, x_2, \dots x_n), y = (y_1, y_2, \dots y_n)$

and $z = (z_1, z_2, \dots z_n) \in R_1 \times R_2 \times \dots . \times R_n$

(i) $\mu_{A_i}(x - y) = \mu_{A_i}((x_1, x_2, \dots x_n) - (y_1, y_2, \dots y_n)$

$= \mu_{A_i}(x_1 - y_{1,} x_2 - y_{2, \dots \dots} x_n - y_n)$

$= min \{\mu_{A_1}(x_1 - y_1), \mu_{A_2}(x_2 - y_2), \dots . \mu_{A_n}(x_n - y_n)\}$

$= min \{min \{\mu_{A_1}(x), \mu_{A_1}(y)\}, min\{ \mu_{A_2}(x), \mu_{A_2}(y)\}, \dots$

$min\{ \mu_{A_n}(x), \mu_{A_n}(y)\}$

$$= \ min\ \{min\ \{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots\ldots \mu_{A_n}(x_n)\},$$

$$min\ \{\mu_{A_1}(y_1), \mu_{A_2}(y_2), \ldots\ldots \mu_{A_n}(y_n)\}\}$$

$$= min\ \{\mu_{A_1} \times \mu_{A_2} \times \ldots .\times \mu_{A_n}(x_1, x_2, \ldots . x_n),$$

$$\mu_{A_1} \times \mu_{A_2} \times \ldots .\times \mu_{A_n}(y_1, y_2, \ldots . y_n)\}$$

$$= min\ \{\mu_{A_i}(x), \mu_{A_i}(y)\}.$$

(ii) $\lambda_{A_i}(x - y) \quad = \ \lambda_{A_i}((x_1, x_2, \ldots x_n) - (y_1, y_2, \ldots y_n)$

$$= \ \lambda_{A_i}(x_1 - y_{1,} x_2 - y_{2,\ldots\ldots} x_n - y_n)$$

$$= max\ \{\lambda_{A_1}(x_1 - y_1), \lambda_{A_2}(x_2 - y_2), \ldots . \lambda_{A_n}(x_n - y_n)\}$$

$$= max\ \{max\ \{\lambda_{A_1}(x), \lambda_{A_1}(y), max\big\{\ \lambda_{A_2}(x), \lambda_{A_2}(y)\big\}, \ldots$$

$$max\big\{\ \lambda_{A_n}(x), \lambda_{A_n}(y)\big\}$$

$$= \ max\ \{max\ \{\lambda_{A_1}(x_1), \lambda_{A_2}(x_2), \ldots\ldots \lambda_{A_n}(x_n)\},$$

$$max\ \{\lambda_{A_1}(y_1), \lambda_{A_2}(y_2), \ldots\ldots \lambda_{A_n}(y_n)\}\}$$

$$= max\ \{\lambda_{A_1} \times \lambda_{A_2} \times \ldots .\times \lambda_{A_n}(x_1, x_2, \ldots . x_n),$$

$$\lambda_{A_1} \times\ \lambda_{A_2} \times \ldots .\times \lambda_{A_n}(y_1, y_2, \ldots . y_n)\}$$

$$= max\ \{\lambda_{A_i}(x), \lambda_{A_i}(y)\}.$$

(iii) $\mu_{A_i}(y + x - y) \ = \ \mu_{A_i}((y_1, y_2, \ldots y_n) + (x_1, x_2, \ldots x_n) - (y_1, y_2, \ldots y_n)$

$$= \ \mu_{A_i}(y_1 + x_1 - y_{1,} y_2 + x_2 - y_{2,\ldots\ldots} y_n + x_n - y_n)$$

$$= min\,\{\mu_{A_1}(y_1 + x_1 - y_1), \mu_{A_2}(y_2 + x_2 - y_2), \ldots.$$

$$\mu_{A_n}(y_n + x_n - y_n)\}$$

$$\geq\ min\,\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots..\mu_{A_n}(x_n)\}$$

$$= \mu_{A_1} \times \mu_{A_2} \times \ldots.\times \mu_{A_n}(x_1, x_2, \ldots.x_n),$$

$$= \mu_{A_i}(x).$$

(iv) $\lambda_{A_i}(y + x - y) = \lambda_{A_i}((y_1, y_2, \ldots y_n) + (x_1, x_2, \ldots x_n) - (y_1, y_2, \ldots y_n)$

$$= \lambda_{A_i}(y_1 + x_1 - y_{1,}y_2 + x_2 - y_{2,\ldots\ldots}y_n + x_n - y_n)$$

$$= \max\,\{\lambda_{A_1}(y_1 + x_1 - y_1), \lambda_{A_2}(y_2 + x_2 - y_2), \ldots.$$

$$\lambda_{A_n}(y_n + x_n - y_n)\}$$

$$\leq\ max\,\{\lambda_{A_1}(x_1), \lambda_{A_2}(x_2), \ldots..\lambda_{A_n}(x_n)\}$$

$$= \lambda_{A_1} \times \lambda_{A_2} \times \ldots.\times \lambda_{A_n}(x_1, x_2, \ldots.x_n),$$

$$= \lambda_{A_i}(x).$$

(v) $\mu_{A_i}(xy) = \mu_{A_i}((x_1, x_2, \ldots x_n)(y_1, y_2, \ldots y_n)$

$$= \mu_{A_i}(x_1y_{1,}x_2, y_{2,\ldots\ldots}x_ny_n)$$

$$= min\,\{\mu_{A_1}(x_1y_1), \mu_{A_2}(x_2y_2), \ldots.\mu_{A_n}(x_ny_n)\}$$

$$\geq min\,\{\mu_{A_1}(y_1), \mu_{A_2}(y_2), \ldots\mu_{A_n}(y_n)\}$$

$$= \mu_{A_1} \times \mu_{A_2} \times \ldots.\times \mu_{A_n}(y_1, y_2, \ldots.y_n)\}$$

$$= \mu_{A_i}(y).$$

(vi) $\mu_{A_i}(xy) = \lambda_{A_i}((x_1, x_2, \dots x_n)(y_1, y_2, \dots y_n)$

$= \lambda_{A_i}(x_1y_{1,}x_2, y_{2,\dots\dots}x_ny_n)$

$= max\,\{\lambda_{A_1}(x_1y_1), \lambda_{A_2}(x_2y_2), \dots . \lambda_{A_n}(x_ny_n)\}$

$\leq max\,\{\lambda_{A_1}(y_1), \lambda_{A_2}(y_2), \dots \lambda_{A_n}(y_n)\}$

$= \lambda_{A_1} \times \lambda_{A_2} \times \dots . \times \lambda_{A_n}(y_1, y_2, \dots . y_n)\}$

$= \lambda_{A_i}(y).$

(vii) $\mu_{A_i}\big((x+z)y - xy\big) = \mu_{A_i}\big((x_1, x_2, \dots x_n)(z_1, z_2, \dots z_n)\big)(y_1, y_2, \dots y_n)$

$- (x_1, x_2, \dots x_n)(y_1, y_2, \dots y_n)$

$= \mu_{A_i}\Big(\big((x_1+z_1)y_1 - x_1y_1\big), \big(x_2+z_2)y_2 - x_2y_2\big) \dots .$

$\big((x_n+z_n)y_n - x_ny_n\big)))$

$= min\big\{\mu_{A_1}\big((x_1+z_1)y_1 - x_1y_1\big), \mu_{A_2}(x_2+z_2)y_2x_2y_2\big),$

$\dots, \mu_{A_n}(x_n+z_n)y_n - x_ny_n)\}$

$\geq min\,\{\mu_{A_1}(z_1), \mu_{A_2}(z_2), \dots . \mu_{A_n}(z_n)\}$

$= \mu_{A_1} \times \mu_{A_2} \times \mu_{A_n}(z_1, z_2, z_n)$

$= \mu_{A_i}(z)$

(viii) $\lambda_{A_i}\big((x+z)y - xy\big) = \lambda_{A_i}((x_1, x_2, \dots x_n) + (z_1, z_2, \dots z_n))(y_1, y_2, \dots y_n)$

$-(x_1, x_2, \dots x_n)(y_1, y_2, \dots y_n)$

$$= \lambda_{A_i}\left(\left((x_1+z_1)y_1 - x_1y_1\right), (x_2+z_2)y_2 - x_2y_2\right) \ldots.$$

$$\left((x_n+z_n)y_n - x_ny_n\right)))$$

$$= max\{\lambda_{A_1}\left((x_1+z_1)y_1 - x_1y_1\right)\lambda_{A_2}(x_2+z_2)y_2 - x_2y_2),$$

$$\ldots., \lambda_{A_n}(x_n+z_n)y_n - x_ny_n)\}$$

$$\leq max\ \{\lambda_{A_1}(z_1), \lambda_{A_2}(z_2), \ldots. \lambda_{A_n}(z_n)\}$$

$$= \lambda_{A_1} \times \lambda_{A_2} \times \lambda_{A_n}(z_1, z_2, z_n)$$

$$= \lambda_{A_i}(z)$$

Therefore, direct product of cubic ideals of near-rings is also cubic ideal of near rings.

Definition:2.8

Let $A = <\mu_A, \lambda_A>$ be a cubic set of R. Then the strongest cubic relation on R is a cubic relation y with A is given by $v(x,y) = \{((x,y), \rho(x,y), \sigma(x,y))/x, y \in R\}$, where ρ is an interval valued fuzzy relation with respect to μ_A defined by $\rho(x,y) = min\{\mu_A(x), \mu_A(y)\}$, and σ is an anti-fuzzy relation with respect to λ_A defined by $\sigma(x,y) = max\ \{\lambda_A(x), \lambda_A(y)\}$.

Theorem : 2.9

Let $A = <\mu_A, \lambda_A>$ be a cubic set of near-ring R and $v(x,y) = \{((x,y), \rho(x,y), \sigma(x,y))/x, y \in R\}$, be a strongest cubic relation with respect to v. Then A is a cubic ideal of R if and only if v is a cubic ideal of $R \times R$.

Proof:

Let us assume that $A = <\mu_A, \lambda_A>$ is a cubic ideal of R. Let $x_1, y_1, x_2, y_2, z_1, z_2 \in R$.

Then for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in R \times R$, we have

(i) $\rho(x, y)$ $= \rho((x_1, x_2) - (y_1, y_2))$

$= \rho(x_1 - y_1, x_2 - y_2)$

$= min\,\{\mu_A(x_1 - y_1) - \mu_A(x_2 - y_2)\}$

$\geq min\{min\,\{\mu_A(x_1), \mu_A(y_1)\}, min\,\{\mu_A(x_2), \mu_A(y_2)\}\}$

$= min\,\{min\{\mu_A(x_1), \mu_A(x_2)\}, min\{\mu_A(y_1), \mu_A(y_2)\}\}$

$= min\,\{\,\rho((x_1, x_2) - \rho(y_1, y_2)\}$

$= min\,\{\,\rho((x), \rho(y)\}$

(ii) $\sigma(x - y)$ $= \sigma((x_1, x_2) - (y_1, y_2))$

$= \ \sigma(x_1 - y_1, x_2 - y_2)$

$= max\,\{\,\lambda_A(x_1 - y_1), \lambda_A(x_2 - y_2)\}$

$\leq max\,\{max\{\lambda_A(x_1), \lambda_A(y_1)\}, max\{\lambda_A(x_2), \lambda_A(y_2)\}\}$

$= max\,\{max\{\lambda_A(x_1), \lambda_A(x_2)\}, max\{\lambda_A(y_1), \lambda_A(y_2)\}\}$

$= max\,\{\sigma(x_1, x_2), \sigma(y_1, y_2)$

$= max\,\{\sigma(x)\,, \sigma(y)\}$

(iii) $\rho(y + x - y) = \rho((y_1, y_2) + (x_1, x_2) - (y_1, y_2))$

$= \rho(y_1 + x_1 - y_1, y_2 + x_2 - y_2)$

$= min\,\{\mu_A(y_1 + x_1 - y_1), \mu_A(y_2 + x_2 - y_2)$

$\geq min\,\{\mu_A(x_1), \mu_A(x_2)\}$

$= \rho(x_1, x_2)$

$= \rho(x).$

(iv) $\sigma(y + x - y) = \sigma((y_1, y_2) + (x_1, x_2) - (y_1, y_2))$

$= \sigma(y_1 + x_1 - y_1, y_2 + x_2 - y_2)$

$= max\,\{\lambda_A(y_1 + x_1 - y_1), \lambda_A(y_2 + x_2 - y_2)$

$\leq max\,\{\lambda_A(x_1), \lambda_A(x_2)\}$

$= \sigma(x_1, x_2)$

$= \sigma(x).$

(v) $\rho(xy) = \rho((x_1, x_2)(y_1, y_2))$

$= \rho(x_1y_1, x_2y_2)$

$= min\,\{\mu_A(x_1y_1), \mu_A(x_2y_2)\}$

$\geq min\,\{\mu_A(y_1), \mu_A(y_2)\}$

$= \rho(y_1, y_2)$

$= \rho(x).$

(vi) $\sigma(xy) = \sigma((x_1, x_2), (y_1, y_2))$

$= \sigma(x_1y_1, x_2y_2)$

$= max\,\{\lambda_A(x_1y_1), \lambda_A(x_2y_2)\}$

$\leq max\ \{\lambda_A(y_1), \lambda_A(y_2)\}$

$= \sigma(y_1, y_2)$

$= \sigma(x)$.

(vii) $\rho(x+z)y - xy) \quad = \rho(((x_1, x_2) + (z_1, z_2))(y_1, y_2) - (x_1, x_2)(y_1, y_2))$

$= \rho((x_1 + z_1)y_1 - x_1y_1, (x_2 + z_2)y_2 - x_2y_2)$

$= \min\{\mu_A\big((x_1 + z_1)y_1 - x_1y_1\big),$

$\mu_A\big((x_2 + z_2)y_2 - x_2y_2\big)\}$

$\geq min\ \{\mu_A(z_1), \mu_A(z_2)\}$

$= \rho(z_1, z_2)$

$= \rho(z)$.

(viii) $\sigma((x+z)y - xy) \quad = \sigma\big(((x_1, x_2) + (z_1, +z_2)\big)(y_1, y_2) - (x_1, x_2)(y_1, y_2))$

$= \sigma((x_1 + z_1)y_1 - x_1y_1, (x_2 + z_2)y_2 - x_2y_2)$

$= max\ \{\lambda_A\big((x_1 + z_1)y_1 - x_1y_1\big),$

$\lambda_A\big((x_2 + z_2)y_2 - x_2y_2\big)\}$

$\geq max\ \{\lambda_A(z_1), \lambda_A(z_1)\}$

$= \sigma(z_1, z_2)$

$= \sigma(z)$.

Conversely,

Assume that v is a cubic ideal of $R \times R$. then $x = (x_1, x_2), y = (y_1, y_2)$, $z = (z_1, z_2) \in R \times R$.

(i) $min\,\{\mu_A(x_1 - y_1), \mu_A(x_2 - y_2)\}$ $= \rho(x_1 - y_1, x_2 - y_2)$

$= \rho((x_1, x_2) - (y_1, y_2))$

$= \rho(x - y)$

$\geq min\,\{\rho(x_1, x_2) - \rho(y_1, y_2)\}$

$= min\,\{min\{\mu_A(x_1), \mu_A(x_2)\},$

$min\{\mu_A(y_1), \mu_A(y_2)\}$

If $\mu_A(x_1 - y_1) \leq \mu_A(x_2 - y_2)$, then $\mu_A(x_1) \leq \mu_A(x_2),\ \mu_A(y_1) \leq \mu_A(y_2)$, we get $\mu_A(x_1 - y_1) \geq min\,\{\mu_A(x_1), \mu_A(y_1)\}$

(ii) $max\,\{\lambda_A(x_1 - y_1), \lambda_A(x_2 - y_2)\}$ $= \sigma(x_1 - y_1, x_2 - y_2)$

$= \sigma((x_1, x_2) - (y_1, y_2))$

$= \sigma(x - y)$

$\leq max\,\{\sigma(x), \sigma(y)\}$

$= max\,\{\sigma(x_1, x_2), \sigma(y_1, y_2)\}$

$= max\,\{max\,\{\lambda_A(x_1), \lambda_A(x_2)\},$

$\max\{\lambda_A(y_1), \lambda_A(y_2)\}$

If $\lambda_A(x_1 - y_1) \geq \lambda_A(x_2 - y_2)$, then $\lambda_A(x_1) \geq \lambda_A(x_2), \lambda_A(y_1) \geq \lambda_A(y_2)$,

we get $\lambda_A(x_1 - y_1) \leq \ max\{\lambda_A(x_1), \lambda_A(y_1)\}$.

(iii) $min\,\{\mu_A(y_1 + x_1 - y_1), \mu_A(y_2 + x_2 - y_2)\}\ = \rho(y_1 + x_1 - y_1, y_2 + x_2 - y_2)$

$= \rho(y_1, y_2) + (x_1, x_2) - (y_1, y_2))$

$= \rho(y + x - y)$

$\geq \rho(x)$

$= \rho(x_1, x_2)$

$= min\,\{\mu_A(x_1), \mu_A(x_2)\}$

If $\mu_A(y_1 + x_1 - y_1) \geq \mu_A(y_2 + x_2 - y_2)$, then $\mu_A(x_1) \geq \mu_A(x_2)$,

we get $\mu_A(y_1 + x_1 - y_1) \geq \mu_A(x_1)$.

(iv) $max\,\{\lambda_A(y_1 + x_1 - y_1), \lambda_A(y_2 + x_2 - y_2)\}\ = \sigma(y_1 + x_1 - y_1, y_2 + x_2 - y_2)$

$= \sigma(y_1, y_2) + (x_1, x_2) - (y_1, y_2))$

$= \sigma(y + x - y)$

$\leq \sigma(x)$

$= \sigma(x_1, x_2)$

$= max\,\{\lambda_A(x_1), \lambda_A(x_2)\}$

If $\lambda_A(y_1 + x_1 - y_1) \geq \lambda_A(y_2 + x_2 - y_2)$, then $\lambda_A(x_1) \geq \lambda_A(x_2)$,

we get $\lambda_A(y_1 + x_1 - y_1) \leq \lambda_A(x_1)$.

(v) $min\,\{\mu_A(x_1y_1), \mu_A(x_2y_2)\}$ $= \rho(x_1y_1, x_2y_2)$

$= \rho((x_1, x_2)(y_1, y_2))$

$= \rho(xy)$

$\geq \rho(y)$

$= \rho(y_1, y_2)$

$= min\{\mu_A(y_1), \mu_A(y_2)\}$

If $\mu_A(x_1y_1) \leq \mu_A(x_2y_2)$, then $\mu_A(x_1) \leq \mu_A(x_2)$ and $\mu_A(y_1) \leq \mu_A(y_2)$,

we get $\mu_A(x_1y_1) \geq \mu_A(y_1)$.

(vi) $max\,\{\lambda_A(x_1y_1), \lambda_A(x_2y_2)\}$ $= \sigma(x_1y_1, x_2y_2)$

$= \sigma((x_1, x_2)(y_1, y_2))$

$= \sigma(xy)$

$\leq \sigma(y)$

$= \sigma(y_1, y_2)$

$= max\{\lambda_A(y_1), \lambda_A(y_2)\}$

If $\lambda_A(x_1y_1) \geq \lambda_A(x_2y_2)$,then $\lambda_A(x_1) \geq \lambda_A(x_2)$ and $\lambda_A(y_1) \geq \lambda_A(y_2)$,

we get $\lambda_A(x_1y_1) \geq \lambda_A(y_1)$.

(vii) $min\{\mu_A(x_1+z_1)y_1 - x_1y_1)\,, \mu_A(x_2+z_2)y_2 - x_2y_2)\}$

$= \rho((x_1+z_1)y_1 - x_1y_1, (x_2+z_2)y_2 - x_2y_2)$

$= \rho(((x_1, x_2) + (z_1, z_2))(y_1, y_2) - (x_1, x_2)(y_1, y_2))$

$= \rho(x + z)y - xy)$

$\geq \rho(z)$

$= \rho(z_1, z_2)$

$= min\,\{\mu_A(z_1), \mu_A(z_2)\}$

If $\mu_A((x_1{+}z_1)y_1 - x_1y_1) \leq \mu_A(x_2{+}z_2)y_2 - x_2y_2)$, then $\mu_A(x_1) \leq \mu_A(x_2)$ and $\mu_A(y_1) \leq \mu_A(y_2)$ and $\mu_A(z_1) \leq \mu_A(z_2)$ we get $\mu_A((x_1{+}z_1)y_1 - x_1y_1) \geq \mu_A(z_1)$.

(viii) $max\{\lambda_A((x_1{+}z_1)y_1 - x_1y_1)\,, \lambda_A((x_2{+}z_2)y_2 - x_2y_2)\}$

$= \sigma((x_1{+}z_1)y_1 - x_1y_1, (x_2{+}z_2)y_2 - x_2y_2)$

$= \sigma(((x_1, x_2) + (z_1, z_2))(y_1, y_2) - (x_1, x_2)(y_1, y_2))$

$= \sigma((x + z)y - xy)$

$\leq \sigma(z)$

$= \sigma(z_1, z_2)$

$= max\,\{\lambda_A(z_1), \lambda_A(z_2)\}$

If $\lambda_A((x_1{+}z_1)y_1 - x_1y_1) \leq \lambda_A(x_2{+}z_2)y_2 - x_2y_2)$, then $\lambda_A(x_1) \geq \lambda_A(x_2)$ and $\lambda_A(y_1) \geq \lambda_A(y_2)$ and $\lambda_A(z_1) \geq \lambda_A(z_2)$ we get $\lambda_A((x_1{+}z_1)y_1 - x_1y_1) \leq \lambda_A(z_1)$.

Therefore, v is a cubic ideal of R.

CHAPTER – III

CUBIC BI-IDEALS IN NEAR RINGS

Definition : 3.1

A cubic set $A = < \mu, \omega >$ is a cubic sub near-ring of N if for all $x, y \in N$.

(i) $\mu(x - y) \geq min\ \{\mu(x), \mu(y)\}$

$\omega(x - y) \leq max\ \{\omega(x), \omega(y)\}$

(ii) $\mu(xy) \geq min\{\mu(x)), \mu(y)\}$

$\omega(xy) \leq max\{\omega(x)), \omega(y)\}$

Definition : 3.2

A cubic set $A = < \mu, \omega >$ is N is a cubic bi-ideal of N if for all $x, y, z \in N$.

(i) $\mu(x - y) \geq min\ \{\mu(x), \mu(y)\}$

$\omega(x - y) \leq max\ \{\omega(x), \omega(y)\}$

(ii) $\mu(xyz) \geq min\{\mu(x), \mu(z)\}$

$\omega(xyz) \leq max\{\omega(x)), \omega(z)\}$

Example : 3.3

Let $N = \{a, b, c, d\}$ be the Klein's four group. Define addition and multiplication in N as follows,

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(N, +.,)$ is a near-ring.

Define a cubic set $A = < \mu, \omega >$, by

$\mu(a) = [0.8, 0.9]$, $\mu(b) = [0.6, 0.7]$,

$\mu(c) = [0.5, 0.5] = \mu(d)$ is an interval-valued fuzzy bi-ideal of N and

$\omega(a) = 0.2$, $\omega(b) = 0.6$, $\omega(c) = 0.8 = \omega(d)$ is a fuzzy bi-ideal of N.

Thus $A = < \mu, \omega >$ is a cubic bi-ideal of N.

Definition : 3.4

Let $A = < \mu_i, \omega_i >$ be cubic ideals of near-rings N_i, for $i = 1,2,3, \dots n$. Then the cubic direct product of A_i $(1,2,3, \dots n)$ is a function

$(\mu_1 \times \mu_2 \times \dots \dots \dots \times \mu_n): R_1 \times R_2 \times \dots \dots \dots \times R_n \rightarrow D[0,1]$,

$(\omega_1 \times \omega_2 \times \dots \dots \dots \times \omega_n): R_1 \times R_2 \times \dots \dots \dots \times R_n \rightarrow D[0,1]$ defined by

$(\mu_1 \times \mu_2 \times \dots \dots \dots \times \mu_n)\ (x_1 \times x_2 \times \dots \dots \dots \times x_n) = min\{\mu_1(x_1), \mu_2(x_2) \dots . \mu_n(x_n)\}$

and $(\omega_1 \times \omega_2 \times \dots \dots \dots \times \omega_n)\ (x_1 \times x_2 \times \dots \dots \dots \times x_n)$

$= max\{\omega_1(x_1), \omega_2(x_2) \dots . \omega_n(x_n)\}$

Definition : 3.5

Let $A = < \mu, \omega >$ be a cubic set of N. Then the strongest cubic relation on N is a cubic relation α with respect to $A = < \mu, \omega >$ given by

$\alpha(x, y) = \{((x, y), \beta(x, y), \gamma(x, y))/x, y \in N\}$, where β is an interval-valued fuzzy relation with respect to μ defined by $\beta(x, y) = min\,\{\mu(x), \mu(y)\}$ and γ is a fuzzy relation with respect to ω defined by $\gamma(x, y) = max\,\{\omega(x), \omega(y)\}$.

Theorem : 3.6

Every cubic bi-ideal in a regular near-ring N is a cubic sub near-ring N.

Proof:

Let $A = < \mu, \omega >$ be a cubic bi-ideal of N and $a, b \in N$. Since N is regular, there exist $x \in N$ such that $a = axa$. Then

$$\mu(ab) = \mu((axa)b)$$

$$= \mu(a(xa)b)$$

$$\geq min\,\{\mu(a), \mu(b) \text{ and}$$

$$\omega(ab) = \omega((axa)b)$$

$$= \omega(a(xa)b)$$

$$\leq max\{\omega(a), \omega(b)\}$$

Thus $A = < \mu, \omega >$ is a cubic sub bi-ideal of N.

Theorem : 3.7

Let N be a strongly regular near-ring. If $A = < \mu, \omega >$ is a cubic bi-ideal in N. Then $\mu(x) = \mu(x^2)$ and $\omega(x) = \omega(x^2)$ for all $x \in N$.

Proof:

Let $A = < \mu, \omega >$ be a cubic bi-ideal of N and $x \in N$. Since N is strongly regular, there exist $y \in N$ such that $x = x^2 y x^2$. Then

$$\begin{aligned} \mu(x) &= \mu(x^2 y x^2) \\ &\geq min\{\mu(x^2), \mu(x^2)\} \\ &= \mu(x^2) \\ &\geq min\,\{\mu(x), \mu(x)\} \\ &= \mu(x) \end{aligned}$$

$$\begin{aligned} \omega(x) &= \omega(x^2 y x^2) \\ &\leq max\{\omega(x^2), \omega(x^2)\} \\ &= \omega(x^2) \\ &\leq max\,\{\omega(x), \omega(x)\} \\ &= \omega(x) \end{aligned}$$

Hence, $\mu(x) = \mu(x^2)$ and $\omega(x) = \omega(x^2)$.

Theorem : 3.8

The direct product of cubic bi-ideals of near-rings is also a cubic bi-ideal of near-ring.

Proof :

Let $A_i = < \mu_i, \omega_i >$ be cubic bi-ideals of near-rings N_i, for $i = 1,2, \dots \dots \dots n$

Let $x = (x_1, x_2, \dots \dots \dots \dots . x_n)$, $y = (y_1, y_2, \dots \dots \dots \dots . y_n)$ and

$z = (z_1, z_2, \dots \dots \dots \dots . z_n) \in R_1 \times R_2 \times \dots \dots \dots R_n$

$$\begin{aligned}
\mu_i(x - y) &= \mu_i((x_1, x_2, \dots \dots \dots \dots . x_n) - (y_1, y_2, \dots \dots \dots \dots . y_n)) \\
&= \mu_i(x_1 - y_1, x_2 - y_2, \dots \dots \dots . x_n - y_n) \\
&= min\,\{\mu_1(x_1 - y_1), \mu_2(x_2 - y_2), \dots \dots \dots \mu_n(x_n - y_n)\} \\
&\geq min\,\{min\{\mu_1(x_1), \mu_1(y_1)\}, min\{\mu_2(x_2), \mu_2(y_2)\}, \dots \dots \dots . \\
&\qquad min\{\mu_n(x_n), \mu_n(y_n)\} \\
&= min\,\{min\{\mu_1(x_1), \mu_2(x_2), \dots \dots \dots \mu_n(x_n)\}, \\
&\qquad min\{\mu_1(y_1), \mu_2(y_2)\}, \dots \dots \dots . \mu_n(y_n)\} \\
&= min\,\{(\mu_1 \times \mu_2 \times \dots \dots \dots \times \mu_n)(x_1, x_2, \dots \dots \dots \dots . x_n), \\
&\qquad (\mu_1 \times \mu_2 \times \dots \dots \dots \times \mu_n)(y_1, y_2, \dots \dots \dots \dots . y_n)\}
\end{aligned}$$

$$\mu_i(x - y) = min\{\, \mu_1(x), \mu_1(y)\}$$

$$\begin{aligned}
\omega_i(x - y) &= \omega_i((x_1, x_2, \dots \dots \dots \dots . x_n) - (y_1, y_2, \dots \dots \dots \dots . y_n)) \\
&= \omega_i((x_1 - y_1), (x_2 - y_2), \dots \dots \dots \dots (x_n - y_n)) \\
&= max\,\{\omega_1(x_1 - y_1), \omega_2(x_2 - y_2), \dots \dots \dots \dots \omega_n(x_n - y_n)\} \\
&\leq max\,\{max\{\omega_1(x_1), \omega_1(y_1)\}, max\{\omega_2(x_2), \omega_2(y_2)\}, \dots \dots \dots ., \\
&\qquad max\{\omega_n(x_n), \omega_n(y_n)\}
\end{aligned}$$

$$= max\ \{max\{\omega_1(x_1), \omega_2(x_2), \ldots \ldots \ldots \omega_n(x_n)\},$$

$$max\{\omega_1(y_1), \omega_2(y_2)\}, \ldots \ldots \ldots . \omega_n(y_n)\}$$

$$= max\ \{(\omega_1 \times \omega_2 \times \ldots \ldots \ldots \times \omega_n)(x_1, x_2, \ldots \ldots \ldots \ldots . x_n),$$

$$(\omega_1 \times \omega_2 \times \ldots \ldots \ldots \times \omega_n)(y_1, y_2, \ldots \ldots \ldots \ldots . y_n)\}$$

$$\omega_i(x - y) = max\ \{\omega_i(x), \omega_i(y)$$

$$\mu_i(xyz) = \mu_i((x_1, x_2, \ldots \ldots \ldots \ldots . x_n)(y_1, y_2, \ldots \ldots \ldots \ldots . y_n)(z_1, z_2, \ldots \ldots \ldots \ldots . z_n))$$

$$= \mu_i(x_1 y_1 z_1, x_2 y_2 z_2, \ldots \ldots \ldots . x_n y_n z_n)$$

$$= min\ \{\mu_1(x_1 y_1 z_1), \mu_2(x_2 y_2 z_2), \ldots \ldots \ldots \mu_n(x_n y_n z_n)\}$$

$$\geq min\ \{min\ \{\mu_1(x_1), \mu_1(z_1)\}, min\{\mu_2(x_2), \mu_2(z_2)\}, \ldots \ldots \ldots,$$

$$min\ \{\mu_n(x_n), \mu_n(z_n)\}\}$$

$$= min\ \{(\mu_1 \times \mu_2 \times \ldots \ldots \times \mu_n)\ (x_1, x_2, \ldots \ldots \ldots \ldots . x_n),$$

$$(\mu_1 \times \mu_2 \times \ldots \ldots \times \mu_n)\ (z_1, z_2, \ldots \ldots \ldots \ldots . z_n)\}$$

$$\mu_i(xyz) = min\ \{\mu_i(x), \mu_i(x)\}$$

$$\omega_i(xyz) = \omega_i((x_1, x_2, \ldots \ldots \ldots . . x_n)(y_1, y_2, \ldots \ldots \ldots . . y_n)(z_1, z_2, \ldots \ldots \ldots . . z_n)$$

$$= \omega_i(x_1 y_1 z_1, x_2 y_2 z_2, \ldots \ldots \ldots . . x_n y_n z_n)$$

$$= max\ \{\omega_1(x_1 y_1 z_1), \omega_2(x_2 y_2 z_2), \ldots \ldots \ldots\ \omega_n(x_n y_n z_n)\}$$

$$\leq max\ \{max\ \{\omega_1(x_1), \omega_1(z_1)\}, max\{\omega_2(x_2), \omega_2(z_2)\}, \ldots \ldots \ldots$$

$$max\{\omega_n(x_n), \omega_n(z_n)\}\}$$

$$= max\{(\omega_1 \times \omega_2 \times \dots\dots.\times \omega_n)(x_1, x_2, \dots\dots\dots..x_n),$$

$$(\omega_1 \times \omega_2 \times \dots\dots.\times \omega_n)\ (z_1, z_2, \dots\dots\dots..z_n)$$

$\omega_i(xyz) \quad = max\ \{\omega_i(x), \omega_i(z)\}$

Hence, the direct product of cubic bi-ideals of near-rings is also a cubic bi-ideal of near-ring.

Theorem : 3.9

Let $A = < \mu, \omega >$ be a cubic set of a near-ring N and $\alpha(x, y) = \{((x, y), \beta(x, y), \gamma(x, y))|x, y \in N\}$ be a strongest cubic relation with respect to α. Then $A = < \mu, \omega >$ is a cubic bi-ideal of N if and only if α is a cubic bi-ideal of $N \times N$.

Proof:

Assume that $A = < \mu, \omega >$ is a cubic bi-ideal of N.

Let $x_1, x_2, y_1, y_2, z_1, z_2 \in N$.

Then $x = (x_1, x_2),\ y = (y_1, y_2),\ z = (z_1, z_2) \in\ N \times N$

We have

$$\begin{aligned}
\beta(x - y) \quad &= \ \beta((x_1, x_2) - (y_1, y_2)) \\
&= \ \beta(x_1 - y_1, x_2 - y_2) \\
&= min\ \{\mu(x_1 - y_1), \mu(x_2 - y_2)\} \\
&\geq min\ \{min\{\mu(x_1), \mu(y_1)\}, min\{\mu(x_2), \mu(y_2)\} \\
&= min\ \{min\{\mu(x_1), \mu(x_2)\}, min\{\mu(y_1), \mu(y_2)\}
\end{aligned}$$

$= min\ \{\ \beta(x_1, x_2), \beta(y_1, y_2)\}$

$= min\ \{\ \beta(x), \beta(y)\}$

$\gamma(x - y) \qquad = \ \gamma((x_1, x_2), (y_1, y_2))$

$= \ \gamma(x_1 - y_1, x_2 - y_2)$

$= max\ \{\omega(x_1 - y_1), \omega(x_2 - y_2)\}$

$\leq max\ \{max\{\omega(x_1), \omega(y_1)\}, max\{\omega(x_2), \omega(y_2)\}\}$

$= max\ \{max\{\omega(x_1), \omega(x_2)\}, max\{\omega(y_1), \omega(y_2)\}\}$

$= max\ \{\ \gamma(x_1, x_2), \gamma(y_1, y_2)\}$

$= max\ \{\ \gamma(x), \gamma(y)\}$

$\beta(xyz) \qquad = \ \beta((x_1, x_2)(y_1, y_2)(z_1, z_2))$

$= \ \beta(x_1y_1z_1, x_2y_2z_2)$

$= min\ \{\mu(x_1y_1z_1), \mu(x_2y_2z_2)\}$

$\geq min\ \{min\{\mu(x_1), \mu(z_1)\}, min\{\mu(x_2), \mu(z_2)\}$

$= min\ \{min\{\mu(x_1), \mu(x_2)\}, min\{\mu(z_1), \mu(z_2)\}$

$= min\ \{\ \beta(x_1, x_2), \beta(z_1, z_2)\}$

$= min\ \{\ \beta(x), \beta(z)\}$

$\gamma(xyz) \qquad = \ \gamma((x_1, x_2)(y_1, y_2)(z_1, z_2))$

$= \ \gamma(x_1y_1z_1, x_2y_2z_2)$

$= max\ \{\omega(x_1y_1z_1), \omega(x_2y_2z_2)\}$

$\leq max\ \{max\{\omega(x_1), \omega(z_1)\}, max\{\omega(x_2), \omega(z_2)\}$

$= max\ \{max\{\omega(x_1), \omega(x_2)\}, max\{\omega(z_1), \omega(z_2)\}$

$= max\ \{\ \gamma(x_1, x_2), \gamma(z_1, z_2)\}$

$= max\ \{\ \gamma(x), \gamma(z)\}$

Hence α is a cubic bi-ideal of $N \times N$.

Conversely

Assume that α is a cubic bi-ideal of $N \times N$, then

$x = (x_1, x_2),\ \ y = (y_1, y_2),\ \ z = (z_1, z_2) \in N \times N$

$min\ \{\ \mu(x_1 - y_1), \mu(x_2 - y_2)\} \quad = \beta(x_1 - y_1, x_2 - y_2)$

$= \beta((x_1, x_2) - (y_1, y_2))$

$= \beta(x - y)$

$\geq min\ \{\beta(x), \beta(y)\}$

$= min\ \{\beta(x_1, x_2), \beta(y_1, y_2)\}$

$= min\ \{min\{\ \mu(x_1), \mu(x_2)\},$

$min\{\ \mu(z_1), \mu(z_2)\}\}$

If $\mu(x_1 - y_1) \leq \mu(x_2 - y_2)$, then $\mu(x_1) \leq \mu(x_2)$ and $\mu(y_1) \leq \mu(y_2)$

we get $\mu(x_1 - y_1) \geq min\ \{\ \mu(x_1), \mu(y_1)\}$

$$\begin{aligned}
max\,\{\omega((x_1-y_1),\omega(x_2-y_2)\} &= \gamma(x_1-y_1,x_2-y_2)\\
&= \gamma((x_1,x_2)-(y_1,y_2))\\
&= \gamma(x-y)\\
&\leq max\,\{\gamma(x),\gamma(y)\}\\
&= max\,\{\gamma(x_1,x_2),\gamma(y_1,y_2)\}\\
&= max\,\{max\{\omega(x_1),\omega(x_2)\},\\
&\qquad max\,\{\omega(y_1),\omega(y_2)\}\}
\end{aligned}$$

If $\omega(x_1-y_1)\geq\omega(x_2-y_2)$, then $\omega(x_1)\geq\omega(x_2)$ and $\omega(y_1)\geq\omega(y_2)$

we get $\omega(x_1-y_1)\leq max\,\{\,\omega(x_1),\omega(y_1)\}$.

$$\begin{aligned}
min\,\{\,\mu(x_1y_1z_1),\mu(x_2y_2z_2)\} &= \beta(x_1y_1z_1,x_2y_2z_2)\\
&= \beta((x_1,x_2)(y_1,y_2)(z_1,z_2))\\
&= \beta(xyz)\\
&\geq min\,\{\beta(x),\beta(z)\}\\
&= min\,\{\beta(x_1,x_2),\beta(z_1,z_2)\}\\
&= min\,\{min\{\mu(x_1),\mu(x_2)\},min\{\mu(z_1),\mu(z_2)\}\}
\end{aligned}$$

If $\mu(x_1y_1z_1)\leq\mu(x_2y_2z_2)$, then $\mu(x_1)\leq\ \mu(x_2)$, $\ \mu(y_1)\leq\ \mu(y_2)$ and $\mu(z_1)\leq\ \mu(z_2)$

we get $\mu(x_1y_1z_1)\geq min\,\{\,\mu(x_1),\mu(z_1)\}$

$max\,\{\,\omega(x_1y_1z_1),\omega(x_2y_2z_2)\}$ $\quad = \gamma(x_1y_1z_1, x_2y_2z_2)$

$= \gamma(x_1,x_2)(y_1,y_2)(z_1,z_2)$

$= \gamma(xyz)$

$\leq max\,\{\gamma(x),\gamma(z)\}$

$= max\,\{\gamma(x_1,x_2),\gamma(z_1,z_2)\}$

$= max\,\{max\{\,\omega(x_1),\omega(x_2)\},$

$max\,\{\omega(z_1),\omega(z_2)\}$

If $\omega(x_1y_1z_1) \geq \omega(x_2y_2z_2)$, then $\omega(x_1) \geq \omega(x_2),\; \omega(y_1) \geq \omega(y_2)$ and

$\omega(z_1) \geq \omega(z_2)$.

We get $\omega(x_1y_1z_1) \leq max\,\{\omega(x_1),\omega(z_2)\}$

Hence $A = <\mu,\omega>$ is a cubic bi-ideal of N.

Theorem : 3.10

If $A = <\mu,\omega>$ be any cubic set of N. Then $A = <\mu,\omega>$ is a cubic bi-ideal of N if and only if the cubic level set $U(A;\,t,\,n)$ is a bi-ideal of N, when it is non-empty.

Proof:

Assume that $A = <\mu,\omega>$ be a cubic bi-ideal of N.

Let $x,y,z \in U(A;\,t,\,n)$ for all $t \in D[0,1]$ and $n \in [0,1]$

Then $\mu(x) \geq t,\; \mu(y) \geq t,\; \mu(z) \geq t$ and $\omega(x) \leq n,\; \omega(y) \leq n,\; \omega(z) \leq n$

Now suppose $x, y \in U(A;\ t,\ n)$ then by definition of cubic bi-ideal

$\mu(x-y) \geq min\,\{\mu(x), \mu(y)\} \geq min\,\{t, t\} \geq t$ and

$\omega(x-y) \leq max\,\{\omega(x), \omega(y)\} \leq max\,\{n, n\} \leq n$

Hence $x - y \in U(A;\ t,\ n)$

Suppose, $x,\ z \in U(A;\ t,\ n)$ and $y \in N$ then

$\mu(xyz) \geq min\,\{\mu(x), \mu(z)\} \geq min\,\{t, t\} \geq t$ and

$\omega(xyz) \leq max\,\{\omega(x), \omega(z)\} \leq max\,\{n, n\} \leq n$

Hence $xyz \in U(A;\ t,\ n)$

Therefore, $U(A;\ t,\ n)$ is a bi-ideal of N.

Conversely,

Let $t \in D[0,1]$ and $n \in [0,1]$ be such that $U(A; t, n) \neq \emptyset$ and $U(A; t, n)$ is a bi-ideal of N.

Suppose we assume that

$\mu(x-y) \ngeq min\,\{\mu(x), \mu(y)\}$ (or)

$\omega(x-y) \nleq max\{\omega(x), \omega(y)\}$

If $\mu(x-y) \ngeq min\,\{\mu(x), \mu(y)\}$ then there exist $t_1 \in D[0,1]$ such that

$\mu(x-y) < t_1 < min\,\{\,\mu(x), \mu(y)\}$ hence $x, y \in U(A;\ t_1, max\{\,\omega(x), \omega(y)\})$,

but $x - y \notin U(A;\ t_1, max\,\{\,\omega(x), \omega(y)\}$ which is contradiction

If $\omega(x-y)\ \nleq max\{\omega(x), \omega(y)\}$ then there exist $n_1 \in [0,1]$ such that

$\omega(x-y) > n_1 > max\ \{\omega(x), \omega(y)\}$ hence $x, y \in U(A; min\{\mu(x), \mu(y)\}, n_1)$,

but $x - y \notin U(A; min\{\mu(x), \mu(y)\}, n_1)$ which is contradiction

Hence $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$and

$\omega(x-y) \leq max\ \{\omega(x), \omega(y)\}$

Suppose assume that

$\mu(xyz) \ngeq min\ \{\mu(x), \mu(z)\}$ (or)

$\omega(xyz) \nleq max\{\omega(x), \omega(z)\}$

If $\mu(xyz) \ngeq min\ \{\mu(x), \mu(z)$ then there exist $t_1 \in D[0,1]$ such that

$\mu(xyz) < t_1 < min\ \{\ \mu(x), \mu(z)\}$ hence $x, z \in U(A;\ t_1, max\{\ \omega(x), \omega(y)\}$),

but $xyz \notin U(A;\ t_1, max\ \{\ \omega(x), \omega(z)\}$ which is contradiction.

If $\omega(xyz) \nleq max\ \{\omega(x), \omega(z)$ then there exist $n_1 \in D[0,1]$ such that

$\omega(xyz)\ >\ n_1\ > max\ \{\ \omega(x), \omega(z)\}$ hence $x, z \in U(A; min\{\ \mu(x), \mu(y)\}$

but $xyz \notin U(A; min\ \{\mu(x), \mu(z)\}$ which is contradiction.

Hence $\mu(xyz) \geq min\ \{\mu(x), \mu(z)\}$

$\omega(xyz) \leq max\{\omega(x), \omega(z)\}$

Therefore $A = < \mu, \omega >$ is a cubic bi-ideal of N.

Theorem : 3.11

Let H be a non-empty subset of N. Then H is a bi-ideal of N if and only if the characteristic cubic set $\chi_H = < \mu_{\chi_H}, \omega_{\chi_H} >$ of H in N is a cubic bi-ideal of N.

Proof:

Assume that H is a bi-ideal of N.

Let $x, y \in N$.

Suppose that $\mu_{\chi_H}(x-y) \ < \ min\,\{\mu_{\chi_H}(x), \mu_{\chi_H}(y)\}$ and

$$\omega_{\chi_H}(x-y) \ > \ max\,\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\}$$

It follows that $\mu_{\chi_H}(x-y) = 0,\ min\{\mu_{\chi_H}(x), \mu_{\chi_H}(y)\} = 1$

$$\omega_{\chi_H}(x-y) = 1,\ max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\} = 0$$

This implies that $x, y \in H$ but $x - y \notin H$ a contradiction to H being a near-ring of N.

Suppose that $\mu_{\chi_H}(xyz) < min\{\mu_{\chi_H}(x), \mu_{\chi_H}(z)\}\, and$

$$\omega_{\chi_H}(xyz) > max\{\omega_{\chi_H}(x), \omega_{\chi_H}(z)\}$$

If follows that $\mu_{\chi_H}(xyz) = 0,\ min\{\mu_{\chi_H}(x), \mu_{\chi_H}(z)\} = 1$

$$\omega_{\chi_H}(xyz) = 1,\ max\{\omega_{\chi_H}(x), \omega_{\chi_H}(z)\} = 0$$

This implies that $x, z \in H$ but $xyz \notin H$ a contradiction to H.

Hence $\chi_H = < \mu_{\chi_H}, \omega_{\chi_H} >$ is a cubic bi-ideal of N.

Conversely,

Assume that $\chi_H = < \mu_{\chi_H}, \omega_{\chi_H} >$ is a cubic bi-ideal of N, for any subset H of N.

Let $x, y \in H$ then $\mu_{\chi_H}(x) = \mu_{\chi_H}(y) = 1$ and $\omega_{\chi_H}(x) = \omega_{\chi_H}(y) = 0$

Since χ_H is a cubic bi-ideal of N.

$\mu_{\chi_H}(x-y) \geq min\,\{\mu_{\chi_H}(x), \mu_{\chi_H}(y) = 1\ and$

$\omega_{\chi_H}(x-y) \leq max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\} = 0$

This implies that $x - y \in H$

Let $x, z \in H$ and $y \in N$ then

$\mu_{\chi_H}(x) = \mu_{\chi_H}(z) = 1$ and $\omega_{\chi_H}(x) = \omega_{\chi_H}(z) = 0$

$\mu_{\chi_H}(xyz) \geq min\,\{\mu_{\chi_H}(x), \mu_{\chi_H}(z)\}$ and

$\omega_{\chi_H}(xyz) \leq max\{\omega_{\chi_H}(x), \omega_{\chi_H}(z)\} = 0$

This implies that $xyz \in H$

Hence H is a bi-ideal of N.

Theorem : 3.12

If $A = < \mu, \omega >$ is a cubic bi-ideal of N, then $A^c = < (\mu)^c,\ (\omega)^c >$ is also a cubic bi-ideal of N.

Proof:

Let $x, y \in N$ and $A = < \mu, \omega >$ is a cubic bi-ideal of N, then

(i) $(\mu)^c(x-y) \quad = 1 - (\mu - y)$

$\leq 1 - min\,\{\mu(x), \mu(y)\}$

$= max\,\{1 - \mu(x), 1 - \mu(y)\}$

$= max\,\{(\mu)^c(x), (\mu)^c(y)\}$ and

$(\omega)^c(x-y) \quad = 1 - \omega(x-y)$

$\geq 1 - max\,\{\omega(x), \omega(y)\}$

$= min\,\{1 - \omega(x), 1 - \omega(y)\}$

$= min\,\{(\omega)^c(x), (\omega)^c(y)\}$

(ii) $(\mu)^c(xyz) = 1 - \mu(xyz)$

$\leq 1 - min\,\{\mu(x), \mu(z)\}$

$= max\,\{1 - \mu(x), 1 - \mu(z)\}$

$= max\,\{(\mu)^c(x), (\mu)^c(z)\}$

$(\omega)^c(xyz) = 1 - \omega(xyz)$

$\geq 1 - max\,\{\omega(x), \omega(z)\}$

$= min\,\{1 - \omega(x), 1 - \omega(z)\}$

$= min\,\{(\omega)^c(x), (\omega)^c(z)\}$

Therefore, $A^c = < (\mu)^c, (\omega)^c >$ is a cubic bi-ideal of N.

CHAPTER – IV

CUBIC WEAK BI-IDEALS OF NEAR RINGS

Definition : 4.1

A cubic set $A = < \mu, \omega >$ of R is called the cubic subgroup of R, if

a) $\mu(x - y) \geq min\,\{\mu(x), \mu(y)\}$

b) $\omega(x - y) \leq max\,\{\omega(x), \omega(y)\}\ \ \forall\, x, y \in R.$

Definition : 4.2

A cubic subgroup $A = < \mu, \omega >$ of R is called the cubic weak bi-ideal of R, if

a) $\mu(xyz) \geq min\,\{\mu(x), \mu(y), \mu(z)\}$

b) $\omega(xyz) \leq max\,\{\omega(x), \omega(y), \omega(z)\}\ \ \forall\, x, y, z \in R.$

Example : 4.3

Let $R = \{a, b, c, d\}$ be a near-ring with two binary operations + and . are defined as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(R, +, .)$ is a near-ring.

Let $\mu : R \rightarrow D[0,1]$ be an interval valued fuzzy subset defined by $\mu(a) = [0.8, 0.9]$, $\mu(b) = [0.6, 0.7]$

and $\mu(c) = [0.4, 0.5] = \mu(d)$. Then μ is an interval valued fuzzy weak bi-ideal of R.

Let $\omega : R \to [0,1]$ be a fuzzy subset defined by $\omega(a) = 0.2,\ \omega(b) = 0.4$ and

$\omega(c) = 0.8 = \omega(d)$. Then ω is a fuzzy weak bi-ideal of R.

Hence $A = (\mu, \omega)$ is a cubic weak bi-ideal of R.

Definition : 4.4

Let A_i be cubic weak bi-ideals of near-rings R_i, for $i = 1,2, \dots n$. Then the cubic direct product of $A_i (i = 1,2, \dots n)$ is a function

$\mu_1 \times \mu_2 \times \dots \times \mu_n : R_1 \times R_2 \times, \dots R_n \to D[0,1]$,

$\omega_1 \times \omega_2 \times \dots \times \omega_n : R_1 \times R_2 \times, \dots R_n \to [0,1]$ defined by

$(\mu_1 \times \mu_2 \times \dots \times \mu_n)(x_1, x_2, \dots . x_n) = min\,\{\mu_1(x_1), \mu_2(x_2), \dots \mu_n(x_n)\}$ and

$(\omega_1 \times \omega_2 \times \dots \times \omega_n)(x_1, x_2, \dots . x_n) = max\,\{\omega_1(x_1), \omega_2(x_2), \dots \omega_n(x_n)\}$

Definition : 4.5

Let $A_1 = < \mu, \omega >$ and $A_2 = < \mu, \omega >$ be two cubic subsets of R. Then $A_1 A_2$ is cubic subsets of R defined by:

$$(A_1A_2)(x) = \begin{cases} (\mu_1\mu_2)(x) = \begin{cases} \sup\limits_{x=yz} min\{\mu(y), \mu(z)\} & if\ x = yz\ \forall x, y, z \in R \\ [0,0] & otherwise \end{cases} \\ (\omega_1\omega_2)(x) = \begin{cases} \inf\limits_{x=yz} max\{\omega(y), \omega(z)\} & if\ x = yz\ \forall x, y, z \in R \\ 1 & otherwise \end{cases} \end{cases}$$

Theorem : 4.6

Let $A = < \mu, \omega >$ be a cubic subgroup of R. Then $A = < \mu, \omega >$ is a cubic weak bi-ideal of R if and only if $AAA \sqsubseteq A. (i.e., \mu\mu\mu \subseteq \mu\ and\ \omega\omega\omega \supseteq \omega)$

Proof:

Assume that $A = < \mu, \omega >$ is a cubic weak bi-ideal of R.

Let $x, y, z, p, q \in R$ such that $x = yz$ and $y = pq$. Then

$$\mu\mu\mu(x) = \sup_{x=yz}\{min\ \{(\mu\mu)(y), \mu(z)\}\}$$

$$= \sup_{x=yz}\{min\ \{\sup_{y=pq} min\ \{\mu(p), \mu(q), \mu(z)\}\}$$

$$= \sup_{x=yz} \sup_{y=pq}\{min\ \{min\ \{\mu(p), \mu(q), \mu(z)\}\}$$

$$= \sup_{x=pqz}\{min\{\mu(p), \mu(q), \mu(z)\}\}$$

$$\leq \sup_{x=pqz} \mu(pqz)$$

$$= \mu(x)$$

If x cannot be expressed as $x = yz$ then $\mu\mu\mu(x) = 0 \leq \mu(x)$.

In both cases $\mu\mu\mu \subseteq \mu$.

$$\omega\omega\omega(x) = \inf_{x=yz}\{max\ \{(\omega\omega)(y), \omega\omega(z)\}\}$$

$$= \inf_{x=yz}\{max\ \{\inf_{y=pq} max\ \{\omega(p), \omega(q), \omega(z)\}\}$$

$$= \inf_{x=yz} \inf_{y=pq}\{max\ \{max\{\omega(p), \omega(q), \omega(z)\}\}$$

$$= \inf_{x=pqz}\{max\{\omega(p), \omega(q), \omega(z)\}\}$$

$$\geq \inf_{x=pqz} \omega(pqz)$$

$$= \omega(x)$$

If x cannot be expressed as $x = yz$ then $\omega\omega\omega(x) = 1 > \omega(x)$.

In both cases $\omega\omega\omega \supseteq \omega$.

Hence $AAA \sqsubseteq A$.

Conversely,

Assume that $AAA \sqsubseteq A$ holds.

To prove that $A = < \mu, \omega >$ is a cubic weak bi-ideal of R.

For any $x, y, z, a \in R$ such that $a = xyz$ then

$$\mu(xyz) = \mu(a) \geq (\mu\mu\mu)(a)$$

$$= \sup_{a=bc} min\,\{(\mu\mu)(b), \mu(c)\}$$

$$= \sup_{a=bc} \{min\,\{\sup_{b=pq} min\{\mu(p), \mu(q)\}, \mu(c)\}\}$$

$$= \sup_{a=pqc} \{min\{\mu(p), \mu(q)\}, \mu(c)\}$$

$$\mu(xyz) \geq min\,\{\mu(x), \mu(y), \mu(z)\}$$

$$\omega(xyz) = \omega(a) \leq (\omega\omega\omega)(a)$$

$$= \inf_{a=bc} max\,\{(\omega\omega)(b), \omega(c)\}$$

$$= \inf_{a=bc} \{max\,\{\inf_{b=pq} max\,\{\omega(p), \omega(q), \omega(c)\}\}$$

$$= \inf_{a=pqc} \{max\,\{(\omega(p), \omega(q), \omega(c)\}$$

$$\omega(xyz) \leq max\,\{\omega(x), \omega(x), \omega(x)\}$$

Hence $A =< \mu, \omega >$ is a cubic weak bi-ideal of R.

Theorem : 4.7

Let $A_1$and A_2 be two cubic weak bi-ideal of R then the product A_1A_2 is a cubic weak bi-ideals of R.

Proof:

Let $A_1 = <\mu_1, \omega_1>$ and $A_2 = <\mu_2, \omega_2>$ be two cubic weak bi-ideals of R.

Since μ_1 and μ_2 are interval-valued fuzzy weak bi-ideals of R then

$$(\mu_1\mu_2)(x-y) = \sup_{x-y=pq} min\,\{\mu_1(p), \mu_2(q)\}$$

$$\geq \sup_{x-y=p_1q_1-p_2q_2\leq(p_1-p_2)(q_1-q_2)} min\,\{\mu_1(p_1-p_2), \mu_2(q_1-q_2)\}$$

$$\geq sup\, min\,\{min\,\{\mu_1(p_1), \mu_1(p_2)\}, min\,\{\mu_2(q_1)\mu_2(q_2)\}\}$$

$$= sup\, min\,\{min\,\{\mu_1(p_1), \mu_2(q_1)\}, min\,\{\mu_1(p_2)\mu_2(q_2)\}\}$$

$$= min\,\{\sup_{x=p_1q_1} min\,\{\mu_1(p_1), \mu_2(q_1)\}, \sup_{y=p_2q_2} min\,\{\mu_1(p_2), \mu_2(q_2)\}\}$$

$$= min\,\{(\mu_1\mu_2)(x), (\mu_1\mu_2)(y)\}$$

It follows that $(\mu_1\mu_2)$ is an interval-valued fuzzy subgroup of R. Further

$$(\mu_1\mu_2)(\mu_1\mu_2)(\mu_1\mu_2) = \mu_1\mu_2(\mu_1\mu_2\mu_1)\mu_2$$

$$\subseteq \mu_1\mu_2(\mu_2\mu_2\mu_2)\mu_2$$

$$\subseteq \mu_1(\mu_2\mu_2\mu_2)$$

$$\subseteq (\mu_1\mu_2)$$

Therefore $(\mu_1\mu_2)$ is an interval-valued fuzzy weak bi-ideals of R.

Since ω_1, ω_2 are fuzzy weak bi-ideals of R, then

$$(\omega_1\omega_2)(x-y) = \inf_{x-y=pq} max\ \{\omega_1(p), \omega_2(q)\}$$

$$\leq \inf_{x-y=p_1q_1-p_2q_2\leq(p_1-p_2)(q_1-q_2)} max\ \{\omega_1(p_1-p_2), \omega_2(q_1-q_2)\}$$

$$\leq inf\ max\ \{\ max\{\omega_1(p_1), \omega_1(p_2)\}, max\ \{\omega_2(q_1)\omega_2(q_2)\}\}$$

$$= inf\ max\ \{\ max\ \{\omega_1(p_1), \omega_2(q_1)\}, max\ \{\omega_1(p_2)\omega_2(q_2)\}\}$$

$$= max\ \{\inf_{x=p_1q_1} max\ \{\omega_1(p_1), \omega_2(q_1)\}, \inf_{y=p_2q_2} max\ \{\omega_1(p_2), \omega_2(q_2)\}\}$$

$$= max\ \{(\omega_1\omega_2)(x), (\omega_1\omega_2)(y)\}$$

It follows that $(\omega_1\omega_2)$ is an interval-valued fuzzy subgroup of R. Further

$$\begin{aligned}(\omega_1\omega_2)(\omega_1\omega_2)(\omega_1\omega_2) &= \omega_1\omega_2(\omega_1\omega_2\omega_1)\omega_2\\ &\supseteq \omega_1\omega_2(\omega_2\omega_2\omega_2)\omega_2\\ &\supseteq \omega_1(\omega_2\omega_2\omega_2)\\ &\supseteq (\omega_1\omega_2)\end{aligned}$$

Therefore $(\omega_1\omega_2)$ is an interval-valued fuzzy weak bi-ideals of R.

Hence $A_1A_2 = <(\mu_1\mu_2), (\omega_1\omega_2)>$ is a cubic weak bi-ideal of R.

Remarks : 4.8

Let A_1 and A_2 be two cubic weak bi-ideals of R then the product A_2A_1 is also a cubic weak bi-ideal of R.

Theorem : 4.9

Let $A = <\mu, \omega>$ be a cubic weak bi-ideal of R, then the set

$$R_A = \{x \in R | A(x) = A(0)\}\ (i.e., R_A = \{x \in R | \mu(x) = \mu(0)\ and\ \omega(x) = \omega(0)\})$$

is a weak bi-ideal of R.

Proof:

Let $A = < \mu, \omega >$ be a cubic weak bi-ideal of R.

Let $x, y \in R_A$

Then $A(x) = A(0)$ and $A(y) = A(0)$ $(i.e.,)$ $\mu(x) = \mu(0),\ \omega(x) = \omega(0)$

and $\mu(y) = \mu(0),\ \omega(y) = \omega(0)$.

Since μ is an interval-valued fuzzy weak bi-ideal of R, we have $\mu(x) = \mu(0)$ and $\mu(y) = \mu(0)$.

$\mu(x-y)\ \geq\ min\{\mu(x), \mu(y)\} = min\ \{\mu(0), \mu(0)\}$ and ω is a fuzzy weak

bi-ideal of R,

we have $\omega(x) = \omega(0)$ and $\omega(y) = \omega(0)$ then

$\omega(x-y)\ \leq\ max\{\omega(x), \omega(y)\} = max\{\omega(0), \omega(0)\} = \omega(0)$

Thus $x - y \in R_A$

For every $x, y, z \in R_A$. Then $A(x) = A(0),\ A(y) = A(0)$ and $A(z) = A(0)$.

Since μ is an interval-valued fuzzy weak bi-ideal of R, we have $\mu(x) = \mu(0)$,

$\mu(y) = \mu(0)$ and $\mu(z) = \mu(0)$

then $\mu(xyz) \geq min\{\mu(x), \mu(y), \mu(z)\} = min\{\mu(0), \mu(0),\ \mu(0)\} = \mu(0)$

and ω is a fuzzy weak bi-ideal of R, we have $\omega(x) = \omega(0), \omega(y) = \omega(0)$ and

$\omega(z) = \omega(0)$ and

$\omega(xyz) \leq max\{\omega(x), \omega(y), \omega(z)\} = max\{\omega(0), \omega(0), \omega(0)\} = \omega(0)$.

Thus $xyz \in R_A$

Hence R_A is a cubic weak bi-ideal of R.

Theorem : 4.10

The direct product of cubic weak bi-ideal of near-rings is also a cubic weak bi-ideal.

Proof:

Let $A_i = < \mu_i, \omega_i >$ be cubic weak bi-ideals of near-rings R_i for $i = 1,2, \dots n$.

Let $x = (x_1, x_2, \dots x_n), \; y = (y_1, y_2, \dots y_n)$ and

$z = (z_1, z_2, \dots z_n) \in R_1 \times R_2 \times \dots \times R_n$.

$$\mu_i(x - y) = \mu_i((x_1, x_2, \dots x_n) - (y_1, y_2, \dots y_n))$$

$$= \mu_i(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$$

$$= min\{\mu_1(x_1 - y_1), \mu_2(x_2 - y_2), \dots, \mu_n(x_n - y_n)\}$$

$$\geq min\{min\{\mu_1(x_1), \mu_1(y_1)\}, min\{\mu_2(x_2), \mu_2(y_2)\}, \dots,$$

$$min\{\mu_n(x_n), \mu_n(y_n)\}\}$$

$$= min\{min\{\mu_1(x_1), \mu_2(x_2), \dots \mu_n(x_n)\}, min\{\mu_1(y_1), \mu_2(y_2), \dots \mu_n(y_n)\}\}$$

$$= min\{(\mu_1 \times \mu_2 \times \dots \mu_n)(x_1, x_2, \dots x_n), (\mu_1 \times \mu_2 \times \dots \mu_n)(y_1, y_2, \dots y_n)\}$$

$$= min\{\mu_i(x), \mu_i(y)\}$$

$$\omega_i(x - y) = \omega_i((x_1, x_2, \dots x_n) - (y_1, y_2, \dots y_n))$$

$$= \omega_i(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$$

$$= max\ \{\omega_1(x_1 - y_1), \omega_2(x_2 - y_2), \dots, \omega_n(x_n - y_n)\}$$

$$\leq max\ \{max\{\omega_1(\mathrm{x}_1), \omega_1(\mathrm{y}_1)\}, max\{\omega_2(x_2), \omega_2(y_2)\}, \dots,$$

$$max\{\omega_n(x_n), \omega_n(y_n)\}\}$$

$$= max\ \{max\{\omega_1(x_1), \omega_2(x_2), \dots \omega_n(x_n)\},$$

$$max\{\omega_1(y_1), \omega_2(y_2), \dots \omega_n(y_n)\}\}$$

$$= max\ \{(\omega_1 \times \omega_2 \times \dots \omega_n)(x_1, x_2, \dots x_n),$$

$$(\omega_1 \times \omega_2 \times \dots \omega_n)(y_1, y_2, \dots y_n)\}$$

$$= max\ \{\omega_i(x), \omega_i(y)\}$$

and $\mu_i(xyz) = \mu_i((x_1, x_2, \dots x_n)(y_1, y_2, \dots y_n)(z_1, z_2, \dots z_n))$

$$= \mu_i(x_1 y_1 z_1, x_2 y_2 y_2, \dots, x_n y_n z_n)$$

$$= min\ \{\mu_1(x_1 y_1 z_1), \mu_2(x_2 y_2 y_2), \dots, \mu_n(x_n y_n z_n)$$

$$\geq min\ \{min\{\mu_1(x_1), \mu_1(y_1), \mu_1(z_1)\}, min\{\mu_2(x_2), \mu_2(y_2), \mu_2(z_2)\}, \dots,$$

$$min\{\mu_n(x_n), \mu_n(y_n), \mu_n(z_n)\}\}$$

$$= min\ \{min\{\mu_1(x_1), \mu_2(x_2), \dots \mu_n(x_n)\},$$

$$min\{\mu_1(y_1), \mu_2(y_2), \dots \mu_n(y_n)\}, min\{\mu_1(z_1), \mu_2(z_2), \dots \mu_n(z_n)\}$$

$$= min\ \{(\mu_1 \times \mu_2 \times \dots \mu_n)(x_1, x_2, \dots x_n), (\mu_1 \times \mu_2 \times \dots \mu_n)(y_1, y_2, \dots y_n)$$

$$(\mu_1 \times \mu_2 \times \dots \mu_n)(z_1, z_2, \dots z_n)\}$$

$$= min\ \{\mu_i(x), \mu_i(y), \mu_i(z)\}$$

$$\omega_i(xyz) \qquad = \omega_i((x_1, x_2, \dots x_n)(y_1, y_2, \dots y_n)(z_1, z_2, \dots z_n))$$

$$= \omega_i(x_1 y_1 z_1, x_2 y_2 y_2, \dots, x_n y_n z_n)$$

$$= max\ \{\omega_1(x_1 y_1 z_1), \omega_2(x_2 y_2 y_2), \dots, \omega_n(x_n y_n z_n)$$

$$\leq max\ \{max\{\omega_1(x_1), \omega_1(y_1), \omega_1(z_1)\},$$

$$max\{\omega_2(x_2), \omega_2(y_2), \omega_2(z_2)\}, \dots, max\{\omega_n(x_n), \omega_n(y_n), \omega_n(z_n)\}\}$$

$$= max\ \{max\{\omega_1(x_1), \omega_2(x_2), \dots \omega_n(x_n)\},$$

$$max\{\omega_1(y_1), \omega_2(y_2), \dots \omega_n(y_n)\}, max\{\omega_1(z_1), \omega_2(z_2), \dots \omega_n(z_n)\}\}$$

$$= max\ \{(\omega_1 \times \omega_2 \times \dots \omega_n)(x_1, x_2, \dots x_n),$$

$$(\omega_1 \times \omega_2 \times \dots \omega_n)(y_1, y_2, \dots y_n), (\omega_1 \times \omega_2 \times \dots \omega_n)(z_1, z_2, \dots z_n)\}$$

$$= max\ \{\omega_i(x), \omega_i(y), \omega_i(z)\}$$

A STUDY ON TYPES OF DIFFERENCE

LABELING OF GRAPHS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

N.AROCKIA SHEGALI

Reg.No: 19SPMT03

Under the guidance of

Dr. P.ANBARASI RODRIGO M.Sc., B.Ed., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON TYPES OF DIFFERENCE LABELING OF GRAPHS"** submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli**in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year**2020-2021** by

N.AROCKIA SHEGALI (Reg. No: 19SPMT03)

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

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CONTENT

INTRODUCTION

Graph theory is a major area of combinatorics, and during recent decades, graph theory has developed into a major area of mathematics. In addition to its growing interest and importance as a mathematical subject, it has applications to many fields. The paper written by Leonhard Euler on the *Seven Bridges of Konigsberg* and published in 1736 is regarded as the first paper in the history of graph theory.

If the vertices of the graph are assigned values subject to certain conditions is known as graph labeling. Graph labeling was first introduced in the mid sixties. A dynamic survey on graph labeling is regularly updated by Gallian [7] and it is published by Electronic Journal of Combinatory. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in past three decades.

Graph labeling problems have three important characteristics.

1. A set of numbers from which vertex labels are chosen.

2. A rule that assigns a value to each edge.

3. A condition that these values must satisfy.

The brief summary of definitions and other information which are necessary for the present investigation are given below. Beginning with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For all other terminology and notations in graph theory I follow West [16].

Labeled graphs have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts. A systematic presentation of diverse applications of graph labeling is given by J.C.Bermond [3] and a detailed study of variety of applications is given by Bloom and Golomb [4].

1 .PRELIMINARIES

Definition :1.1

A **graph** $G = (V(G), E(G))$ consists of a set of objects $V(G) = \{v_1, v_2, \dots\}$ called **vertices**, and another set $E(G) = \{e_1, e_2, \dots\}$ whose elements are called **edges**, such that each edge e_k is identified with an unordered pair (v_i, v_j)ofvertices. The set $V(G)$ is called the **vertex set** of G and $E(G)$ its **edge set** .The number of vertices in $V(G)$ is called the **order** of G and the number of edges in $E(G)$ is called the size of G. A graph G of order p and size q is called a (p, q)graph.

Definition :1.2

The vertices v_i, v_j associated with edge e_k are called the **end vertices** of e_k. An edge having the same vertex as both its end vertices is called a **self-loop** or loops. Two or more edges associated with a given pair of vertices are called **parallel edges**. A graph that has neither self-loops nor parallel edges is called a **simple graph**.

Definition :1.3

A graph with a finite number of vertices as well as a finite number of edges is called a **finite graph**.

Definition :1.4

When a vertex v_i is an end vertex of some edge e_j, $v_i$and e_j are said to be **incident** with(on or to)each other. Two non parallel **edges** are said to be **adjacent** if they are incident on a common vertex. Similarly two vertices are said to be adjacent if they are the end vertices of the same edge. A vertex of degree one is called a **pendant vertex** or an end vertex.

Definition :1.5

The **degree** of a vertex v in a graph G is defined as the number of edges of G incident on v, with self-loops counted twice.

Definition :1.6

A **subgraph** of a graph $G = (V(G), E(G))$ is a graph $H = (V(H), E(H))$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition :1.7

If e is an edge in graph G, then $\boldsymbol{G - e}$denotes a subgraph of G obtained by deleting e from G. Deletion of an edge does not imply deletion of its end vertices.

Definition :1.8

A **walk** is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. However, a vertex may appear more than once.

Definition :1.9

Vertices with which a walk begins and ends are called its **terminal** vertices. A walk that begins and ends at the same vertex is called a **closed walk**. A walk that is not closed is called an **open walk**.

Definition :1.10

An open walk in which no vertex appears more than once is called a **path**. A path on n vertices is denoted by P_n. The number of edges in a path is called the **length** of a path.

Definition :1.11

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a **cycle**. A cycle on n vertices is denoted by C_n .

Definition :1.12

A graph G is said to be **connected** if there is at least one path between every pair of vertices in G.

Definition :1.13

A **tree** is a connected graph without any cycles.

Definition :1.14

A simple graph in which there exists an edge between every pair of vertices is called a **complete graph**. A complete graph on n vertices is denoted by K_n.

Definition :1.15

A **bipartite graph** is one whose vertex set can be partitioned into two subsets X and Y so that each edge has one end in X and the other end in Y; such a partition (X, Y) is called a **bipartition** of the graph.

Definition :1.16

A **complete bipartite graph** is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. If $|X| = m$ and $|Y| = n$, then a complete bipartite graph with bipartition (X, Y) is denoted by $K_{m,n}$.

Definition :1.17

The graph $K_{1,n}$ is called a **star**. The vertex of $K_{1,n}$ with degree n is called the **apex** or central vertex.

Definition :1.18

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then G_1 and G_2 are said to be **disjoint** if they have no vertex in common. If G_1 and G_2 are disjoint graphs, then the **join** of G_1 and G_2 is denoted by $G_1 + G_2$ and is defined as

$V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \epsilon V_1, v \epsilon V_2\}$.

Definition :1.19

For n ≥ 4, the **wheel** on n vertices, denoted by W_n, is defined to be the graph $K_1 + C_{n-1}$.

Definition :1.20

The **corona** $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2. The graph $P_n \odot K_1$ is called a **comb**.

Definition :1.21

The **bistar** $B_{m,n}$ is the graph obtained by making the two central vertices of $K_{1,m}$ and $K_{1,n}$ adjacent.

Definition :1.22

The **friendship graph** F_n can be constructed by joining n copies of the cycle graph $C_3$with a common vertex.

Definition :1.23

Let G be a graph with fixed vertex v. The comb of G is the graph $(P_m\ ; G)$ obtained from m copies of G and the path $P_m : u_1, u_2, \dots, u_m$ by joining u_i with vertex v of the i^{th} copy of G by means of an edge for $1 \leq i \leq m$.

Definition :1.24

Ladder graph L_n is a planar undirected graph with $2n$ vertices and $n + 2(n-1)$ edges. The ladder graph can be obtained as the cartesian product of two path graphs, one of which has only one edge : $L_n = P_n * P_1$.

Definition :1.25

The **gear graph**, also sometimes known as a bipartite wheel graph, is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle . The gear graph has nodes and edges.

Definition :1.26

A **triangular snake** is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block-cut point graph is a path. Equivalently it is obtained from a path $P_n : u_1, u_2, \dots, u_{n+1}$ by joining $u_i$and u_{i+1} to a new vertex v_i for $1 \leq i \leq n$. A triangular snake has $2n + 1$ vertices and $3n$ edges, where n is the number of blocks in the triangular snake. It is denoted by T_n.

Definition :1.27

A **fan graph** $F_{m,n}$ is defined as the graph join $K_m + P_n$, where K_m is the empty graph on m nodes and $P_n$is the path graph on n nodes. The case $m = 1$ corresponds to the usual fan graphs, while $m = 2$ corresponds to the double fan, etc.

Definition :1.28

Double star $K_{(I,n,n)}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n + 1$ vertices and $2n$ edges.

Definition :1.29

A **sun graph** S_n is a cycle on t vertices with an edge terminating in a vertex of degree 1 attached to each vertex on the cycle.

Definition :1.30

A **butterfly graph** is a planar undirected graph with 5 vertices and 6 edges. It can be constructed by joining 2 copies of the cycle graph C_3 with a common vertex and is therefore isomorphic to the friendship graph F_2.

Definition :1.31

For each point v of a graph G take a new vertex v' and join v' to those points of G adjacent to v. The graph thus obtained is called the **splitting graph** of G and is denoted as $S'(G)$.

Definition :1.32

A **book graph** with n pages is defined as the Cartesian product of the complete bipartite graph $K_{1,n}$ and a path of length 1 and is denoted by B_n.

Definition :1.33

A **crown graph** R_n is formed by adding to the n points $v_1, v_2, \ldots, v_n$ of a cycle C_n , n more pendant points $u_1, u_2, \ldots, u_n$ and n more lines $u_i v_i$, $i = 1,2,3, \ldots., n$ for $n \geq 3$.

Definition :1.34

A **pentagonal snake** is a connected graph all of whose blocks are pentagons. A pentagonal snake is a pentagonal cactus whose block-cut point graph is a path. Equivalently it is obtained from a path $P_n : u_1 u_2 \ldots u_{n+1}$ by joining u_i and u_{i+1} to a new vertices v_i , w_i , x_i for $1 \leq i \leq n$. A pentagonal snake has $2n + 3$ vertices and $5n$ edges, where n is the number of blocks in the pentagonal snake. i.e. The pentagonal snake is obtained from the path P_n by replacing each edge of thepath by a pentagon C_n.

Definition :1.35

The **middle graph** of G, denoted by $M(G)$, is $V\ (G) \cup E(G)$ such that two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds.

(i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in$V$ (G), y is in $E(G)$ and x, y are incident in G.

Definition :1.36

A **shell graph** S_n is the graph obtained by taking $n - 3$ concurrent chords in cycle C_n . The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan f_{n-1}. i.e. $S_n = f_{n-1} = P_{n-1} + K_1$.

Definition :1.37

Let $G = (V, E)$ be a graph. A **difference labeling** of G is an injection f from V to the set of non- negative integer with weight function f^*on E given by $f^*(uv) = |f(u) - f(v)|$ for every edge in $G. A$ graph with a difference labeling defined on it is called a labeled graph .

Definition :1.38

If the vertices of the graph are assigned values subject to certain conditions is known as **Graph labeling** .

CHAPTER 2

2. SQUARE DIFFERENCE LABELING OF SOME GRAPHS

2.1 Introduction

Definition : 2.1.1

Let $G = (V(G), E(G))$ be a graph. Then G is said to be a square difference labeling if there exists a bijection $f : V(G) \rightarrow \{0,1,2, \ldots, p-1\}$ such that the induced function $f^* : E(G) \rightarrow N$ given by

$$f^*(uv) = |[f(u)]^2 - [f(v)]^2|$$

for every $uv \in E(G)$ are all distinct.

Definition : 2.1.2

Any graph which admits square difference labeling is called square difference graph**.**

2.2 Main Results

Here we prove that the sun, triangular snake , Butterfly, book, splitting graph, cycle , graph $K_2 + mK_1$, path and bistars admit square difference labelling.

Theorem : 2.2.1

The Sun graph S_n is a square difference graph.

Proof :

Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle S_n and $u_1, u_2, \ldots, u_n$ be the end vertices of each edge attached to $v_1, v_2, \ldots, v_n$.

Here $V(S_n) = \{0,1, \ldots., n-1\}$

$$E(S_n) = \{u_i v_i / 1 \leq i \leq n\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{ v_i v_1 / i \in n\}$$

Define $f: V(S_n) \rightarrow \{0,1, \ldots., p-1\}$ as follows

$$f(u_i) = 2i - 2 \quad , \qquad 1 \leq i \leq n$$

$$f(v_i) = 2i - 1 \quad , \qquad 1 \leq i \leq n$$

Then$f^*: E(G) \rightarrow N$ given by

$$f^*(u_{i+1} v_{i+1}) = v_{i+1}^2 - u_{i+1}^2 \, , \qquad 0 \leq i \leq n-1$$

$$f^*(v_i v_{i+1}) = v_{i+1}^2 - v_i^2, \qquad 1 \leq i \leq n-1$$

$$f^*(v_i v_1) = v_i^2 - v_1^2 \qquad , \qquad i \in n$$

Clearly we have,

$$f^*(v_i v_{i+1}) = 8i \qquad , \qquad 1 \leq i \leq n-1$$

$$f^*(u_{i+1} v_{i+1}) = 4i + 1 \, , \qquad 0 \leq i \leq n-1$$

$$f^*(v_i v_1) = 4i(i-1), \qquad i \in n$$

Hence the edge labels are distinct.

Therefore the sun graph S_n is a square difference graph.

Example : 2.2.1

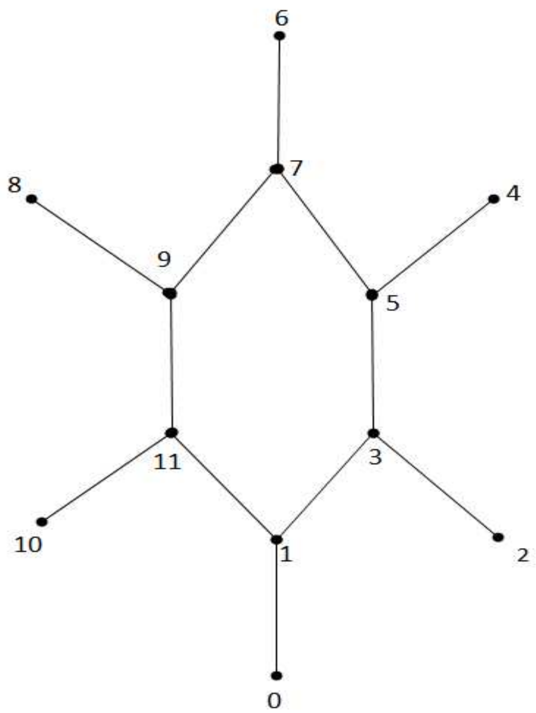

Figure 1 : Square difference labeling of S_6.

Theorem : 2.2.2

The triangular snake T_n is a square difference graph.

Proof :

We define a function $f : V(G) \rightarrow \{ 0, 1, 2, \ldots, n-1 \}$ by

$f(u_i) = 2i$, i is even

$f(v_i) = 2i - 1$, i is odd

Then $f^*: E(G) \rightarrow N$ given by

$$f^*(u_i u_{i+1}) = u_{i+1}^2 - u_i^2 \quad , \quad 1 \leq i \leq n$$

$$f^*(u_{i+1}v_{i+1}) = v_{i+1}^2 - u_{i+1}^2 \quad , \quad 0 \le i \le n-1$$

$$f^*(v_i u_{i+1}) = u_{i+1}^2 - v_i^2 \quad , \quad 1 \le i \le n$$

We have ,

$$f^*(u_i u_{i+1}) = 4(2i-1) \ , \quad 1 \le i \le n$$

$$f^*(u_{i+1}v_{i+1}) = 4i+1 \quad , \quad 0 \le i \le n-1$$

$$f^*(v_i u_{i+1}) = 4i-1 \quad , \quad 1 \le i \le n$$

Hence the edge labels are distinct.

Therefore the triangular snake T_n is a square difference graph.

Example : 2.2.2

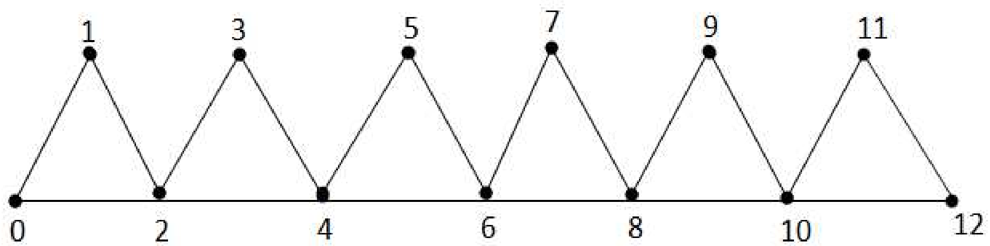

Figure 2 : Square difference labeling of T_6.

Theorem : 2.2.3

The Butterfly graph Bf_n is a square difference graph.

Proof :

Let the cycles be denoted by $G_1, G_2, \dots \dots, G_n$. All the cycles meet at one vertex, let it be u.

Let us define a function $f : V(G) \to \{ 0, 1, 2, \dots, n-1 \}$ as follows

$$f(u) = 0$$

$$f(v_i) = i, \qquad i = 1,2,3, \ldots$$

Then $f^*: E(G) \rightarrow N$ given by

$f^*(uv_i) = v_i^2 - u^2, \quad i = 1,2,3, \ldots$

Clearly ,

$$f^*(uv_i) = i^2, \qquad i = 1,2,3, \ldots$$

Hence the edge labels are distinct.

Therefore the Butterfly graph Bf_n is a square difference graph.

Example : 2.2.3

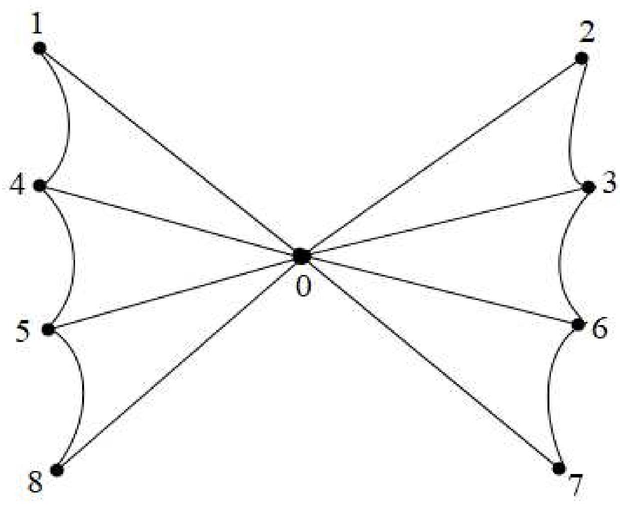

Figure 3 : Square difference labeling of Bf_9

Theorem : 2.2.4

The splitting graph of path $S'(P_n)$ is a square difference graph.

Proof :

Let the vertex set

$$V\big(S'(P_n)\big) = \{v_i/1 \leq i \leq n\} \cup \{v_i'/1 \leq i \leq n\}$$

and the edge set

$$E\big(S'(P_n)\big) = \{v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{v_i v_{i+1}'/1 \leq i \leq n-1\} \cup$$

$$\{v_i' v_{i+1}/1 \leq i \leq n-1\}$$

where $v_1', v_2', \dots, v_n'$ are the new vertices joined corresponding to $v_1, v_2, \dots, v_n$of the pathP_n.

Define $f : V\big(S'(P_n)\big) \rightarrow \{0,1, \dots, n-1\}$ by

$$f(v_i) = 2i - 2, \quad 1 \leq i \leq n$$

$$f(v_i') = 2i - 1, \quad 1 \leq i \leq n$$

Then $f^*: E(G) \rightarrow N$ given by

$$f^*(v_i v_{i+1}) = v_{i+1}^2 - v_i^2, \quad 1 \leq i \leq n-1$$

$$f^*(v_i v_{i+1}') = v_{i+1}'^{\,2} - v_i^2, \quad 1 \leq i \leq n-1$$

$$f^*(v_i' v_{i+1}) = v_{i+1}^2 - v_i'^{\,2}, \quad 1 \leq i \leq n-1$$

Clearly we have,

$$f^*(v_i v_{i+1}) = 4(2i-1), \quad 1 \leq i \leq n-1$$

$$f^*(v_i' v_{i+1}) = 4i - 1, \quad 1 \leq i \leq n-1$$

$$f^*(v_i v_{i+1}') = 12i - 3, \quad 1 \leq i \leq n-1$$

Hence the edge labels are distinct.

Therefore the splitting graph of path $S'(P_n)$ is a square difference graph.

Example : 2.2.4

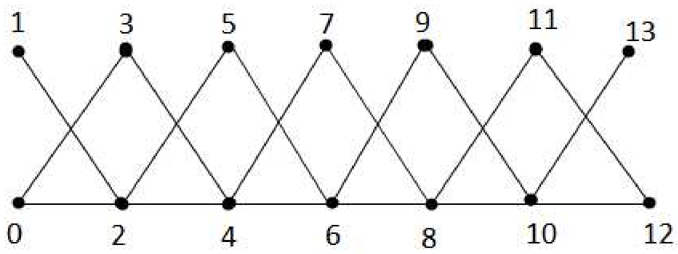

Figure 4 : Square difference labeling of $S'(P_7)$

Theorem : 2.2.5

The book graph B_n is a square difference graph.

Proof :

Let $V(B_n) = \{u, v\} \cup \{u_i, v_i / 1 \leq i \leq n\}$

and $E(B_n) = \{uv\} \cup \{uu_i / 1 \leq i \leq n\} \cup \{vv_i / 1 \leq i \leq n\} \cup \{u_i v_i / 1 \leq i \leq n\}$

Define $f : V(B_n) \rightarrow \{0,1, \dots, n-1\}$ by

$$f(u) = 1$$

$$f(v) = 0$$

$$f(u_i) = 2i + 1, \quad 1 \leq i \leq n$$

$$f(v_i) = 2i, \quad 1 \leq i \leq n$$

Then $f^*: E(G) \to N$ given by

$$f^*(uv) = 1$$

$$f^*(uu_i) = u_i^2 - u^2, \qquad 1 \leq i \leq n$$

$$f^*(vv_i) = v_i^2 - v^2, \qquad 1 \leq i \leq n$$

$$f^*(u_iv_i) = u_i^2 - v_i^2, \qquad 1 \leq i \leq n$$

Clearly ,

$$f^*(uv) = 1$$

$$f^*(uu_i) = 8\,\frac{i(i+1)}{2}, \qquad 1 \leq i \leq n$$

$$f^*(vv_i) = 4i^2 \quad, \qquad 1 \leq i \leq n$$

$$f^*(u_iv_i) = 4i + 1, \qquad 1 \leq i \leq n$$

Hence the edge labels are distinct.

Therefore the book graph B_n is a square difference graph.

Example : 2.2.5

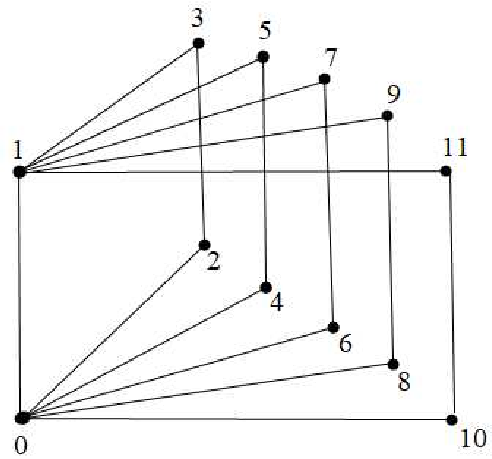

Figure 5 : Square difference labeling of B_5

Theorem : 2.2.6

The cycles C_k are square difference graph.

Proof:

Let C_k be a cycle of length k and let $C_k = (u_1u_2, \dots, u_ku_1)$

Define $f : V(C_k) \to \{0,1, \dots k-1\}$ by

$$f(u_i) = i - 1 \ , \qquad 1 \le i \le k$$

The induced function $f^*: E(C_k) \to N$ by

$$f^*(u_iu_{i+1}) = \ u_{i+1}^2 - u_i^2 \ , \qquad 1 \le i \le k-1$$

$$f^*(u_ku_1) = u_k^2 - u_1^2$$

Clearly,

$$f^*(u_iu_{i+1}) = 2i - 1 \quad , \quad 1 \le i \le k-1$$

$$f^*(u_k u_1) = (k-1)^2$$

Hence the edge labels are distinct.

Therefore the cycle graph C_k is a square difference graph.

Example : 2.2.6

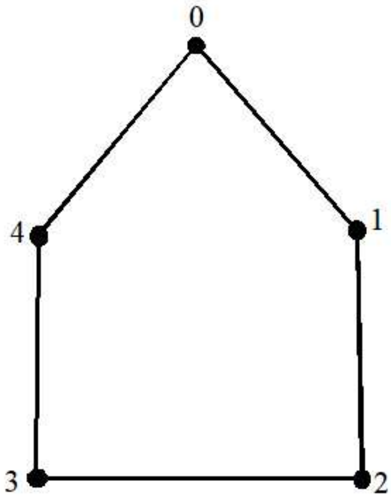

Figure 6 : Square difference labeling of C_5.

Theorem : 2.2.7

The graph P_n^2 is a square difference graph.

Proof:

Let $P_n: u_1, u_2, \dots, u_n$ be a path. Let $G = P_n^2$

Here $p = |V(G)| = n$ and $q = |E(G)| = 2n - 3$

Define a vertex labeling $f: V(G) \to \{0,1,2,\dots,p-1\}$ by

$$f(u_i) = i - 1, \quad 1 \leq i \leq n$$

And the induced edge labeling function $f^* : E(G) \to N$ by

$$f^*(u_iu_{i+1}) = u_{i+1}^2 - u_i^2, \qquad 1 \le i \le n-1$$

$$f^*(u_iu_{i+2}) = u_{i+2}^2 - u_i^2, \qquad 1 \le i \le n-2$$

Clearly,

$$f^*(u_iu_{i+1}) = 2i-1, \qquad 1 \le i \le n-1$$

$$f^*(u_iu_{i+2}) = 4i, \qquad 1 \le i \le n-2$$

Hence the edge labels are distinct.

Therefore the graph P_n^2 is a square difference graph.

Example : 2.2.7

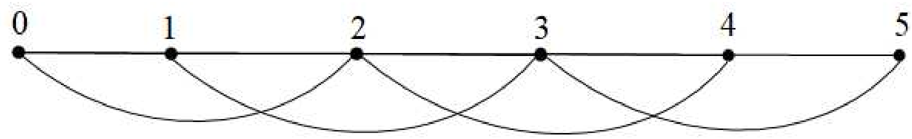

Figure 7 : Square difference labeling of P_n^2

Theorem : 2.2.8

The graph $G = K_2 + mK_1$ is a square difference graph.

Proof :

Let $V(G) = \{u, v, w_2, w_3, \dots, w_{m+1}\}$ and $E(G) = \{uv\} \cup \{uw_i, vw_i : 2 \le i \le m+1\}$

Then G is of order $m+2$ and size $2m+1$.

Let us define $f: V(G) \to \{0,1,\dots,m+2\}$ by

$$f(u) = 0$$

$$f(v) = 1$$

$$f(w_i) = i, \quad 2 \leq i \leq m+1$$

The induced function $f^*: E(G) \rightarrow N$ by

$$f^*(uv) = v^2 - u^2$$

$$f^*(uw_i) = w_i^2 - u^2, \qquad 2 \leq i \leq m+1$$

$$f^*(vw_i) = w_i^2 - v^2, \qquad 2 \leq i \leq m+1$$

Clearly ,

$$f^*(uv) = 1$$

$$f^*(uw_i) = i^2\,, \qquad 2 \leq i \leq m+1$$

$$f^*(vw_i) = i^2 - 1\,, \quad 2 \leq i \leq m+1$$

Hence the edge labels are distinct.

Therefore the graph $G = K_2 + mK_1$ is a square difference graph.

Example : 2.2.8

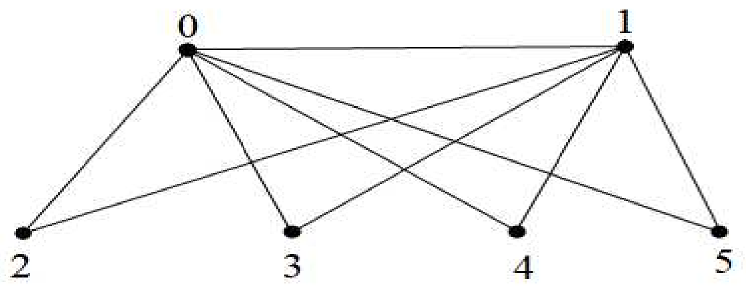

Figure 8: Square difference labeling of $k_2 + 4k_1$.

Theorem : 2.2.9

The graph $K_1 + C_n$ is a square difference graph.

Proof:

The graph $K_1 + C_n$ has $n + 1$ vertices and $2n$ edges. Let u be vertex of K_1 and be the vertices of $v_1, v_2, \dots, v_n$ the cycle.

Define a vertex labeling $f : V(K_1 + C_n) \to \{0,1,\dots,n-1\}$ by

$$f(u) = 0$$

$$f(v_i) = i, \qquad 1 \le i \le n$$

And the induced edge labeling function $f^* : E(G) \to N$ by

$$f^*(uv_i) = v_i^2 - u^2, \qquad 1 \le i \le n$$

$$f^*(v_i v_{i+1}) = v_{i+1}^2 - v_i^2, \quad 1 \le i \le n-1$$

$$f^*(v_1 v_n) = v_n^2 - v_1^2$$

Clearly,

$$f^*(uv_i) = i^2\,, \qquad 1 \le i \le n$$

$$f^*(v_i v_{i+1}) = 2i + 1\,, \quad 1 \le i \le n-1$$

$$f^*(v_1 v_n) = i^2 - 1\,, \quad i = n$$

Hence the edge labels are distinct.

Therefore $K_1 + C_n$ the graph is a square difference graph.

Example : 2.2.9

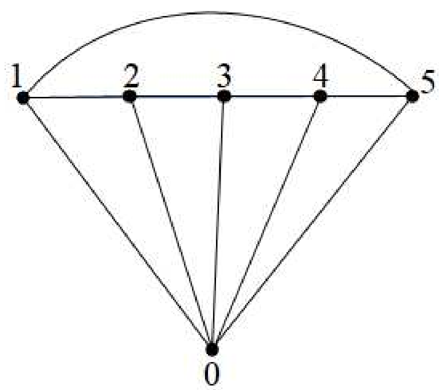

Figure 9: Square difference labeling of $K_1 + C_5$.

Theorem : 2.2.10

The bistars $B_{n,n}$ are square difference graphs .

Proof :

Let u and v be the apex vertices of the bistar. Let $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be the pendant vertices of the bistar $B_{n,n}$.

The vertex set is $V(B_{n,n}) = \{u\ , u_i\ :\ 1 \leq i \leq n\}$and the edge set is

$E(B_{n,n}) = \{uu_i\ :\ 1 \leq i \leq n\} \cup \{vv_i\ :\ 1 \leq i \leq n\} \cup \{uv\}$.

Hence the order of the graph is $p = |V\ (B_{n,n})| = 2n + 2$ and the size is

$q = |E(B_{n,n})| =\ 2n + 1.$

Label the vertex u by 0 , v by 1, $u_1, u_2, \dots, u_n$ by $3, 5, 7, \dots 2n +\ 1$ and $v_1, v_2, \dots, v_n$ by $2, 4, 6, \dots, 2n$. The elements of the first edge set are $3^2, 5^2, 7^2, \dots, (2n + 1)^2$ and the second edge set are $2^2, 4^2, 6^2, \dots, n^2$.

The edge weight of uv is 1. Since these edge sets are disjoint it satisfies the definition of square difference labeling.

Hence the bistars $B_{n,n}$ are square difference graphs.

Example2.2.10

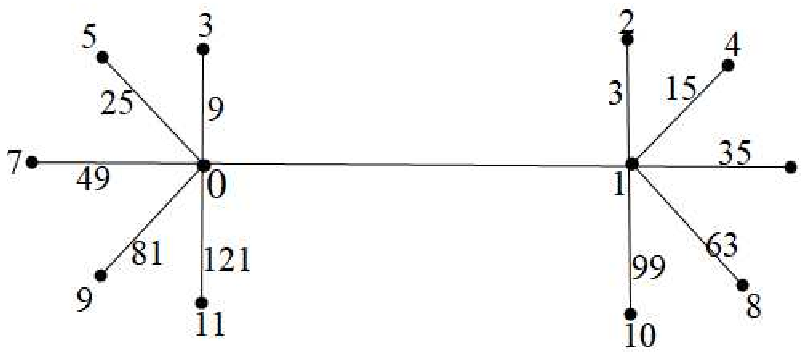

Figure 10: Square difference labeling of a bistar $(B_{n,n})$

CHAPTER 3

3 .CUBE DIFFERENCE LABELING OF SOME GRAPHS

3.1 Introduction

Definition : 3.1.1

Let $G = (V(G), E(G))$ be a graph. Then G is said to be a cube difference labeling if there exists a bijection $f: V(G) \rightarrow \{0,1,2,\ldots,p-1\}$ such that the induced function $f^* : E(G) \rightarrow N$ given by

$$f^*(uv) = |[f(u)]^3 - [f(v)]^3|$$

for every $uv \in E(G)$ are all distinct.

Definition : 3.1.2

Any graph which admits cube difference labeling is called cube difference graph.

3.2 Main Results

Definition : 3.2.1

A coconut tree graph $CT(m,n)$ is the graph obtained from the path $P_n$by appending m new pendant edges at an end vertex of P_n .

Definition : 3.2.2

A Helm $H_n\ ,n\ \geq\ 3$ is the graph obtained from a crown R_n by adding a new vertex joined to every vertex of the unique cycle of the crown .

Theorem : 3.2.3

The wheel graphs W_n admit cube difference labeling.

Proof :

Let $v_0$be the central vertex and$v_1, v_2, \dots, v_n$be the rim vertices of the cycle C_n.

Define

$$f(v_0) = 0$$

and $$f(v_i) = i\,, \qquad 1 \le i \le n$$

and the induced edge function $f^*: E(G) \rightarrow N$ define by

$$f^*(uv) = |[f(u)]^3 - [f(v)]^3|.$$

The edge sets are obtained by defining

$$f^*(v_i) = i^3\,, \qquad 1 \le i \le n$$

and $f^*(v_i v_{i+1}) = 3i^2 - 3i + 1\,, \quad 1 \le i \le n$

The edge sets are disjoint ,the edge weights are distinct and in increasing order.

So the wheels W_n admit the cube difference labeling. Hence the wheels W_n are cube difference graphs.

Example : 3.2.3

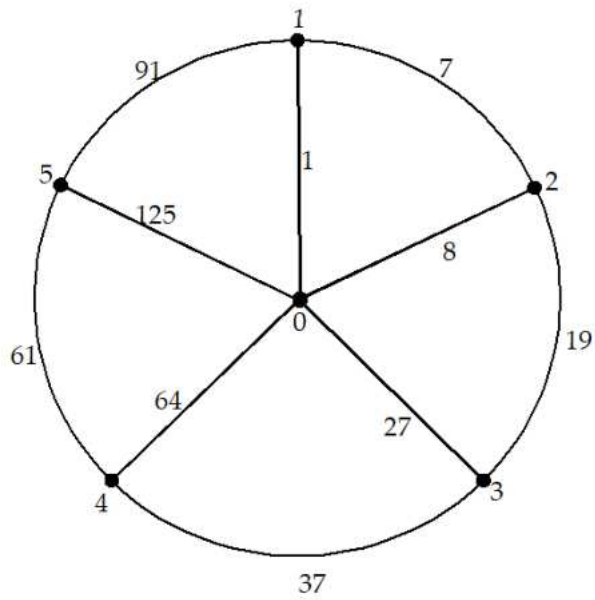

Figure 1 : Cube difference graph of W_5

Theorem : 3.2.4

The paths P_n are cube difference graphs.

Proof :

Let $P_n : u_1, u_2, \dots, u_n$ be the path and let $e_i = u_i u_{i+1} (1 \leq i \leq n-1)$ be the edges. Here the order of the graph is n and the size of the graph is $n-1$. Define $f : V(G) \rightarrow \{0,1,2,\dots,n-1\}$ by

$$f(u_i) = i - 1, \quad 1 \leq i \leq n$$

and the induced edge function $f^*: E(G) \rightarrow N$ define by $f^*(uv) = |[f(u)]^3 - [f(v)]^3|$.

Then the edge labeling are

$$f^*(e_i) = 3i^2 - 3i + 1, \quad 1 \leq i \leq n-1$$

The edge weights of the edge set are in the increasing order. So the paths admit cube difference labeling. Hence the paths P_n are cube difference graphs.

Example : 3.2.4

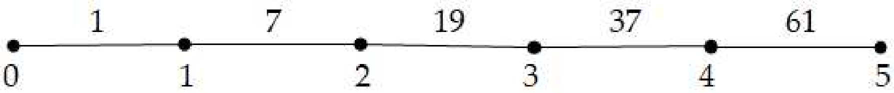

Figure 2: Cube difference graph of P_6.

Theorem : 3.2.5

The cycles C_n are cube difference graphs.

Proof :

Let $C_n : u_1 u_2 \dots u_n u_1$ be the cycle. Then the edges are $e_i = u_i u_{i+1}, (1 \leq n - 1)$ and $e_n = u_n u_1$. Here the order and the size of the graphs are n.

Define $f : V(G) \rightarrow \{0,1,2,\dots,n-1\}$ by

$$f(u_i) = i - 1, \quad 1 \leq i \leq n$$

and the induced edge function $f^*: E(G) \rightarrow N$ define by

$f^*(uv) = |[f(u)]^3 - [f(v)]^3|$.

That is

$$f^*(v_i) = 3i^2 - 3i + 1, \quad 1 \leq i \leq n - 1$$

and $f^*(v_n) = (n-1)^3$

The edge weights are distinct and in increasing order. So the cycles admit the cube difference labeling. Hence the cycles C_n are cube difference graphs.

Example : 3.2.5

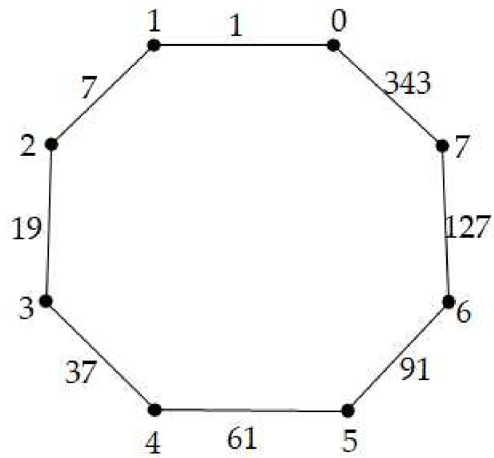

Figure 3: The Cube difference graph of C_8.

Theorem : 3.2.6

The star graphs $K_{1,n}$ admit cube difference labeling.

Proof :

Let v be the apex vertex and let $v_1, v_2, \dots, v_n$ be the pendant vertices of the star $K_{1,n}$. The edge set is $\{vv_i : 1 \leq i \leq n\}$. Here the order of the graph is $n+1$ and the size of the graph is n. Define $f : V(G) \rightarrow \{0,1,2, \dots, n\}$ by

$$f(v) = 0$$

$$f(v_i) = i\,, \quad 1 \leq i \leq n$$

and the induced edge function $f^*: E(G) \rightarrow N$ is defined by

$$f^*(uv) = |[f(u)]^3 - [f(v)]^3|$$

That is

$$f^*(v_i) = i^3 , \qquad 1 \leq i \leq n$$

The edge weights are distinct and in increasing order. So the stars $K_{1,n}$ admit the cube difference labeling. Hence the stars $K_{1,n}$ are cube difference graphs.

Example 3.2.6

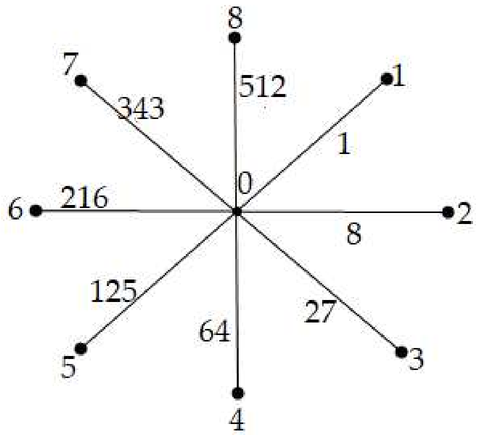

Figure 4 : Cube difference graph of $K_{1,8}$.

Theorem : 3.2.7

The shell graphs $S_{n,n-3}$, $n \geq 4$ admit cube difference labeling.

Proof :

Let G be the shell graph $S_{n,n-3}$. Let $u_1, u_2, \ldots, u_n$ be the vertices and let $e_i = u_i u_{i+1}$, $(1 \leq i \leq n-1)$,$e_n = u_1 u_n$ and $e_j = u_1 u_{j-3}$ $(n+1 \leq j \leq 2n-3)$ be the edges of G. Here the order the graph $p = n$ and the size of graph $q = 2n - 2$

For $1 \leq i \leq n$ define

$$f(v_i) = i - 1$$

Then the edge labeling are

$$f^*(e_i) = 3i^2 - 3i + 1, \quad 1 \le i \le n - 1$$

$$f^*(e_n) = (n - 1)^3$$

and $f^*(e_j) = (j - (n - 1))^3, n + 1 \le j \le 2n - 3$

Here all the three edge sets are disjoint. They have no elements in common. Hence the shell graphs $S_{n,n-3}$ are cube difference graphs.

Example : 3.2.7

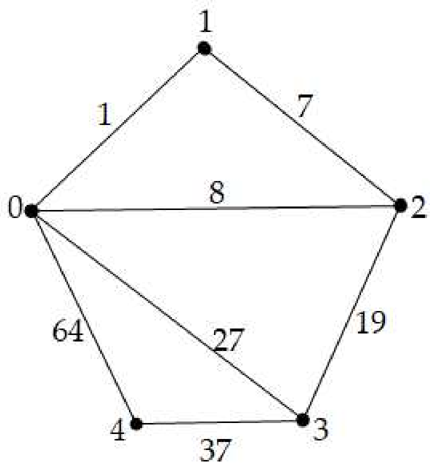

Figure 5 : Cube difference labeling of $S_{5,2}$ ·

Theorem : 3.2.8

Coconut trees admit cube difference labeling.

Proof :

Let $v_0, v_1, \ldots, v_i$ be the vertices of a path, having path length $i(i \ge 1)$ and $v_{i+1}, v_{i+2}, \ldots, v_n$ be the pendant vertices, being adjacent with v_0. For $0 \le j \le i$, Define

$$f(v_0) = 0$$

$$f(v_i) = i\,,\ \ 1 \le i \le k$$

$$f(v_k) = k\,,\ \ \ i+1 \le k \le n$$

The edge labeling are defined by

$$f^*(v_k) = k^3\,,\ \ i+1 \le k \le n$$

$$f^*(v_i v_{i+1}) = 3i^2 - 3i + 1\,,\qquad 1 \le i \le k$$

The edge weights are distinct and in increasing order. So the coconut trees admit the cube difference labeling. Hence the coconut trees are cube difference graphs.

Example : 3.2.8

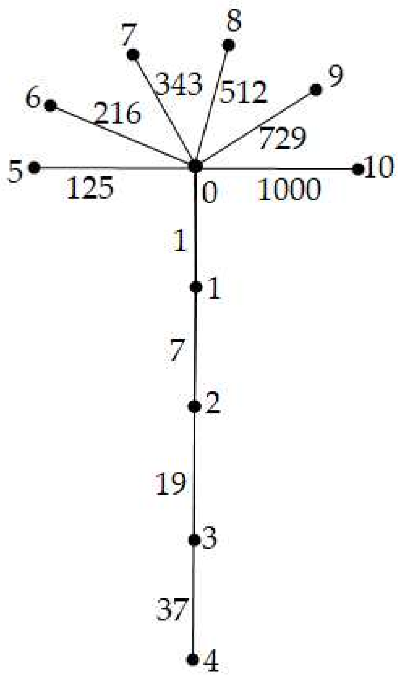

Figure 6 : Cube difference labeling of a coconut tree

Theorem : 3.2.9

The dragon graphs $D_n(m)$ are cube difference graph.

Proof :

Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle part of dragon graph $D_n(m)$ and let $n+1, n+2, \ldots, n+m$ be the vertices of the path part of the dragon graph. The number of vertices of the dragon graph $D_n(m)$ is $n+m$ and the number of edges is $n+m-1$. The vertex set of the dragon graph $D_n(m)$ is $\{u_i : 1 \leq i \leq n+m\}$and the edge sets are $\{u_i u_{i+1}: 1 \leq i \leq n+m-1\}$ and $\{u_i u_m\}$.

For $1 \leq i \leq n+m$, define

$$f(u_i) = i-1\,, \quad 1 \leq i \leq n+m-1$$

Then the edge labeling are

$$f^*(u_i u_{i+1}) = 3i^2 - 3i + 1\,, \quad 1 \leq i \leq n+m-1$$

and $f^*(u_1 u_m) = (m+n-1)^3$

The elements of the first edge set are $1,7,19, \ldots, (m+n)^3 - (m+n-1)^3$ i.e the difference between each number increases by multiples of 6. The singleton set consists of m^3.Therefore these two edges sets are distinct. So the dragon graphs $D_n(m)$ admit the cube difference labeling. Hence the dragon $D_n(m)$graphs are cube difference graphs.

Example : 3.2.9

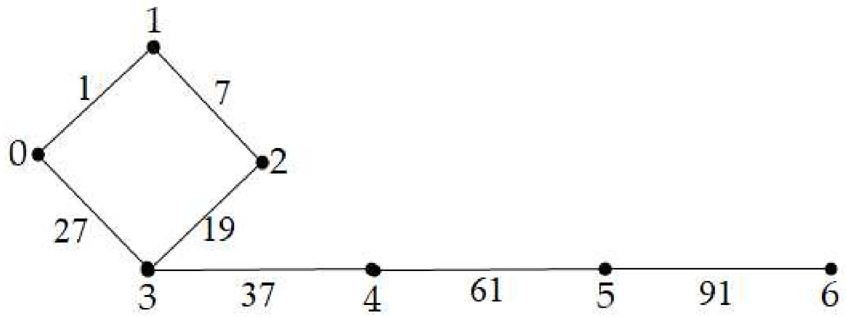

Figure 7 : Cube difference labeling of $D_4(3)$

Theorem : 3.2.10

The helm graphs $H_n$admit cube difference labeling for $n \geq 3$.

Proof:

Let u be the central vertex , let $u_1, u_2, \dots, u_n$ be the vertices of C_n and let $v_1, v_2, \dots, v_n$be the pendant vertices adjacent to $v_1, v_2, \dots, v_n$ respectively. Let the edges be $e_1, e_2, \dots, e_{3n}$ .Therefore the edge sets are $\{uu_i : 1 \leq i \leq n\}$, $\{u_iu_{i+1} : 1 \leq i \leq n-1\}$,

$\{u_iv_i : 1 \leq i \leq n-1\}$ and $\{u_1u_{n-1\}}$.

Define

$$f(v) = 0$$

and $f(v_i) = i\,,\quad 1 \leq i \leq 2n-2$

Then the edge labeling are

$$f^*(uu_i) = i^3\,,\quad 1 \leq i \leq n$$

$$f^*(u_iv_i) = (n+i-1)^3 - n^3\,, 1 \leq i \leq n-1$$

$$f^*(u_iu_{i+1}) = 3i^2 - 3i + 1\,,\quad 1 \leq i \leq n-2$$

and $\quad f^*(u_1u_{n-1}) = (n-1)^3$

The weights of the edge sets are in the increasing order and the sets will never intersect at anywhere. So the helm graphs admit cube difference labeling. Hence the helm graphs $H_n$are cube difference graphs.

Example : 3.2.10

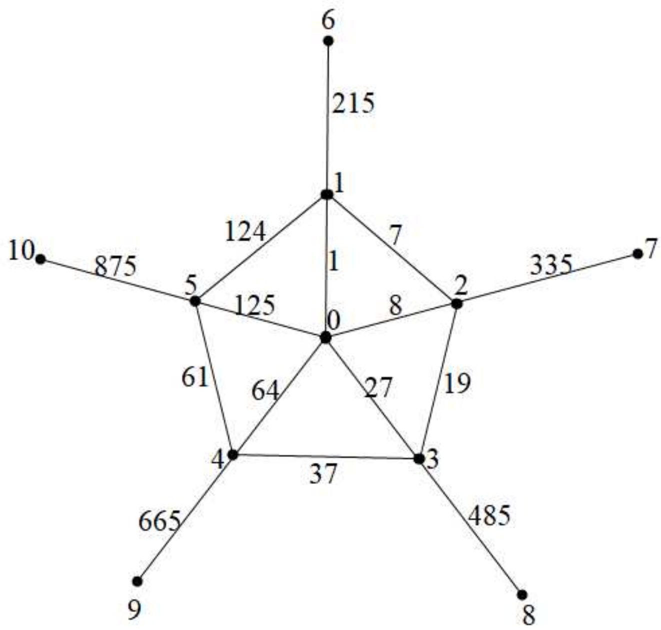

Figure 8 : Cube difference labeling ofH_5 .

Theorem : 3.2.11

The pentagonal snakes (KC_5) are cube difference graphs .

Proof :

Let the graph be $G = (KC_5)$. Let be the vertices $u_1, u_2, \ldots, u_n$ be the vertices of the graph (KC_5) .

$$|V(KC_5)| = 4k + 1\,, \quad 1 \le k$$

$$|E(KC_5)| = 5k\,, \quad 1 \le k$$

Define $f : V(KC_5) \rightarrow \{0,1,2, \ldots ,5k\}$ by

$$f(v_i) = i - 1\,, \qquad 1 \le i \le 5k\,, 1 \le k$$

The edge labeling are

$$f^*(u_i u_{i+1}) = 3i^2 + 3i + 1\,,\ \ 0 \le i \le k\,, 1 \le nk - 3$$

$$f^*(u_{4i+1} u_{4i+5}) = 192i^2 - 192i + 64\,,\ \ 0 \le i \le k\,, \qquad 1 \le k - 1$$

Since both the sequence of the edge labeling are in increasing order. The pentagonal snakes (KC_5) admit the cube difference labeling. Hence the pentagonal snakes (KC_5) are cube difference graphs .

Example : 3.2.11

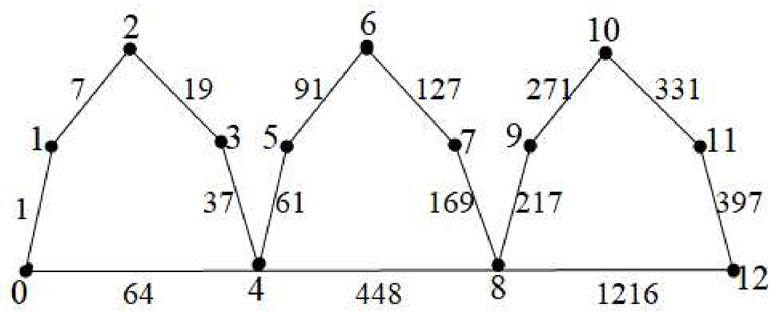

Figure 9 : Cube difference graph of KC_5 .

Theorem : 3.2.12

The middle graph of a path $M(P_n)$are cube difference graphs .

Proof :

Let $u_1, u_2, \dots, u_n$be the vertices of a path of length $n - 1$.

Let $u_{n+1}, u_{n+2}, \dots, u_{2n-1}$be the vertices of the mid points of the edges of the path .

$$|V(M(P_n))| = 2n - 1\,, \quad n \geq 3$$

$$|E(M(P_n))| = 2n + 2, \quad n \geq 3$$

Define $f : V(M(P_n)) \to \{0,1,2,\dots,2n-2\}$ by

$$f(u_i) = i - 1\,,\ 1 \leq i \leq 2n - 1$$

The edge labeling are

$$f^*(u_i u_{i+1}) = 3i^2 + 3i + 1\,,\ 0 \leq i \leq 2n - 1$$

$$f^*(u_{2i} u_{2i+2}) = 24i^2 + 48i + 26\,,\ 0 \leq i \leq n - 2$$

Since all the edge labeling are in increasing order. The middle graphs of paths $M(P_n)$ satisfies the cube difference labeling. Hence the middle graphs of the paths $M(P_n)$are cube difference graphs.

Example : 3.2.12

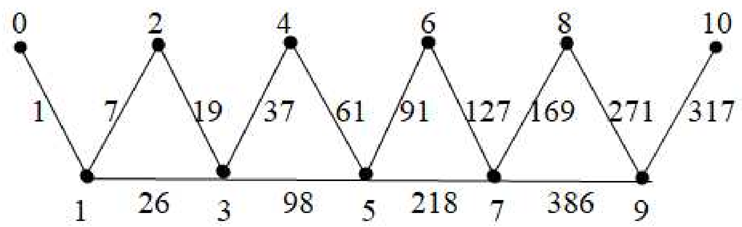

Figure 10 : Cube difference graph of $M(P_6)$

CHAPTER 4

4 . SOME SPECIAL TYPES OF DIFFERENCE GRAPHS

4.1 Absolute Difference of Square Sum and Sum Mean Prime Labeling of Some star Graphs

Introduction

Definition : 4.1.1

Let $G = (V(G), E(G))$ be a graph with p vertices and q edges . Define a bijection $f : V(G) \rightarrow \{1,2,\dots,p\}$ by $f(v_i) = i$, for every i from 1 to p and define a $1-1$ mapping $f^*_{adsssmp} : E(G) \rightarrow$ set of natural numbers N by

$$f^*_{adsssmp}(uv) = \frac{1}{2}\left|f(u)^2 + f(v)^2 - \{f(u) + f(v)\}\right| .$$

The induced function $f^*_{adsssmp}$ is said to be an absolute difference of square sum and sum mean prime labeling, if for each vertex of degree at least 2, the **gcin** of the labels of the incident edges is 1.

Definition : 4.1.2

A graph which admits absolute difference of square sum and sum mean prime labeling is called an absolute difference of square sum and sum mean prime graph.

4.2 Main Results

Theorem : 4.2.1

Double graph of star $K_{1,n} (n > 2)$admits absolute difference of square sum and sum mean prime labeling.

Proof:

Let $G = D(K_{1,n})$ and let $V_1, V_2, \dots, V_{2n+2}$ are the vertices of G.

Here $|V(G)| = 2n + 2$ and $|E(G)| = 4n$.

Define a function $f : V \to \{1,2, \dots ,2n + 2\}$by

$f(v_i) = i$, $\quad i = 1,2, \dots ,2n + 2$.

Clearly f is a bijection.

For the vertex labeling f, the induced edge labeling $f^*_{adsssmp}$ is defined as follows

$$f^*_{adsssmp}(v_1 v_{i+2}) = \frac{i(i+3)}{2} + 1 , \qquad i = 1,2, \dots , n$$

$$f^*_{adsssmp}(v_2 v_{i+2}) = \frac{i(i+3)}{2} + 2 , \qquad i = 1,2, \dots , n f^*_{adsssmp}(v_1 v_{n+i+2})$$

$$= \frac{(n+i+2)^2 - n - i}{2} - 1 , \qquad i = 1,2, \dots , n$$

$$f^*_{adsssmp}(v_2 v_{n+i+2}) = \frac{(n+i+2)^2 - n - i}{2} , \qquad i = 1,2, \dots , n$$

Clearly $f^*_{adsssmp}$ is an injection.

gcin of(V_2) = gcd of $\{f^*_{adsssmp}(v_2 v_3) , f^*_{adsssmp}(v_2 v_4)\}$

= gcd of $\{4 ,7\} = 1$.

gcin of(V_1) =gcd of $\{f^*_{adsssmp}(v_1 v_5) , f^*_{adsssmp}(v_1\, v_3)\}$

= gcd of $\{10 ,3\} = 1$.

gcin of(V_{i+2}) =gcd of $\{f^*_{adsssmp}(v_2 v_{i+2}) , f^*_{adsssmp}(v_1\, v_{i+2})\}$

= gcd of $\{\frac{i(i+3)}{2} + 2 , \frac{i(i+3)}{2} + 1\}$

$$= 1 , \quad i = 1,2, \dots , n.$$

gcin of (V_{n+i+2}) =gcd of $\{f^*_{adsssmp}(v_2 v_{n+i+2})\, , f^*_{adsssmp}(v_1\, v_{n+i+2})\}$

=gcd of $\{\frac{(n+i+2)^2-n-i}{2}\, , \frac{(n+i+2)^2-n-i}{2} - 1$

$$= 1\, , \qquad i = 1,2, \dots, n.$$

So, **gcin** of each vertex of degree greater than one is 1. Hence $D(K_{1,n})$ admits absolute difference of square sum and sum mean prime labeling .

Example : 4.2.1

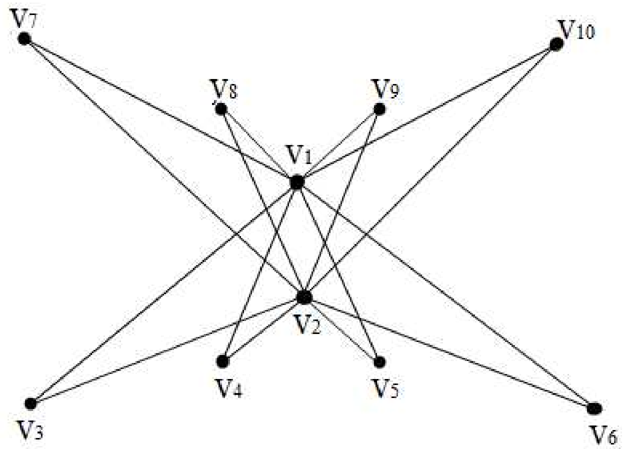

Figure 1 : $G = D(K_{1,4})$

Theorem : 4.2.2

Splitting graph of star $K_{1,n}$ $(n > 2)$ admits absolute difference of square sum and sum mean prime labeling.

Proof :

Let $G = S'(K_{1,n})$and let $V_1, V_2, \dots, V_{2n+2}$ are the vertices of G. Here $|V(G)| = 2n + 2$ and $|E(G)| = 3n$.

Define a function $f : V \rightarrow \{1,2, \dots ,2n + 2\}$ by

$f(v_i) = i\,, \qquad i = 1,2, \dots ,2n + 2\,.$

Clearly f is a bijection.

For the vertex labeling f, the induced edge labeling $f^*_{adsssmp}$ is defined as follows

$$f^*_{adsssmp}(v_1 v_{i+2}) = \frac{i(i+3)}{2} + 1\,, \qquad i = 1,2, \dots , n$$

$$f^*_{adsssmp}(v_2 v_{i+2}) = \frac{i(i+3)}{2} + 2\,, \qquad i = 1,2, \dots , n$$

$$f^*_{adsssmp}(v_1 v_{n+i+2}) = \frac{(n+i+2)^2 - n - i}{2} - 1\,, \qquad i = 1,2, \dots , n$$

Clearly $f^*_{adsssmp}$ is an injection.

gcin of (V_{i+2}) =gcd of $\{f^*_{adsssmp}(v_2 v_{i+2})\,, f^*_{adsssmp}(v_1\, v_{i+2})\}$

$=$ gcd of $\{\frac{i(i+3)}{2} + 2\,, \frac{i(i+3)}{2} + 1\}$

$$= 1\,, \qquad i = 1,2, \dots , n.$$

So, **gcin** of each vertex of degree greater than one is 1. Hence $S'(K_{1,n})$admits absolute difference of square sum and sum mean prime labeling.

Example : 4.2.2

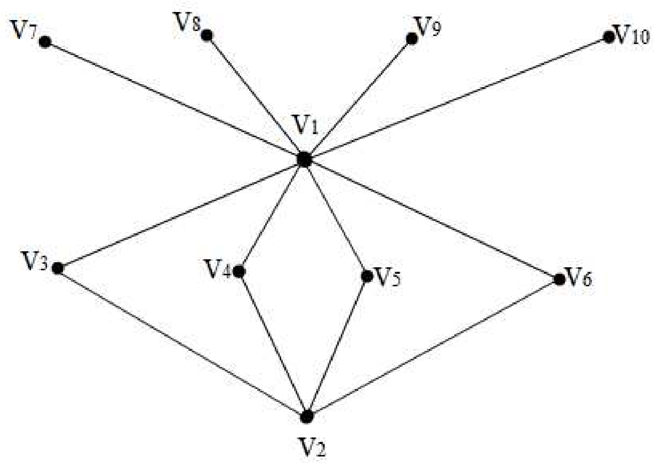

Figure 2 : $G = S'(K_{1,4})$

Theorem : 4.2.3

Let G be the graph obtained by joining each vertex of star $K_{1,n}$ to vertices of path P_2 by edges $(n > 2)$ admits absolute difference of square sum and sum mean prime labeling.

Proof :

Let G be the graph and let $V_1, V_2, \dots, V_{n+2}$ are the vertices of G. Here $|V(G)| = n + 2$ and $|E(G)| = 2n + 1$.

Define a function $f : V \to \{1,2,\dots,n+2\}$ by

$f(v_i) = i, \qquad i = 1,2,\dots,n+2$.

Clearly f is a bijection.

For the vertex labeling f, the induced edge labeling $f^*_{adsssmp}$ is defined as follows

$$f^*_{adsssmp}(v_1v_{i+2}) = \frac{i(i+3)}{2} + 1, \qquad i = 1,2,\dots,n$$

$$f^*_{adsssmp}(v_2v_{i+2}) = \frac{i(i+3)}{2} + 2, \qquad i = 1,2,\dots,n$$

$f^*_{adsssmp}(v_1v_2) = 3.$

Clearly $f^*_{adsssmp}$ is an injection.

gcin of (V_2) = gcd of $\{f^*_{adsssmp}(v_1v_2), f^*_{adsssmp}(v_2v_3)\}$

= gcd of $\{1, 4\} = 1$.

gcin of (V_1) =gcd of $\{f^*_{adsssmp}(v_1v_2), f^*_{adsssmp}(v_1\, v_3)\}$

= gcd of $\{1, 3\} = 1$.

gcin of (V_{i+2}) =gcd of $\{f^*_{adsssmp}(v_2v_{i+2}), f^*_{adsssmp}(v_1\, v_{i+2})\}$

= gcd of $\{\frac{i(i+3)}{2} + 2, \frac{i(i+3)}{2} + 1\}$

$$= 1, \quad i = 1,2,\dots,n.$$

So, **gcin** of each vertex of degree greater than one is 1. Hence G admits absolute difference of square sum and sum mean prime labeling.

4.3 Absolute Difference of Cubic and Square Sum Labeling of a Class of Trees

Introduction

Definition : 4.3.1

Let $G = (V(G), E(G))$ be a graph. A graph G is said to be absolute difference of the sum of the cubes of the vertices and the sum of the squares of the vertices, if there exist a bijection $f : V(G) \to \{1,2,\dots,p\}$such that the induced function $f^*_{adcss} : E(G) \to$ multiples of 2 is given by

$$f^*_{adcss}(uv) = \ f(u)^3 + f(v)^3 - (f(u)^2 + f(v)^2)$$

is injective.

Definition : 4.3.2

A graph in which every edge associates distinct values with multiples of 2 is called the sum of the cubes of the vertices and the sum of the squares of the vertices. Such a labeling is called an absolute difference of cubic and square sum labeling or an absolute difference of css-labeling.

4.4 Main Results

Definition : 4.4.1

An (n, k) – banana tree, is a graph obtained by connecting one leaf of each of n copies of a k- star graph with a single root vertex that is distinct from all stars.

Definition : 4.4.2

An(n, k) -firecracker is a graph obtained by the concatenation of nk-stars by linking one leaf from each .

Theorem : 4.4.3

The banana tree $B_{(n,k)}$ is the absolute difference of the css-labeling.

Proof :

Let $G = B_{(n,k)}$ and let $V_1, V_2, \dots, V_{nk+1}$ are the vertices of G.

Here $V(G) = nk + 1$ and $E(G) = \ n$

Define a function $f : V \to \{1,2, \dots, nk + 1\}$ by

$f(v_i) = i\,, \qquad i = 1,2,\dots,nk+1\,.$

For the vertex labeling f, the induced edge labeling f^*_{adcss} is defined as follows

$$f^*_{adcss}\left[v_{(j-1)k+1}v_{i+(j-1)k+1}\right] = \{(j-1)k+1\}^2\{(j-1)k\} +$$
$$\{i+(j-1)k+1\}^2\{i+(j-1)k+1\}\,,$$
$$j = 1,2,\dots,n \text{ and } i = 2,4,\dots,(k-1)$$

$$f^*_{adcss}\left[v_{(i-1)k+2}v_{nk+1}\right] = \{(i-1)k+2\}^2\{(i-1)k+1\} + (nk+1)^2(nk)\,,$$
$$i = 1,2,\dots,k$$

All edge values of G are distinct, which are multiples of 2.That is the edge values of G are in the form of an increasing order. Hence $B_{(n,k)}$ admits absolute difference of css-labeling.

Example : 4.4.3

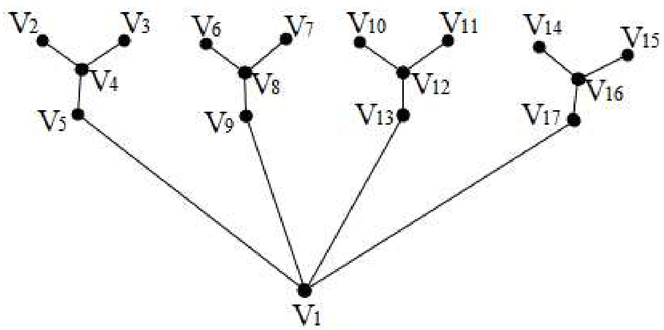

Figure 1 : $G = B_{(4,4)}$

Theorem : 4.4.4

Fire cracker graph $F_{n,k}$ is the absolute difference of the css-labeling.

Proof :

Let $G = F_{n,k}$ and let $V_1, V_2, \dots, V_{nk}$ are the vertices of G.

Here $V(G) = nk$ and $E(G) = nk - 1$.

Define a function $f : V \to \{1,2,\dots,nk\}$by

$f(v_i) = i$, $i = 1,2,\dots,nk$.

For the vertex labeling f, the induced edge labeling f_{adcss}^* is defined as follows

$$f_{adcss}^*(v_{jk+1}v_{jk+i+1}) = (jk+1)^2(jk) + (jk+i+1)^2(jk+i)\ ,$$

$$j = 0,1,2,\dots,n-1$$

$$i = 1,2,3,\dots,k-1$$

$$f_{adcss}^*(v_{ik}v_{(i+1)k}) = (ik)^2(ik-1) + \{(i+1)k\}^2\{i+1)k-1\},$$

$$i = 1,2,\dots,n-1$$

All edge values of G are distinct, which are multiples of 2.That is the edge values of G are in the form of an increasing order. Hence $F_{n,k}$ admits absolute difference of css-labeling.

Example : 4.4.4

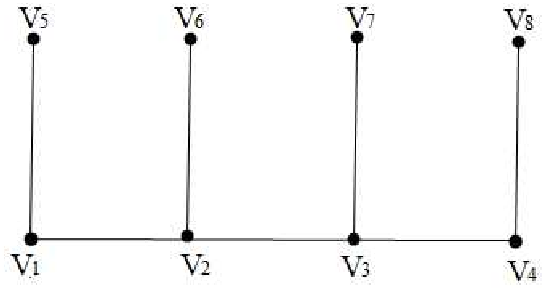

Figure 12 : $G = F_{(4,2)}$

Theorem : 4.4.5

The $(n; k; m)$ −double star tree is the absolute difference of the css-labeling.

Proof :

Let G be the $(n; k; m)$-double star tree and let $V_1, V_2, \dots, V_{n+k+m-2}$ are the vertices of G. Here$|V(G)| = n + k + m - 2$ and $|E(G)| = n + k + m - 3$.

Define a function $f : V \to \{1,2,3, \dots, n + k + m - 2\}$ by

$f(v_i) = i\,, \qquad i = 1,2, \dots, n + k + m - 2\,.\,.$

For the vertex labeling f, the induced edge labeling f^*_{adcss} is defined as follows

$$f^*_{adcss}(v_i v_{i+1}) = (i+1)^2 i + (i)^2 (i-1)\,, i = 1,2, \dots, n-1$$

$$f^*_{adcss}(v_1 v_{n+i}) = (n+i)^2 (n+i-1), i = 1,2, \dots, k-1 f^*_{adcss}(v_n v_{n+k+i-1})$$

$$= (n)^2 (n-1) + (n+k+i-1)^2 (n+k+i-2),$$

$$i = 1,2, \dots, m-1$$

All edge values of G are distinct, which are multiples of 2 .That is the edge values of G are in the form of an increasing ordser. Hence $(n; k; m)$ double star tree admits absolute differencse of css-labelings.

A STUDY ON VERTEX AND SUPER VERTEX-MAGIC LABELING

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

A. ASWILA

Reg. No: 19SPMT04

Under the guidance of

MS. M. PARVATHI BANU M.Sc., M.Phil.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON VERTEX MAGIC AND SUPER VERTEX- MAGIC LABELING"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by

A. ASWILA (Reg. No: 19SPMT04)

M. Parvathi Bai

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of thePrincipal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON VERTEX AND SUPER VERTEX-MAGIC LABELING"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. M. Parvathi Banu M.Sc., M.Phil.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

A. Aswila

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER 1

PRELIMINARIES

Definition: 1.1

A **graph** G consists of a pair ($V(G)$,X (G)) where $V(G)$ $is\ a$ non-empty finite set whose elements are called **points or vertices** and $X(G)$ is called a set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called **lines or edges** of the graph G.

Definition: 1.2

Two vertices u and v in a graph are said to be connected if there is a (u, v) path in G. A graph G is **connected** if any two vertices are connected. A graph which is not connected is said to be **disconnected.**

Definition: 1.3

If $x = \{u, v\} \in X(G)$**,** the line x is said to join u and v. We write $x = uv$ and say that the points u and v are **adjacent**. We also say that the point u and the line x are **incident** with each other.

Definition: 1.4

A simple graph G is said to be a **complete graph** if every vertex is adjacent to all other vertices. A complete graph with n vertices is denoted by $k_{n.}$

Definition: 1.5

Two vertices u and v are said to be adjacent if $(u, v) \in E$ are called **adjacent vertices**. If two edges are said to be adjacent if they have a vertex in common are called **adjacent lines.**

Definition: 1.6

A vertices of a graph $G(V, E)$ is said to be **odd vertices** if its degree is an odd number . A vertices of a graph G is be **even vertices i**f its degree is an even.

Definition: 1.7

A **walk** is a alternating sequence of vertices and edges of a graph such that $v_{0,}e_{1,}v_{1,}e_{2,,} \dots \dots e_k v_k$ end with vertices .

Definition: 1.8

A graph is called a **bigraph or bipartite graph** if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. $(V_{1,}V_2)$ is called a **bipartition** of G . If further G contains every line joining the points of V_1 to the point of V_2 then G is called a **complete bigraph**.

Definition: 1.9

A **bijection** of a mapping that is both one-to-one and onto. A function which relates each member of a set S to a separate and distinct member of another set T, where each member in T also has a corresponding member in S.

Definition: 1.10

A closed trial in which the origin and internal vertices are distinct is called a **cycle**. A cycle of length n is called a $n - cycle$ and is denoted by c_n.

Definition: 1.11

A graph G is called **labeled** if its p points are distinguished from one another by name such as $v_1, v_2, \dots \dots., v_p$.

Definition: 1.12

A **cycle graph** is connected graph where every vertex is adjacent to two other distinct vertices.

Definition: 1.13

If (X, Y) is a bipartition of a graph G such that every vertex in X is adjacent to every vertex in Y. Then the graph G is called a **complete bipartite graph**.

If $|X| = m$ and $|Y| = n$, then the complete bipartite graph is denoted by $k_{m,n}$.

Definition: 1.14

A complete bipartite graph $k_{1,n-1}$ is called a **star** graph with n vertices. It is denoted by S_n

CHAPTER 2

A LOWER BOUND AND UPPER BOUND

FOR THE MAGIC NUMBER

Definition: 2.1

If a graph G with v vertices and e edges is labeled with numbers 1 through $v+e$ such that every vertex and its incident edges adds up to the same number, the G is a vertex–magic graph. We call this number the magic number.

Definition: 2.2

If a graph G with v vertices and e edges is labeled with numbers 1 through $v+e$ such that every edge and its two adjacent vertices adds up to same magic number, then G is an edge-magic graph.

Example:

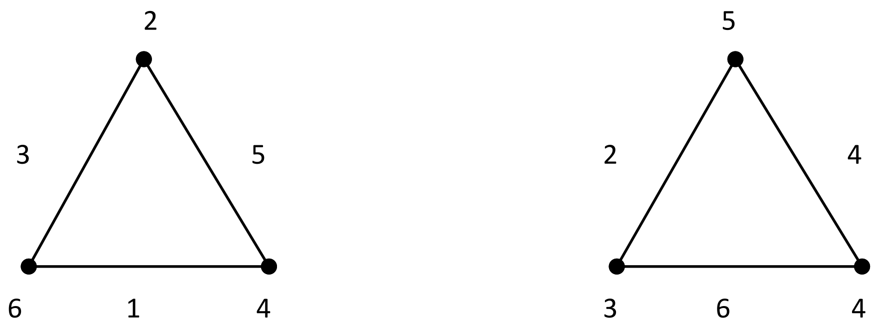

Figure 2.1

A cycle graph with a magic number of 10. If the labels are rotated clockwise, an edge-magic graph is Created with a magic number of 10.

2.1. A LOWER BOUND FOR THE MAGIC NUMBER

Lemma: 2.1.1

If G is a vertex-magic graph with v vertices and e edges, then

$\frac{(v+e)(v+e+1)}{2v} + \frac{E\ sum}{v} = k.$

Proof:

Let V_{sum} be the sum of all vertex labels and E_{sum} be the sum of all edge labels of graph G. Since each edge is incident to two vertices, each edge label will be counted towards it two adjacent vertices magic numbers, Therefore,

$$V_{sum} + 2E_{sum} = vk.$$

Since edges and vertices are labeled 1 through $v + e$,

$$V_{sum} + E_{sum} = 1 + 2 + \dots\dots\dots\dots.. + (v + e) = \frac{(v+e)(v+e+1)}{2}$$

Therefore,
$$\frac{(v+e)(v+e+1)}{2v} + E_{sum} = vk$$

$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k$$

Theorem: 2.1.2

Let G be a graph with v vertices and e edges. If G is a vertex-magic graph, then the magic number, k, is bounded such that

$$\frac{e(e+1)+(v+e+1)(v+e)}{2v} \leq k \leq e + \frac{e(e+1)+(v+e+1)(v+e)}{2v}$$

Proof:

From lemma 2.1.1 we can conclude that $E_{sum} = vk - \frac{(v+e+1)(v+e)}{2}$

The minimum E_{sum} occurs when numbers 1 through e are assigned to the edges

Therefore,

$$\frac{e(e+1)}{2} \leq E_{sum}$$

The maximum E_{sum} occurs when numbers $v+1$ through $v+e$ are assigned to the edges. Hence,

$$E_{sum} \leq \sum_{i=1}^{e} v + i$$

$$= \sum_{i=1}^{e} v + \sum_{i=1}^{e} i$$

$$= ve + \frac{e(e+1)}{2}$$

Hence,

$$\frac{e(e+1)}{2} \leq E_{sum} \leq ve + \frac{e(e+1)}{2}$$

$$\frac{e(e+1)}{2} \leq vk - \frac{(v+e+1)(v+e)}{2} \leq ve + \frac{e(e+1)}{2}$$

$$\frac{e(e+1)+(v+e+1)(v+e)}{2v} \leq k \leq e + \frac{e(e+1)+(v+e+1)(v+e)}{2v}$$

Corollary: 2.1.3

Let be G a cycle graph with v vertices. If G is a vertex–magic graph, then the magic number, k, is bounded such that

$$\frac{5}{2}v + \frac{3}{2} \leq k \textit{ where } v \textit{ is odd}$$

and

$$\frac{5}{2}v + 2 \leq k \textit{ where } v \textit{ is even}$$

Proof:

For every cycle graph, $v = e$. By substituting v in for e in Theorem 2.1.1, We obtain,

$$k \geq \frac{v(v+1)+(2v+1)(2v)}{2v}$$

$$= 2v + 1 + \frac{v+1}{2}$$

$$= \frac{5}{2}v + \frac{3}{2}$$

When v is odd , $v + 1$ is divisible by 2, so the minimum magic number for a cycle graph with an odd number of vertices is $\frac{5}{2}v + \frac{3}{2}$. For an example 2.1.4 of how to apply this bound to a vertex-magic.

This formula, $k = 2v + 1 + \frac{v+1}{2}$ is possible only if v is odd, and a better bound can be found if v is even.

The minimum vertex labelling for a cycle graph occurs when numbers 1 through v are assigned to the edges. From Lemma 2.1.1, $2v + 1 + \frac{E_{sum}}{v} = k$. This implies that E_{sum} must be divisible by v . The edge labelings be in order to have a minimum vertex labelings for an even cycle graph where E_{sum} is divisible by v.

If v is even, can edges be labeled with numbers 1 through v. If this labeling is used, then $k = 2v + 1 + \frac{v+1}{2}$. We know that $v + 1$ must be divisible by 2, however if v is even, then $v + 1$ cannot be divisible by 2. Therefore, the numbers 1 through cannot be used for the edges to find a minimum labeling.

Although the edge labeling of 1 through v cannot be used for a minimum labeling, consider adding $\frac{v}{2}$ to one of the labels. Example 2.1.5 and 2.1.6 E_{sum} can be adjusted so that the magic number is an integer. Adding $\frac{v}{2}$ to E_{sum} if v is even will create a valid edge labeling. Note that this new E_{sum} is divisible by v.

$$E_{sum} = 1 + 2 + 3 \ldots \ldots \ldots + v + \frac{v}{2},$$

$$= \frac{v(v+1)}{2} + \frac{v}{2}$$

As before,

$$k = 2v + 1 + \frac{E_{sum}}{v},$$

$$\geq 2v + 1 + \frac{\frac{v(v+1)+v}{2}}{v}$$

$$= 2v + 1 + \frac{v+2}{2},$$

$$= \frac{5}{2}v + 2$$

Example 2.1.4:

Let G be graph with $v = 15\ vertices$. If G is a vertex magic what would the minimum magic number.

The smallest E_{sum} comes from labeling the edges with numbers 1 through 15. By corollary 2.1.3 , the magic number for this particular graph would be

$\frac{5}{2}(15) + \frac{3}{2} = 39$

Example 2.1.5:

Let G be a cycle graph with $v = 4\ vertices$. what are some combinations of vertex labelings that could be used in creating a vertex-magic graph with a minimum k.

Being by making $E_{sum} = 1 + 2 + 3 + 4$. Since $v = 4,$ and 4 does not divide $E_{sum} = 10,$ then E_{sum} must be adjusted such that E_{sum} is divisible by v. The edge labels cannot be shifted down because the smallest numbers are already on the edges. The closest multiple 4 after 10 is 12. Therefore, E_{sum} must be shifted from 10 to 12 by adding 2. There are numerous ways of doing this. By adding 2 to 4, E_{sum} becomes $1 + 2 + 3 + 6$. since the sum of the edges equals 12, $\{1,2,3,6\}$ is a valid edge labeling for finding a possible minimum k.

Example 2.1.6:

Let G be a cycle graph with $v = 6\ vertices$. What is a valid E_{sum} that can be used to make G Vertex- magic with a minimum k.

By using the same technique, the edge labelings can be altered for $v = 6,$

Let $E_{sum} = 1 + 2 + 3 + 4 + 5 + 6$. Since this sum 21 is not divisible by 6, we must add 3 to obtain 24 which is divisible by 6. Therefore, the minimum magic number is

24 when $v = 6$.

2.2 AN UPPER BOUND FOR THE MAGIC NUMBER

Corollary: 2.2.1

Let G be a graph with v vertices. If G is a vertex-magic graph, then the magic number, k, is bounded such that

$$k \leq \frac{7}{2}v + \frac{3}{2} \; where \; v \; is \; odd,$$

And

$$k \leq \frac{7}{2}v + 1 \; where \; v \; is \; even$$

Proof:

The upper bound would occur on any graph when the largest numbers, $v + 1$ through $2v$, are placed on the edges. Therefore,

$$
\begin{aligned}
E_{sum} &= \sum_{i=1}^{v} v + i, \\
&= v^2 + \frac{v(v+1)}{2}, \\
&= \frac{2v^2 + v(v+1)}{2} \\
&= \frac{2v^2 + v^2 + v}{2} \\
&= \frac{3v^2 + v}{2} \\
&= \frac{v(3v+1)}{2}.
\end{aligned}
$$

Also,

$$k = 2v + 1 + \frac{E_{sum}}{v},$$

$$\leq 2v + 1 + \frac{v(3v+1)}{2v},$$

$$= 2v + 1 + \frac{v+1}{2},$$

$$= \frac{7}{2}v + \frac{3}{2}$$

As in the case of the lower bound, 2 must divide $v + 1$ and v must divide E_{sum} . For any cycle graph with an odd number of vertices, this holds true when the edges are the largest numbers. So, the maximum possible k for any cycle graph with an odd number of vertices is $\frac{7}{2}v + \frac{3}{2}$. However, a cycle graph with an even number of vertices is more difficult.

When looking at the upper bound for k of a cycle graph with an even number of vertices, assume that the numbers $v + 1$ through $2v$ are placed on the edges. Since $v + 1$ is not divisible by 2, one must alter the edge labelings. To find the lower bound for the magic number when v is even, the number E_{sum} was increased, but in this case, the numbers cannot be increased because no label can be greater than $2v$. So, in order to find an E_{sum} divisible by v, the original E_{sum} must be decreased, In example 2.2.2 and 2.2.3

For any cycle graph with an even number of vertices, the number $\frac{v}{2}$ must be subtracted from the original sum. Therefore,

$$E_{sum} = \frac{v(3v+1)}{2} - \frac{v}{2}$$

$$= \frac{3v^2+v}{2} - \frac{v}{2}$$

$$= \frac{3v^2}{2} + \frac{v}{2} - \frac{v}{2}$$

$$= \frac{3v^2}{2}$$

$$= \frac{v(3v)}{2}$$

Thus,

$$k \quad = 2v + 1 + \frac{E_{sum}}{v}$$

$$\leq 2v + 1 + \frac{(3v)}{2}$$

$$= \frac{7}{2}v + 1$$

Example 2.2.2

Let G be a cycle graph with $v = 4\, vertices.$ What are some possible combinations of vertex labelings such that G has a maximum k.

Label the graph such that $E_{sum} = 8 + 7 + 6 + 5 = 26$. The largest number less than 26 that is divisible by 4 is 24. Therefore 2, or $\frac{v}{2}$, must be subtracted from E_{sum} making the magic number 15. One possible edge labeling is $\{8,7,6,3\}$ By changing the 5 to a 3, E_{sum} becomes $8 + 7 + 6 + 3$ which is divisible by v. Therefore, the edges labelings could possibly be used to label this graph with the maximum k.

Example 2.2.3

Let G be a cycle graph with $v = 6$. What is the maximum E_{sum} for G such that G is vertex-magic.

If G is labeled with the largest numbers on the edges, G has an E_{sum} of 21. The next E_{sum} less than 21 that divides 6 is 18. Thus, 3 must be subtracted from the original E_{sum} to make the new E_{sum} equal to 18. This edge labeling has a magic number of 22.

CHAPTER 3

MAXIMUM AND MINIMUM MAGIC NUMBER FOR ODD AND EVEN CYCLES GRAPHS

3.1 Maximum and Minimum Magic Number For Odd Cycle Graphs

Theorem: 3.1.1

Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling with the numbers 1 to v located on the vertices and a magic number of $\frac{7}{2}v + \frac{3}{2}$, the upper bound for the magic number.

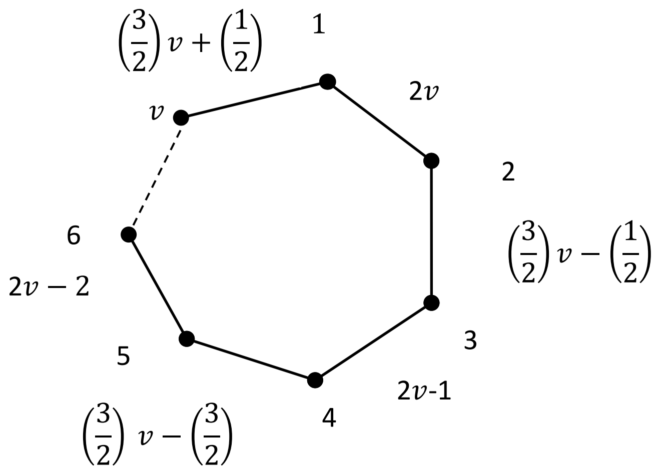

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Figure 3.1

Vertices of a general cycle graph labeled with the smallest numbers

Proof:

Label the vertices with the consecutive numbers 1 through v in a clockwise manner. Assume in this paper that no two edges of the cycle graph cross. An edge is to the right of a vertex if it is adjacent and clockwise to that vertex, and an edge is to the left of a vertex if it is adjacent and clockwise to that vertex, and an edge is to the left of a vertex if it is adjacent and counterclockwise to that vertex. Starting with the edge to the right of the vertex labeled 1, go clockwise around the polygon twice labeling every other edge with consecutive numbers $v + 1$ through $2v$ in descending order starting with $2v$. Any vertex of a cycle graph can be categorized into one of the following two categories

Description of Case	**Vertex label**	**Left edge Label**	**Right Edge Label**
Every other Vertex starting With 1	$2i + 1$ $i = 0, \ldots., \frac{v-1}{2}$	$\frac{3v}{2} + \frac{1}{2} - i$	$2v - i$
Every other Vertex starting With 2	$2i + 2$ $i = 0, \ldots., \frac{v-1}{2}$ -1	$2v - i$	$\frac{3v}{2} - \frac{1}{2} - i$

A vertex in the first case will have a magic number of

$$2i + 1 + \frac{3v}{2} + \frac{1}{2} - i + 2v - i = \frac{7v}{2} + \frac{3}{2}.$$

A vertex in the second case will have a magic number of

$$2i + 2 + 2v - i + \frac{3v}{2} - \frac{1}{2} - i = \frac{7v}{2} + \frac{3}{2}.$$

Thus, this graph is vertex-magic with a magic number of $\frac{7v}{2} + \frac{3}{2}$.

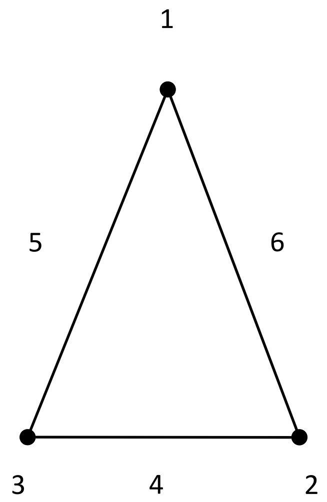

Figure 3.2

A cycle graph with 3 vertices and a magic number of 12.

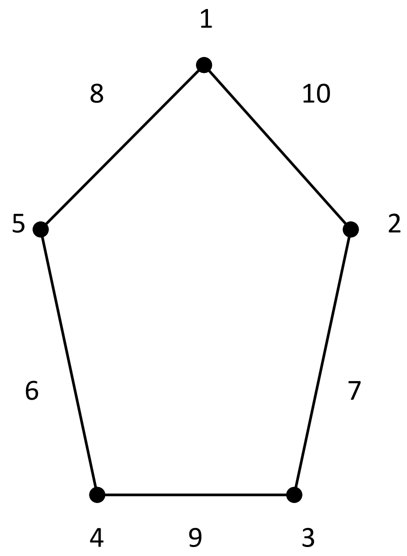

Figure 3.3

A cycle graph with 5 vertices and a magic number of 19.

Now, assign the largest numbers to the vertices. As expected, the maximum magic number is fixed and depends only on the number of vertices.

Theorem: 3.2

Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling with the numbers $v+1$ to $2v$ located on the vertices and a magic number of $\frac{5}{2}v+\frac{3}{2}$, the lower bound for the magic number.

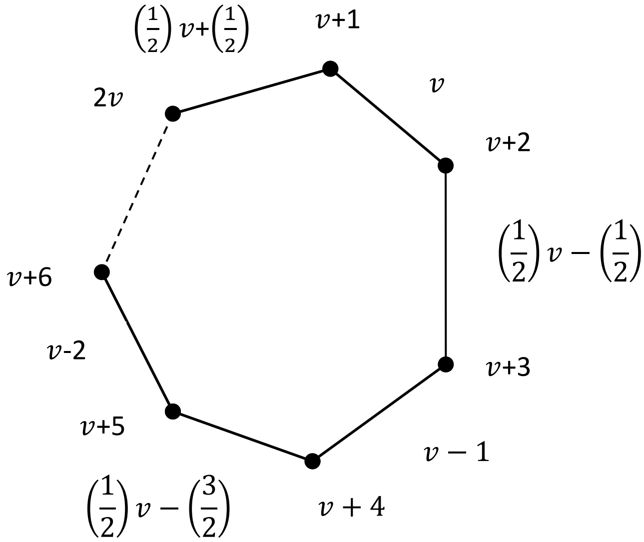

Figure 3.4

The vertices of a cycle graph labeled with the large numbers

Proof:

A cycle graph can be labeled with the largest numbers on the vertices. Label the vertices with the Consecutive numbers numbers $v+1$ through $2v$ in a clockwise manner. Starting with the edge to the right of verte$x$ $v+1$, go clockwise around the polygon twice labeling every other edge with consecutive numbers 1 through v in descending order starting with v. The vertices of the cycle graph can be categorized into one of the following categories.

Description of Case	Vertex label	Left Edge Label	Right Edge label
Every other Vertex starting With $v+1$	$v+1+2i$ $i=0,\ldots,\dfrac{v-1}{2}$	$\frac{v}{2}+\frac{1}{2}-\frac{2i}{2}$	$v-i$
Every other Vertex starting With $v+2$	$v+2+2i$ $i=0,\ldots,\frac{v-1}{2}$ -1	$v-i$	$\dfrac{v}{2}-\dfrac{1}{2}-\dfrac{2i}{2}$

-

A vertex in the first case will have a magic number of

$$v+1+2i+v-i+\frac{v+1-2i}{2}=\frac{5v}{2}+\frac{3}{2}.$$

A vertex in the second case will have a magic number of

$$v+2+2i+v-i+\frac{v-1-2i}{2}=\frac{5v}{2}+\frac{3}{2}.$$

This cycle graph is vertex-magic with a magic number of $\frac{5\,v}{2}+\frac{3}{2}$.

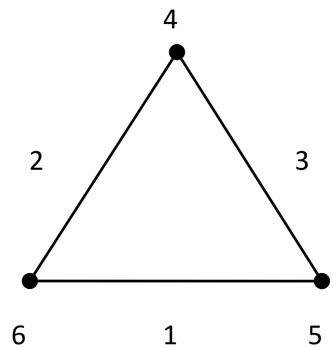

Figure 3.5

A cycle graph with 3 vertices and a magic number of 9.

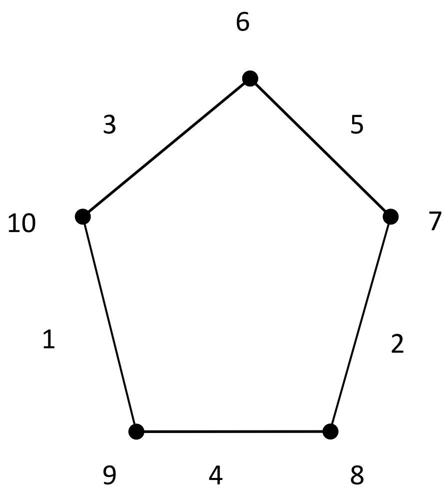

Figure 3.6

A cycle graph with 5 vertices and a magic number of 14.

3. 2. Minimum Magic Number For Even Cycle Graphs

Conjecture 3.2.1:

Let G be a graph with v vertices where v is even. There exists a vertex-magic labeling for G with Minimum magic number $k = \frac{5}{2}v + 2$.

If we can find edge labelings that create a vertex-magic graph, then by adding the two incident edges of a vertex and subtracting that sum from the magic number, the vertex labels can be easily obtained. The edges of a cycle graph with an even number of vertices can be labeled as follows:

Let $v = 2n$. The value of n can be either even or odd, and the constructio of the vertex-magic graph depends on n. If n is even, the following is a construction for how to label the edges.

S

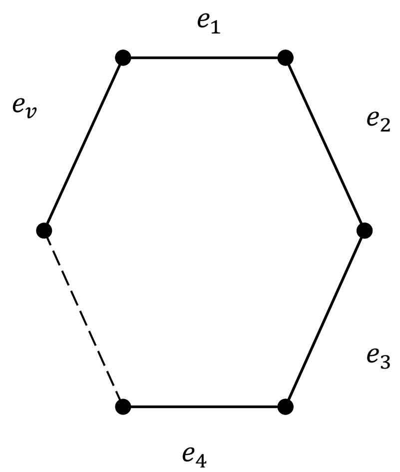

Figure 3.7

The edges of a cycle graph

$$
e_i = \begin{cases} \frac{i+1}{2}, & i = 1, 3, \dots, n+1, \\ 3n, & i = 2, \\ \frac{2n+i}{2}, & i = 4, 6, .. n, \\ \frac{2n+i-1}{2}, & i = n+3, n+5, \dots, 2n-1, \\ \frac{i+2}{2}, & i = n+2, n+4, \dots 2n. \end{cases}
$$

A cycle graph with $v = 8$ and $n = 4$ should have a minimum k of 22. By using the given edge labelings for an even n, one can find the edges and vertices for this graph.

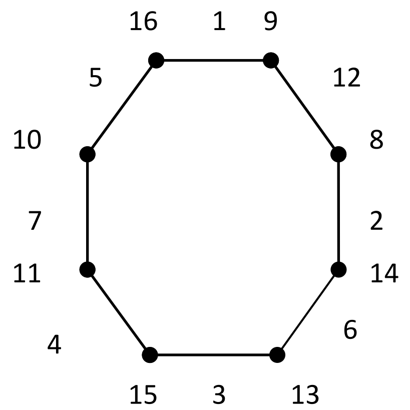

Figure 3.8

A cycle graph with the minimum magic number of 22.

The type of labeling occurs when $n = 6$ or $v = 12$. Using conjecture 3.2.1, we

Calculate that the magic number should be 32.

If n is odd a similar technique can be used to label the cycle graph.

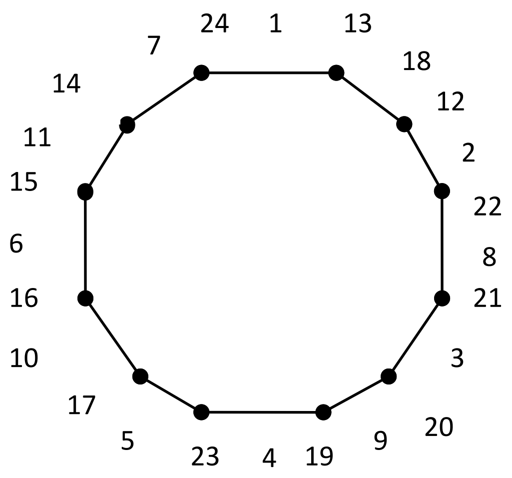

Figure 3.9

A cycle graph with the minimum magic number of 32.

$$e_i = \begin{cases} \frac{i+1}{2}, & i = 1, 3, \ldots, n, \\ 3n, & i = 2, \\ \frac{2n+i+2}{2}, & i = 4,6,\ldots n\text{-}1, \\ \frac{n+3}{2}, & i = n+1, \\ \frac{2n+i}{2}, & i = n+3, n+5, \ldots\ldots, 2n-2 \\ \frac{i+3}{2}, & i = n+2, n+4, \ldots\ldots, 2n-1 \\ n+2, & i = 2n \end{cases}$$

Figure 3.10 and 3.11 Vertex–magic cycles where n is odd. If $v = 6$, then n = 3 and the magic number is 17.

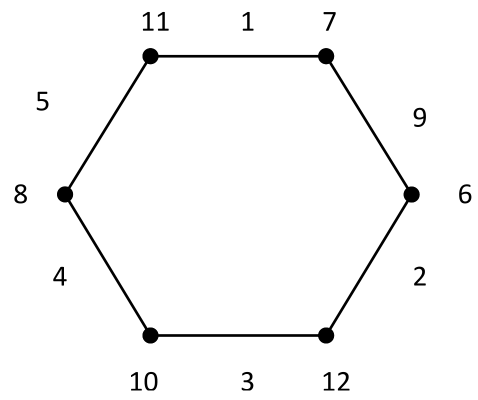

Figure 3.10

A cycle graph with the minimum magic number

Vertex- magic labeling when $v = 10$. Thus n = 5, and the magic number is 27.

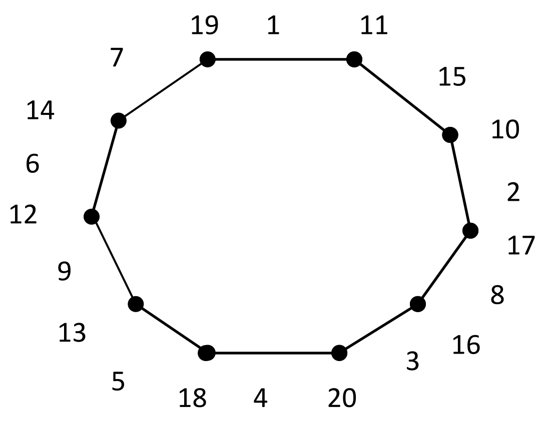

Figure 3.11

A cycle graph with the minimum magic numbers of 27.

We have to just shown how to produce vertex-magic graphs with the minimum magic for both even and odd cycles. We have also shown how to create vertex-magic graphs With the maximum magic number for odd cycles

3.3 Maximum Magic Number For Even Cycle Graphs

Theorem: 3.3.1

Let G be a cycle graph with v vertices where v is even. If Conjecture 3.2.1 holds then there exists a vertex-magic labeling for G with the maximum magic number $k = \frac{7}{2}v + 1$.

To obtain the minimum k in conjecture 3.2.1 for an even the edge labeling should be $1,2,3, \ldots\ldots, v-1, v+\frac{v}{2}$. In order to obtain the maximum k the edge labels should be $2v, 2v-1, \ldots., v+2, v+1-\frac{v}{2}$. These values are exactly $2v+1$ minus the numbers on the edges in the minimum labeling. The following cases produce the edge labelings for a maximum magic vertex graph. Using the maximum k we can subtract the sum of the two edges in order to label each vertex. Again, let $v = 2n$ and let n be even.

$$e_i = \begin{cases} 2v - \frac{i+1}{2} + 1, & i = 1, 3, \ldots, n+1, \\ 2v - 3n + 1, & i = 2, \\ 2v - \frac{2n+i}{2} + 1, & i = 4,6, .. n, \\ 2v - \frac{2n+i-1}{2} + 1, & i = n+3, n+5, \ldots, 2n-1, \\ 2v - \frac{i+2}{2} + 1, & i = n+2, n+4, \ldots.., 2n, \end{cases}$$

Now let n be odd, subtracting the edge labels of the algorithm in conjecture 3.2.1

From $2v$ and adding 1, a new edge labeling with a maximum k can be obtained.

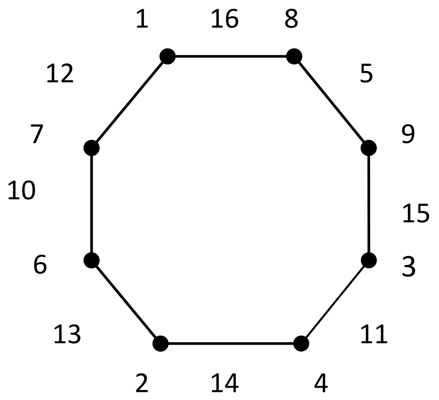

Figure 3.12

A cycle graph with the maximum magic number of 29.

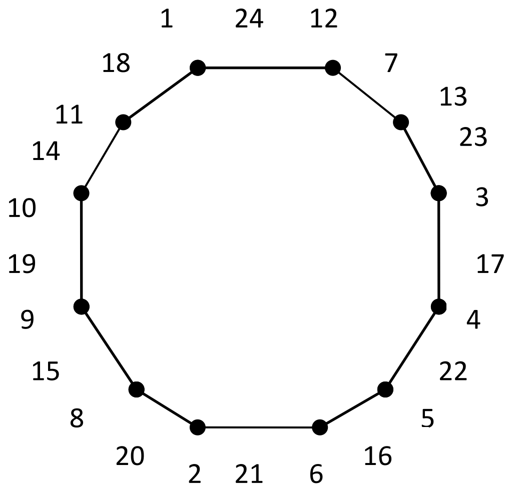

Figure 3.13

A cycle graph with the maximum magic number of 43.

$$e_i = \begin{cases} 2v - \frac{i+1}{2} + 1, & i = 1,3, \dots, n, \\ 2v - 3n + 1\ , & i = 2, \\ 2v - \ \frac{2n+i+2}{2} + 1\ , & i = 4,6, \dots n\text{-}1, \\ 2v - \frac{n+3}{2} + 1, & i = n + 1, \\ 2v - \frac{2n+i}{2} + 1\ , & i = n + 3, n + 5, \dots ,2n\text{-}2 \\ 2v - \frac{i+3}{2} + 1, & i = n + 2, n + 4, \dots ,2n - 1, \\ 2v - (n + 2) + 1, & i = 2n, \end{cases}$$

Different algorithms can be found for other odd cycle graphs such that the magic numbers are within the range for k. Since we have found algorithms for creating vertex-magic graphs such that the magic number is the magic number is the minimum and maximum bound, we can conclude that the bounds are sharp.

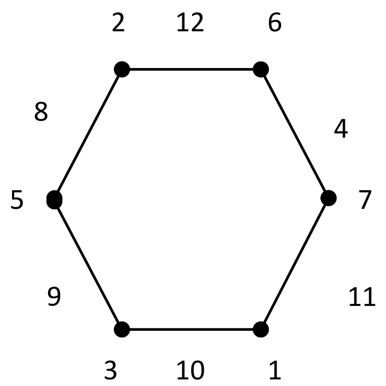

Figure 3.14

A cycle graph with the maximum magic number of 22

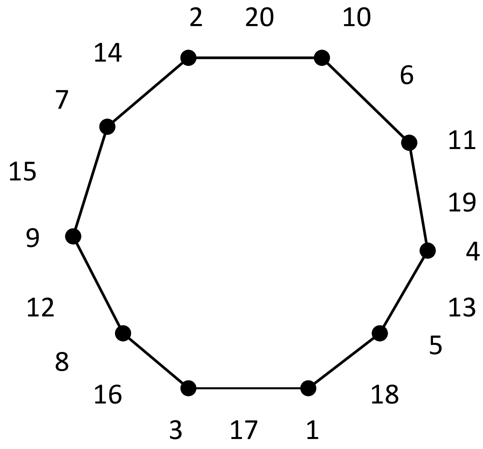

Figure 3.15

A cycle graph with the maximum magic number of 36.

CHAPTER 4

ODD AND EVEN NUMBERS ON THE VERTICES

Theorem: 4.1

Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling for G with the odd numbers from 1 to $2v-1$ located on the vertices and a magic number of $3v+2$.

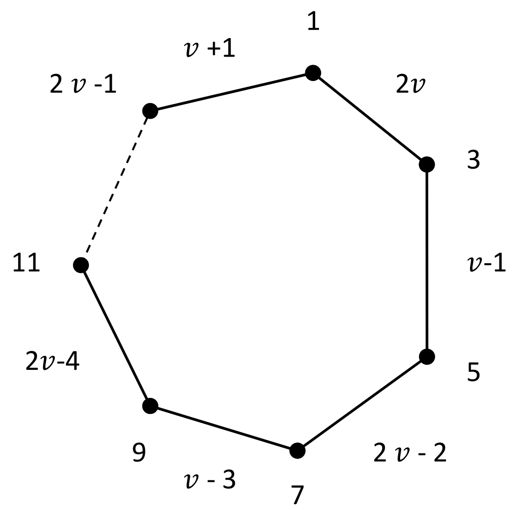

Figure 4.1

The vertices of a general cycle graph labeled with odd numbers

Proof:

In order to label an odd cycle graph with odd numbers on the vertices, the vertices can be labeled with the consecutive odd numbers 1 through $2v-1$ in a clockwise manner. Starting with the edge to the right of the vertex labeled 1, go clockwise around the polygon twice labeling every other edge with consecutive even numbers 2 through $2v$ in descending order starting with $2v$.

In general, any vertex of this labeling can be categorized into one of the following categories.

Description of Case	**Vertex Label**	**Left Edge Label**	**Right Edge Label**
Every other Vertex starting With 1	$4i + 1$ $i = 0, \dots, \frac{v-1}{2}$	$v + 1 - 2i$	$2v - 2i$
Every other Vertex starting With 3	$4i + 3$ $i = 0, \dots, \frac{v-1}{2} - 1$	$2v - 2i$	$v - 1 - 2i$

In the first case, the magic number is $4i + 1 + v + 1 - 2i + 2v - 2i$, or $3v + 2$. In the second case, the magic number is $4i + 3 + 2v - 2i + v - 1 - 2i$, which is also $3v + 2$. Therefore, the graph is vertex-magic with a magic number of $3v + 2$.

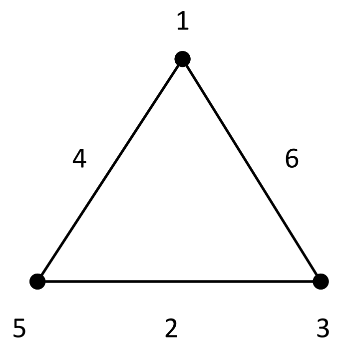

Figure 4.2

A cycle graph with 3 vertices and a magic number of 11.

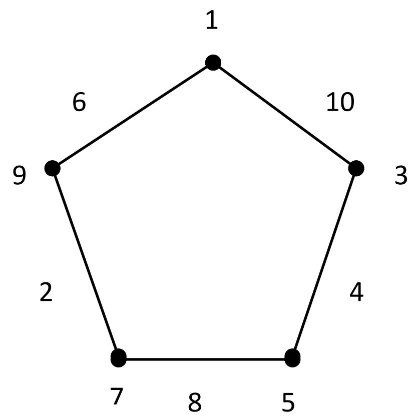

Figure 4.3

A cycle graph with 5 vertices and a magic number of 17

Using similar ideas, one can label an odd cycle graph with the even numbers on the vertices. Like the odd numbers on the vertices, the even number on the vertices produce a pattern that is dependent upon the number of vertices.

Theorem :4.2

Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling for G with the even numbers from 2 to $2v$ located on the vertices and a magic number of $3v + 1$.

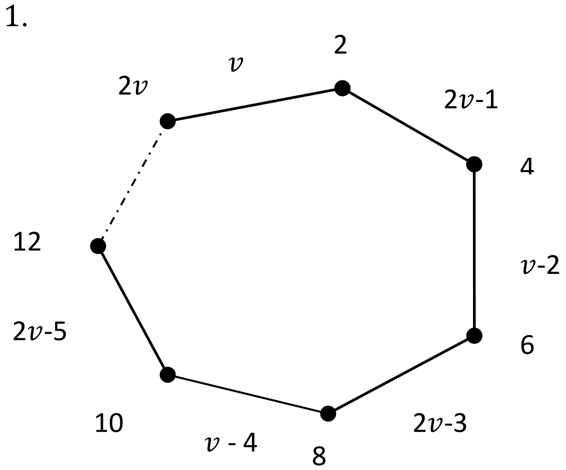

Figure 4.4

The vertices of a general cycle graph labeled with even numbers

Proof:

In order to label an odd cycle graph with even numbers on the vertices, the vertices can be labeled with the consecutive even numbers 2 through $2v$ in a clockwise manner. Starting with the edge to the right of the vertex labeled 2, go clockwise around the polygon twice labeling every other edge with consecutive odd numbers 1 through $2v - 1$ in descending order starting with $2v - 1$. in general , any vertex of this labeling can be categorized into one of the following categories.

Description of Case	**Vertex Label**	**Left Edge Label**	**Right Edge Label**
Every other Vertex starting With 2	$4i + 2$ $i = 0, \ldots.., \frac{v-1}{2}$	$v - 2i$	$2v - 1 - 2i$
Every other Vertex starting With 4	$4i + 4$ $i = 0, \ldots.., \frac{v-1}{2} - 1$	$2v - 1 - 2i$	$v - 2i - 2$

In the first case, the magic number is $4i + 2 + v - 2i + 2v - 1 - 2i$, or $3v + 1$. In the second case, the magic number is $4i + 4 + 2v - 1 - 2i + v - 2i - 2$, or $3v + 1$. Therefore, the graph is vertex-magic with a magic number of $3v + 1$.

Finding the average of the minimum and maximum bounds, we get

$$\frac{\frac{7}{2}v+\frac{3}{2}+\frac{5}{2}v+\frac{3}{2}}{2}=3v+\frac{3}{2}$$

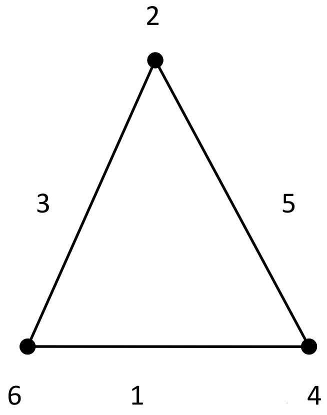

Figure 4.5

A cycle graph with 3 vertices and a magic number of 10.

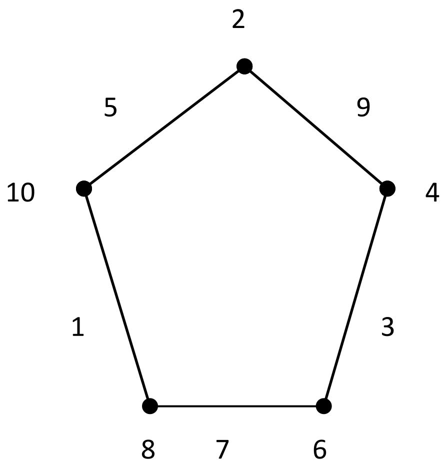

Figure 4.6

A cycle graph with 5 vertices and a magic number of 16.

This value is not a possible for . However the two middle k values,

$3v + \left\lceil\frac{3}{2}\right\rceil = 3v + 1$ and $3v + \left\lceil\frac{3}{2}\right\rceil = 3v + 2$ occur when the even and odd numbers are placed on the vertices respectively.

CHAPTER 5

SUPER VERTEX–MAGIC LABELING

Definition: 5.1

A vertex magic labeling f is called super vertex-magic labeling if

$f(E) = \{1,2,3, \dots, \varepsilon\}$ and $f(V) = =\{\varepsilon + 1, \varepsilon + 2, \dots., \varepsilon\}$. A graph G is called super vertex-magic if there exists a super vertex-magic labeling of G.

Lemma:5.1

If a nontrivial graph G is Super vertex-magic then the magic number k is given by $k = \varepsilon + \frac{v+1}{2} + \frac{\varepsilon(\varepsilon+1)}{v}$

Proof:

Let f be a super vertex magic-labeling of a graph G with the magic number k. Then

$f(E) = \{1,2,3, \dots.\varepsilon,\}$ and $K = f(u) + \sum_{v \in N(u)} f(uv)$, for all $u \in$V.

Then, $$vk = \sum_{u \in V} f(u) + \sum_{u \in V} \sum_{v \in N(u)} f(uv)$$

$$= \sum_{u \in V} f(u) + 2 \sum_{e \in E} f(e)$$

$$= (\varepsilon + 1) + (\varepsilon + 2) + \cdots + (\varepsilon + v) + \varepsilon(\varepsilon + 1)$$

$$= \varepsilon v + \frac{V(V+1)}{2} + \varepsilon(\varepsilon + 1)$$

Thus, $$k = \varepsilon + \frac{(v+1)}{2} + \frac{\varepsilon(\varepsilon+1)}{v}$$

Theorem: 5.2

A path P_n is super vertex- magic if and only if n is odd and $n \geq 3$.

Proof :

Suppose there exists a super vertex-magic labeling f of P_n with the magic number k.

Then by lemma 5.1 $k = n - 1 + \frac{n+1}{2} + \frac{n(n-1)}{n} = \frac{5n-3}{2}$

Since k is an integer, n must be odd.

Let n be an odd integer,

$V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1} / 1 \leq i \leq n-1\}$

Define f: V∪E→ $\{1,2, \ldots, 2n-1\}$ as follows:

$f(v_1) = 2n - 1$

$f(v_i) = n - 2 + i$ for $2 \leq i \leq n$

$f(e_i) = \frac{n-i}{2}$ if i is odd

$= n - \frac{i}{2}$ if i is even.

It is easily seen that f is a super vertex-magic labeling with the magic number $\frac{5n-3}{2}$.

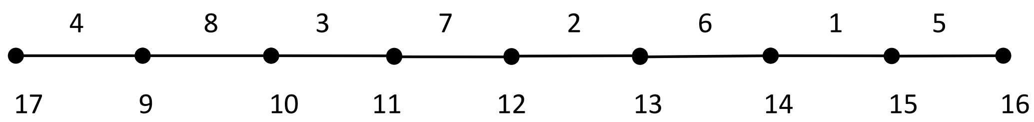

Figure 5.1

Theorem 5.3:

A cycle c_n is super vertex –magic if and only if n is odd.

Proof:

Suppose there exists a super vertex-magic labeling f of c_n with the magic number k.

Then by Lemma 5.1 $k = n + \frac{n+1}{2} + \frac{n(n+1)}{n} = \frac{5n-3}{2}$

Since k is an integer, n must be odd.

Let n be an odd integer,

$\mathrm{V}(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $\mathrm{E}(C_n) = \{v_n v_1\} \cup \{v_i v_{i+1} / 1 \le i \le n-1\}$

Define f: V∪E→ $\{1,2, \ldots, 2n\}$ as follows:

$f(v_i) = 2n + 1 - i$ for $1 \le i \le n$

$f(v_i v_{i+1}) = \frac{i+1}{2}$ if i is odd

$= \frac{n+1+i}{2}$ if i is even

$f(v_n v_1) = \frac{n+1}{2}$

It is easily seen that f is a super vertex-magic labeling with the magic number $\frac{5n+3}{2}$.

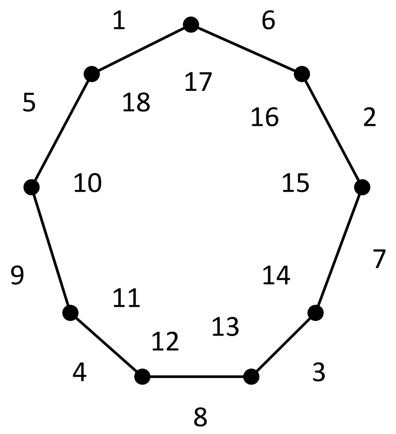

Figure 5.2

Theorem: 5.4

Let G be a graph and g is a bijection from E onto $\{1,2,3,\ldots,\varepsilon\}$. Then g can be extended to a super vertex-magic labeling of G if and only if

$\{w(u) = \sum_{v\in N(u)} g(uv)/u \in V\}$ consists of $|v|$ sequential integers.

Proof:

Assume that $\{w(u). u \in V\}$ consists of $|V|$ sequential integers. Let

$t = \min\{w(u)/u \in V\}$.

Define f: $V \cup E \rightarrow \{1,2,3,\ldots,v+\varepsilon\}$ as $f(xy) = g(xy)$ for xy$\in E$ and

$f(x) = t + v + \varepsilon - w(x)$. Then $f(E) = \{1,2,3,\ldots.,\varepsilon\}$ .and

$f(V) = \{\varepsilon + v, v + \varepsilon - 1, \ldots.\varepsilon + 1\}$. Hence f is a super vertex-magic labeling

With $k = t + v + \varepsilon$.

Suppose g can be extended to a super vertex-magic labeling f of G with a constant k.

Now let

$t = \min\{w(u)/u \in V\}$. Since for every $u \in V$, $f(u) + w(u) = k$, we have

$\mathrm{W}(u) = k - f(u)$. Thus

$\{w(u)/u \in V\} = \{k - \varepsilon - v, k - \varepsilon - v + 1, \dots k - \varepsilon - 1\} = \{t, t+1, \dots . t + v - 1\}$.

Theorem: 5.5

A star graph S_n is super vertex-magic if and only if $n = 2$

Proof:

Let $\mathrm{V}(S_n) = \{c, u_1, u_2, \dots . u_n\}$ and $\mathrm{E}(S_n) = \{cu_i/1 \le i \le n\}$. Let f be super Vertex-magic labeling of S_n. Then by theorem 5.4

$\{w(u)/u \in V\} = \{1,2, \dots \dots ., n+1\}$. Again

$w(c) = \frac{n(n+1)}{2}$ and $\mathrm{w}(u_i) = i$ for $1 \le i \le n$. Hence $n+1 = \frac{n(n+1)}{2}$ Thus $n=2$, When, $n = 2$, $S_n = P_z$ which is super vertex-magic.

5.2 Super Vertex–Magic Labeling On a Disconnected Graph

Theorem: 5.2.1

$mC_{n,}$ is super vertex- magic labeling if and only if both m and n are odd.

Proof:

Suppose there exists a super vertex magic labeling of mC_n, with the magic number k. Then by

Lemma 5.1

k=mn+ $\frac{mn+1}{2}+\frac{mn(mn+1)}{mn}=\frac{5mn+3}{2}$

Thus, k is an integer only when both m and n are odd. Let m and n are odd integers. Assume that the graph mC_n, has vertex set

$V = V_1 \cup V_2 \cup \dots .\cup V_m$. where $V_i = \{v_i^1, v_i^2, \dots., v_i^n\}$, and the edge

$E = E_1 \cup E_2 \cup \dots \dots .\cup E_m$ where $E_i = \{e_i^1, e_i^2, \dots .. e_i^n\}$, and $e_i^j = v_i^j v_i^{j+1}$

for 1$\leq i \leq m$, 1$\leq j \leq n-1$, $e_i^n = v_i^n v_i^1$

Define f: $V \cup E \rightarrow \{1,2,3,\dots,2mn\}$ as follows:

For 1$\leq i \leq \frac{(m-1)}{2}$

$f(v_i^j) = (2n-j)m+1-2i$ for 1$\leq j \leq n-2$

$= mn+i$ for $j=n-1$

$= \frac{1}{2}(4n-1)m+\frac{1}{2}+i$ for $j=n$

$f(e_i^j) = \frac{1}{2}(j-1)m+i$ for $j = 1,3,\dots.,n-2$

$= \frac{1}{2}(n+j)m+\frac{1}{2}+i$ for $j = 2,4,\dots,n-1$

$= \frac{1}{2}(n+1)m+1-2i$ for $j=n$

For

$\frac{m+1}{2} \leq i \leq m$

$f(v_i^j) = (2n+1-j)m+1-2i$ for 1$\leq j \leq n-2$

$= mn + i$ for $j = n$

$= \frac{1}{2}(4n - 3)m + \frac{1}{2} + i$ for $j = n$

$f(e_i^j) = \frac{1}{2}(j - 1)m + i$ for $j = 1,3, \dots, n - 2$

$= \frac{1}{2}(n + j - 2)m + \frac{1}{2} + i$ for $j = 2,4, \dots, n - 1$

$= \frac{1}{2}(n + 3)m + 1 - 2i$ for $j = n$

It is easily that verified that f is a super vertex-magic labeling of mC_n

with $k = \frac{5mn+3}{2}$

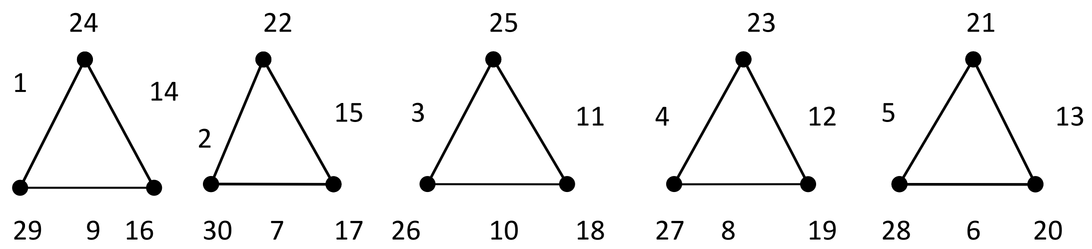

Figure 5.3

CHAPTER 6

APPLICATION OF VERTEX MAGIC LABELING OF GRAPH

A company want to distribute equal number to its departments such that each computer is either by one department or by two departments, here the problem is to find the number of computers allotted to each department and also to find number of computers utilized by one department and number of computers used by two departments. Representing this situation as a graph by considering the departments as vertices and if two departments shares computers then there is an edge between the corresponding vertices. Considering the vertex magic labeling to this graph we are able to get the solution needed.

Problem: 6.1

A company wants to provide exact numbers of computers or workstations to its 5 departments D_1, D_2, D_3, D_4, D_5, departments are utilizing computers in such a manner that the departments D_1 and D_2, D_2 and D_3, D_3 and $D_{4,}$ D_4 and D_5 are sharing few computers. Find the number of computers required for a department. Find the exact number of computers utilized for two departments.

Solution:

For each $i = 1,2,3,4,5,$ we take the department D_i as the vertex v_i and if departments D_i and D_j are sharing computers then we take an edge between v_i and v_j.

The graph corresponding to the given circumstance is given in figure 6.1

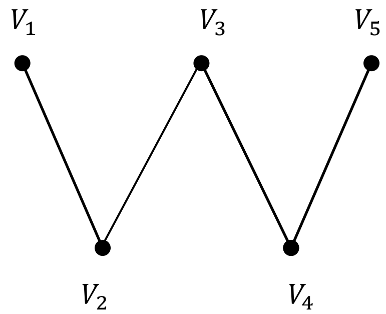

Figure 6.1

Consider vertex Magic Total Labeling (VMTL) of the given graph

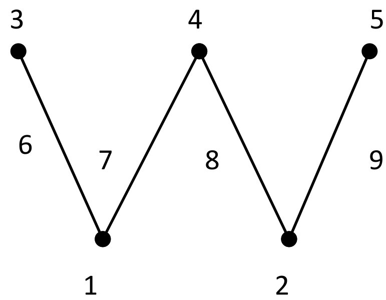

Figure 6.2

The sum of the labels on the vertex and the incident edges is a constant k , this k provides the exact number of computers or workstations required for a particular department. The vertex labels expresses the exact number of computers utilized by one department. Also it gives the edge labels by the another department. Here $k = 3$ so each department utilizes 3 computers. We catalog the vertex and edge labels as in the following

Dept	D_1	D_2	D_3	D_4	D_5
D_1	6	3	0	0	0
D_2	3	7	1	0	0
D_3	0	1	8	4	0
D_4	0	0	4	9	2
D_5	0	0	0	2	5

The entry in the diagonal of the table shows the exact number of computers utilized by number one department. The other table shows the number of computers utilized by the respective pair of computers.

Problem: 6.2

A company wants to provide equal number of computers to its 5 departments Administrators, Human resources, Logistics, Finance and Accounts. In order to reduce the idle time of the computers the company wants few computers are utilized by two departments. The computers shared by the departments are Human resource and Logistics, Human resource and Administrators, Administrators and Accounts and Finance and Logistics. Find the exact number of computers required and also number of computers utilized by each department. Find the number of computers utilized by one department and by two departments.

Solution:

Let us denote the departments Administrators, Human resource, Logistics, Finance and Accounts as the vertices v_1, v_2, v_3, v_4, v_5 respectively. We can take an edge between vertices if there is a sharing of computers between the corresponding departments. The reverse vertex magic total labeling graph is shown in figure.

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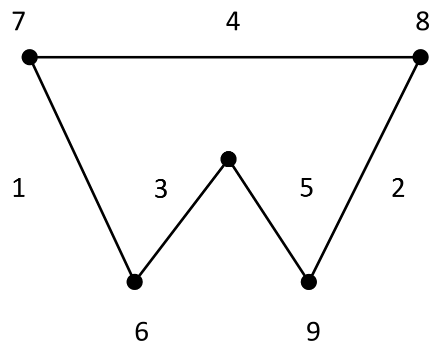

Figure 6.3

This VMT labeling gives the magic constant $k = 2$. Thus 2 computers are utilized by each department.The number of computers utilized by one department and which are utilized by two departments are given in the following table.

Dept	D_1	D_2	D_3	D_4	D_5
D_1	7	1	0	0	4
D_2	1	6	3	0	0
D_3	0	3	10	5	0
D_4	0	0	5	9	2
D_5	4	0	0	2	8

The entry in the diagonal of the table shows the number of computers utilized by one department. The other entries are the number of computers utilized by the respective pair of elements.

Remark:

1. The vertex magic labeling considered in the problems 6.1 and 6.2 are also a super vertex magic labeling(SVM). In this SVM labeling, the edge labels are smaller than the vertex labels. The computers shared by more than one department have to be installed with more software or with more equipment. So the company has to spent more money on the computers which are utilized by more than one department. If a SVM is considered for the graph (if it exists for the graph) which is drawn according to the situation described, then the number of computers which are utilized by more than one department is minimized Hence the amount spent by the company on those computers is minimized.

2. Suppose the company needs computers in large numbers, then consider the multiple of the labeling with any number

A STUDY ON DOMINATION IN GRAPHS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

P.CHRISTINA CAROLIN

Reg.No:19SPMT05

Under the guidance of

Ms. A. FERDINA M.Sc., M.Phil., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON DOMINATION IN GRAPHS"** is submitted to **St. Mary's college (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **P.CHRISTINA CAROLIN (REG.NO: 19SPMT05)**

Ferdina
Signature of the Guide

Signature of the Coordinator

Signature of the Director
Director
Self Supporting Courses
St. Mary's College (Autonomous
Thoothukudi - 628 001.

Lucia Rose
Signature of the Principal
Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON DOMINATION IN GRAPHS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms.A.Ferdina M.Sc.,M.Phil.,SET.,** Assistant Professor ,Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi.**

Station: Thoothukudi

P. Christina carolin
Signature of student

Date: 10.02.21

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Place: Thoothukudi

Date: 10.02.21

CHAPTER 1

PRELIMINARIES

In this chapter we present some basic definitions in graph theory that are needed for subsequent chapters.

Definition:1.1

A **Graph** is an order triple G =(V(G),E(G),I_G) where V(G) is a nonempty set of vertices, E(G) is a set of edges disjoint from V(G) and I(G) is an incidence relation that associates with each element of E(G) an unorder pair of elements of V(G).

Definition:1.2

The graph G is called a **Planar Graph** if it has a digraph in which no two edges intersect at a vertex or a point other than a vertex.

Definition:1.3

The vertex v is incidence with an edge e if it is n points of e. Therefore every edges incidence with its n vertex.

Two **Vertices** are **Adjacent** if they are incident with same edge. Two **Edges** are **Adjacent** if they are incident with same vertex.

Definition:1.4

An edge starting and ending with the same vertex is called a **Loop**.

An edge with distinct ends is called a **Link**.

Definition:1.5

A graph G is said to be a **Finite Graph** if it is a vertex set and edge set are finite

Defintion:1.6

A graph G is called a **Simple Graph** if it has no loops and no two of links join the same pair of vertices.

Defintion:1.7

A simple graph G is saıd to be a **Complete Graph** if every vertex is adjacent to all the other vertices

Defintion:1.8

A Graph G is said to be **Bipartite Graph** if V(G) is partitioned into two sets x and y such that every edge of G has one end in x and another end in Y. The pair (X,Y) is called a Bipartition of V.

Defintion:1.9

If (X,Y) is a bipartition of a graph G such that every vertex in x is adjacent to every vertex in Y. Then the graph G is called a **Complete Bipartite Graph**

Definition:1.10

Two graphs G and H are said to be **Isomorphic** if there are two bijection $\theta:V(G)\rightarrow V(H)$ and $\emptyset:E(G)\rightarrow E(H)$ such that $\emptyset(I_G(e)= uv \leftrightarrow I_H(e))=\theta(u)\ \theta(v)\ \forall\ e \in E(G)$

Definition:1.11

Let $G=(V,E,I_G)$ be a graph. A graph $H=(V',E',I_H)$ is a **Subgraph** of G if $V'\subseteq V$, $E'\subseteq E$ and $I_G\subseteq I_H$.

A Subgraph H of G is a **Proper Subgraph** if $V(H) \subseteq V(G)$.

If H is a subgraph of G,then G is a **Supergraph** of H.

Definition:1.12

The **Degree (or) Valency** of a vertex in a graph G is a number of edges of incident with v, counting each loop twice.

Definition:1.13

A graph G is **Regular** if degree of each vertex is the same.

A graph G is **K-Regular Graph** if the degree of each vertex is k.

Definition:1.14

A finite sequence in which vertices and edges alternatively and which begins and ends with vertices is called a **Walk**.

A walk in which edges are not repeated is called a **Trail.** A walk in which vertices are not repeated is called a **Path**.

Definition:1.15

Two vertices u,v in a graph G are said to be **Connected** if there is (u,v) path in G.

A Graph G is **Connected Graph** if any two vertices are connected.

Definition:1.16

A walk is **Closed Walk** if it has positive length and its origin and terminus are same.

A path is called the **Closed Path** if its origin and terminus are the same.

Definition:1.17

A non-trivial closed of a graph G is called the **Cycle** of G (or) edges are not repeated.

A closed trivial in which the origin and the internal vertices are distinct is called a **Cycle**.

A cycle of length k is called a **K-Cycle**

Definition:1.18

A graph is **Acycle** if it has no cycle.

A connected acycle graph is called **Tree**.

A forest is an acycle graph (or) a collection of tree is called a **Forest**

Definition:1.19

A tree is said to be a **Spanning Tree** of a connected graph G if it is a subgraph of G and it contains all vertices of G.

Definition:1.20

A **Tour** of G is a closed walk that traverses each edge exactly once.

A trail in a graph containing all the edges of G is called an **Euler Trail** of G.

A graph G is said to be **Traversable** if it contains euler trail.

Definition:1.21

Let G=(V,E) be a graph. A set D of vertices in a graph G is called **Dominating Set** of G if every vertex in V-D is adjacent to some vertex in D. The minimum cardinality of the dominating sets of G is called the **Domination number** $\gamma(G)$ of G

Example:1.22

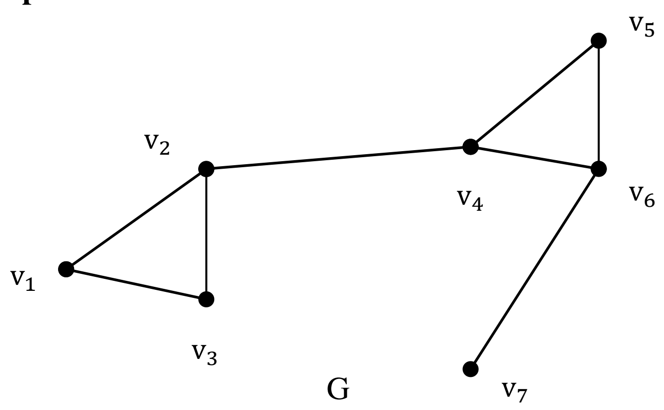


Fig1.1

$\{V_2, V_6\}$ is a dominating set

Example :1.23

If K_2 is a complete graph, then $\gamma(K_2)=1$

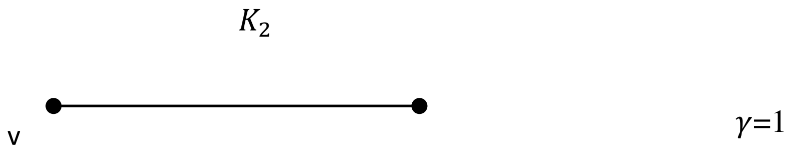
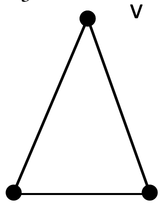


Fig 1.2

{v}is a dominating set of the complete graph K_2.

If K_3 is a complete graph ,then $\gamma(K_3)=1$

K$_3$



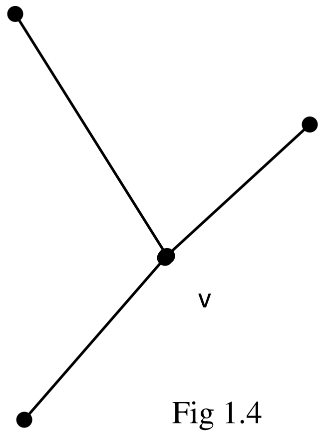
γ=1

Fig 1.3

In general $\gamma(K_n)=1$

Example:1.24

If S_3 is a star graph, then $\gamma(S_3)=1$ $\quad S_3$



γ=1

Fig 1.4

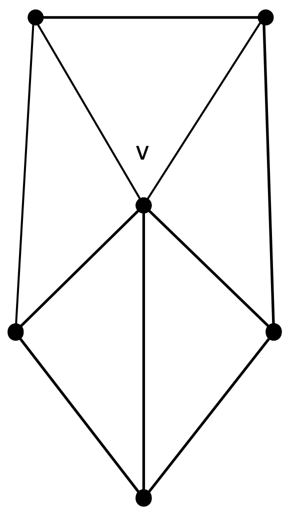
{v} is the minimal dominating set of S_3

$\gamma(S_n)=1, n\geq1$

Example:1.25

If W_n be the wheel graph, then $\gamma(W_n)=1$

W$_5$



γ=1

Fig1.5

{v} is the minimal dominating set of $W_{5,}$ $\gamma(W_5)=1$,

Definition:1.26

A problem is called **NP** (nondeterministic polynomial) if its solution can be guessed and verified in polynomial time. Nondeterministic means that no particular rule is followed to make the guess.

Definition:1.27

NP-complete problem is any of a class of computational problems for which no efficient solution algorithm has been found. Many significant computer science problems belong to this class. Eg. The traveling salesman problems , satisfiability problems and graph covering problems.

Defintion:1.28

A **Pendent Vertex** can also be found to be described as end vertex.In the context of trees, a pendent vertex is usually known as a terminal node ,a leaf node or just leaf .A leaf vertex (pendent vertex) is a vertex with degree one

CHAPTER 2

TOTAL DOMINATION OF GRAPHS

Total domination of graphs consists of characterizations whose total dominating graphs are complete bipartite and Eulerian . In this chapter some properties of total dominating graph are also obtained .

Definition:2.1

A set D of vertices in G is a **Total Dominating Set** of G if every vertex of G is adjacent to some vertex in D.

Definition:2.2

The **Total Domination Number** γ_t (G) of G is the minimum cardinality of a total dominating set in G

Definition:2.3.

A total dominating set D is said to be a Minimal total dominating set if for any vertex v$\in D, D - \{v\}$ is not a total dominating set of G.A total dominating set with minimum cardinality is called **Minimum Total dominating set.**

Example :2.4

The Graph G_1

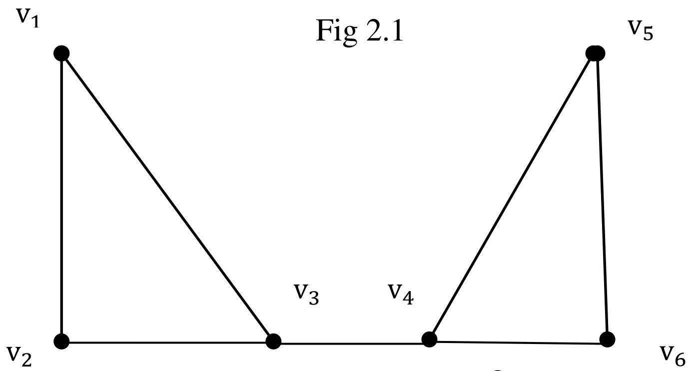


Fig 2.1

The Graph G_2

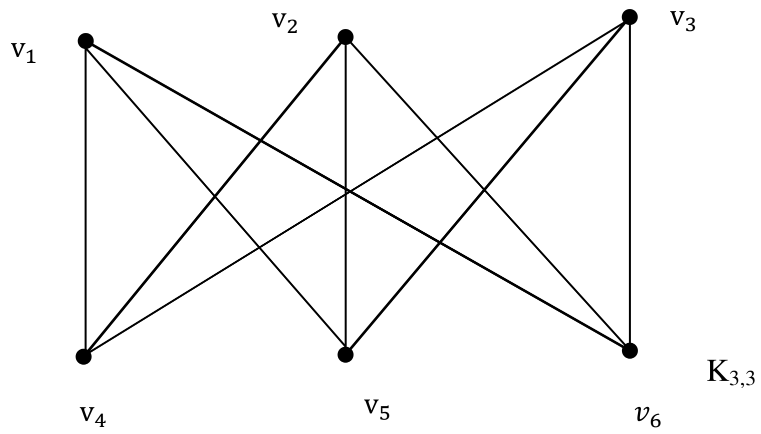


Fig 2.2

1.Total domination number of G_1 is 2,since $\{v_3,v_4\}$ is a minimum total dominating set.

2.Total domination number of G_2 is 2,since $\{v_1,v_4\}$ is a minimum total dominating set

Theorem:2.5

If S is a minimal dominating set of a graph G without isolated vertices,

then **V(G)-S** is a dominating set of G.

Proof:

Let $v \in S$.

Suppose that there exists a vertex w in V(G)-S such that $\Gamma(w) \cap S = \{v\}$.

Hence v is adjacent to some vertex in V(G)-S.

Suppose next that v is adjacent to no vertex in S.

Then v is an isolated vertex of the subgraph <S>.

Since v is not isolated in G the vertex V is adjacent to some vertex of V(G)-S.

Thus V(G)-S is a dominating set of G.

Note:2.6

We note that any graph G without isolated vertices has a total dominating set.

Thus we consider only graphs without isolated vertices.

Defintion:2.7

The **Total Minimal Dominating Graph** M_t(G) of a graph G is the intersection graph defined on the family of all minimal total dominating sets of vertices of G .

Defintion:2.8

The **Common Minimal Total Dominating Graph** CD_t(G) of a graph G is the graph with same vertex set as G with two vertices in CD_t(G) adjacent if there exists a minimal total dominating set in G containing them.

Theorem:2.9

A nontrivial graph is bipartite if and only if all its cycles are even.

Definition:2.10

Let G=(V,E) be a graph. Let S be the set of all minimal total dominating sets of G.The **Total dominating Graph** D_t(G) of G is the graph with the vertex set V∪S in which two vertices u and v are adjacent if u∈V and v is a minimal total dominating set of G containing u.

Example:2.11

A graph G and its total dominating graph D_t(G) are shown

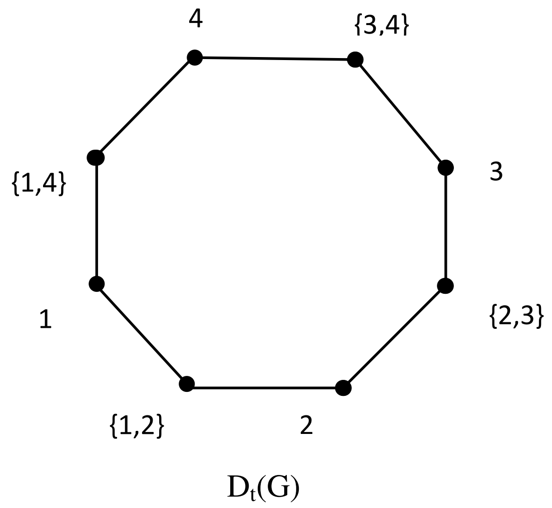


Fig 2.3

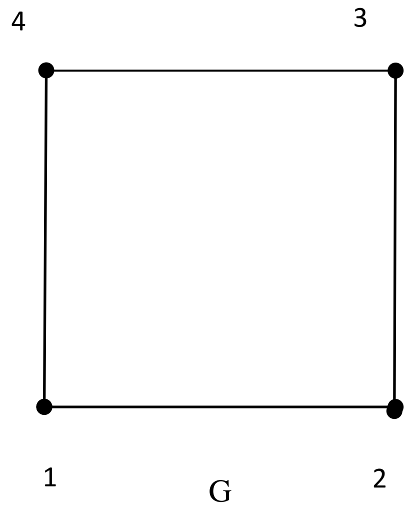


Fig 2.4

Proposition:2.12

If G has a vertex which does not lie in any minimal total dominating set, then D_t(G) is disconnected.

Proof:

Let u be a vertex of a graph G.

If u does not lie in any minimal total dominating set,

then u is an isolated vertex in D_t(G).

Hence D_t(G) is disconnected.

Theorem:2.13

If G is a graph without isolated vertices, then D_t(G) is bipartite.

Proof:

By definition, no two vertices corresponding to vertices of G in D_t(G) are adjacent .

Also no two vertices corresponding to minimal total dominationg sets of G in D_t(G) are adjacent.

Then D_t(G) has no odd cycles.

D_t(G) is bipartite.

Theorem:2.14

The total dominating graph D_t(G) of G is complete bipartite if and only if G=mK_2,m≥1.

Proof:

Suppose D_t(G) is complete bipartite.

Clearly V(D_t(G))=$V_1 \cup V_2$,where V_1 is all vertices of G and V_2 is the set of all minimal total dominating sets of of G.

We now prove that G=mK_2 ,m≥ 1.

On the contrary, assume G≠mK_2.

Then there exists a component G_1 in G which is not K_2.

Let v be a vertex of G_1.

Then v∈G.

We consider the following two cases:

Case1:

Suppose v ∉ D,where D is an minimal total dominating set in G.

Then the corresponding vertex of v is an isolated vertex in D_t(G).

It implies that the corresponding vertices of D and v are not adjacent in D_t(G).

Thus D_t(G) is not complete bipartite, which is a contradiction.

Case:2

Suppose there exist two minimal total dominating sets D_1 and D_2 such that v$\in D_1$ and v $\notin D_2$.

Thus the corresponding vertices of v and D_2 are not adjacent in D_t(G).

Hence D_t(G) is not complete bipartite,which is a contradiction.

From Case 1 and Case2, we conclude that every component of G is K_2.

Thus G=mK_2,m≥1.

Conversely, suppose G=mK_2,m≥1.

Then there exists exactly one minimal total dominating set containing all vertices of G.

Then I V(D_t(G)) I=2m+1.

Thus by definition D_t(G))=$K_{1,2m}$.

Hence D_t(G) is complete bipartite. □

Theorem:2.15

Consider the total dominating graph D_t(G))=$K_{1,2m}$ if and only if G=mK_2,m≥1.

Proof:

Suppose D_t(G))=$K_{1,2m}$,m≥1.

Then D_t(G)) is complete bipartite.

Then, G=mK_2,m≥1.

Conversely, suppose G=mK_2,m≥1.

Then there exists exactly one minimal total dominating set containing all vertices of G.

Thus by definition, D_t(G))=$K_{1,2m}$.

The double star $S_{m,n}$ is the graph obtained from joining the centers of two stars

The centers of $K_{1,m}$ and $K_{1,n}$ are called central vertices of $S_{m,n}$.

Thus $S_{m,n}$ has m+n+2 vertices.

Theorem: 2.16

If $S_{m,n}$is a double star , 1≤ $m \leq n$, then D_t($S_{m,n}$)=(m+n) $K_1 \cup K_2$

Proof:

Let $S_{m,n}$ be a double star, 1≤ $m \leq n$ and u and v be central vertices of $S_{m,n}$.

Then $S_{m,n}$has exactly one minimal total dominating set D containing the central vertices u and v of $S_{m,n}$.

Then D={u,v}.

Thus the vertex set of $D_t(S_{m,n})$ in V∪ D,where V is a vertex set of $S_{m,n}$ and hence $D_t(S_{m,n})$ has m+n+2+1 vertices.

The corresponding vertices of D and u are adjacent

Also the corresponding vertices of D and v are adjacent in $D_t(S_{m,n})$ and all other vertices of $D_t(S_{m,n})$ are isolated vertices.

Thus $D_t(S_{m,n})$ is disconnected and $D_t(S_{m,n})$=(m+n) $K_1 \cup K_2$.

Theorem:2.17

Let G be a nontrivial connected graph.

Let S(G) be the subdivision graph of G .

The graphs D_t(G)) and S(G) are isomorphic if and only if every pair of vertices forms a minimal total dominating set of G.

Proof:

Let G be a nontrivial connected graph.

Suppose D_t(G)) = S(G)

Since G is connected, S(G) is connected.

For each edge $e_i = u_i v_i$ of G, w_i is a new vertex such that $u_i w_i$ and $w_i v_i$

are edges of S(G).

Since D_t(G)= S(G),it implies that every pair of vertices $u_i v_i$ forms a minimal total dominating set of G.

Conversely, suppose every pair of vertices of G forms a minimal total dominating set of G.

Then they are adjacent in G.

Clearly for each minimal total dominating set D of G, the corresponding vertex of D in D_t(G) is adjacent with exactly two vertices .

Hence D_t(G) = S(G).

Corollary:2.18

If $G = K_p, p \geq 2\ or\ K_{m,n}, 1 \leq m \leq n, then\ D_t(G) = S(G)$

Theorem:2.19

A connected graph G is Eulerian if and only if every vertex of G has even degree.

Note:2.20

We characterize total dominating graphs which are Eulerian.

Theorem:2.21

Let G be a nontrivial connected graph.

The total dominating graph D_t(G) of G is Eulerian if and only if the following conditions hold:

i) every minimal total dominating set contains even number of vertices

ii)every vertex of G is in even number of minimal total dominating sets of G.

Proof:

Suppose D_t(G) is Eulerian.

On the contrary ,if condition (i) is not satisfied, then there exists a minimal total dominating set containing odd number of vertices and hence D_t(G) has a vertex of odd degree.

D_t(G) is not eulerian, a contradiction.

Similarly we can prove (ii).

Conversely, suppose the given conditions are satisfied.

Then the degree of each vertex in D_t(G) is even.

Hence , D_t(G) is eulerian.

Theorem:2.22

Let Γ_t(G)=2.If every vertex is in exactly two minimal total dominating sets of G,

Then D_t(G) is Hamiltonian.

Proof:

Clearly γ_t(G)= Γ_t(G) and D_t(G) is connected.

Let v$\in V$ and D be a minimal total dominating set of G.

Then$deg_{D(G)}v = deg_{D(G)}$D=2.

Hence $D\ _t$(G) is connected 2-regular.

Thus D_t(G) is Hamiltonian

Theorem:2.23

Let G be a graph.Then γ_t(G-{e})≥ γ_t(G).

Proof

Let G be a graph and edge e=uv be an edge of graph G.

Let S be $\gamma_t$the set of graph G.

Now consider the graph G-{e}.

Suppose T is $\gamma_t$set of graph G-{e}.

Now T is total dominating set in graph G also.

So, γ_t(G)≤ $|T|\gamma_t$(G-{e})means γ_t(G-{e})≥ γ_t(G).

Corollary:2.24

(i)If G has n vertices and no isolates ,then γ_t(G) ≤ n- $\Delta(G)$+1

(ii)If G is connected and γ_t(G) ≤ n - $\Delta(G)$

Theorem:2.25

Let G be a graph with no isolated vertices. Then $\gamma \leq \frac{n}{2}$.

Proof :

Let D⊆V(G) be a γ set.

Since G has no isolated vertices, every v∈D has at least one neighbor in V-D.

This means that V-D is also a dominating set.

If $|D| > \frac{n}{2}$,then V-D is a smaller dominating set,contradicting the choice of D as a γ set.

Theorem:2.26

Let G be a graph of order n with no isolated vertices. Then $\gamma_t \geq \frac{n}{\Delta}$

Proof:

Let S be a $\gamma_t$set of G.

Every vertex of G is adjacent to some vertex of S.

That is, N(S)=V(G).

Since every v∈S can have at most Δ neighbours,it follows that

$\Delta\gamma_t \geq |V|$=n.

Theorem;2.27

Prove that $\gamma(G) \leq\ \gamma_t$(G)≤ 2 $\gamma(G)$

Proof :

Since every total dominating set is a dominating set the first inequality holds.

To prove the second inequality, let D be a dominating set with γ elements say $v_1,v_2,v_3,\ldots,v_\gamma$.

For each v_i∈D,choose one vertex u_i∈V-D such that v_iu_i∈E.

This is possible since G has no isolated vertices

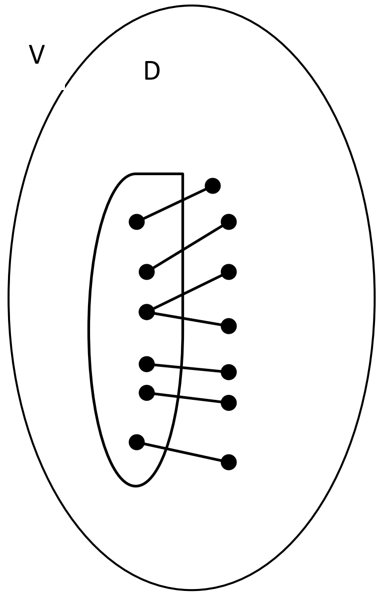


Fig 2.5

The graph illustrating the above theorem

Now the set {v_1,v_2,v_3,…,v_γ,u_1,u_2,…,u_γ} is a total dominating set of G.

Hence $\gamma_t \leq 2\,\gamma$.

Hence $\gamma(G) \leq \gamma_t(G) \leq 2\,\gamma(G)$

Theorem:2.28

For any graph without isolated vertices, prove that γ+i=p

Proof:

Let D be an minimum independent dominating sets with i vertices, V-D contains the remaining p-i vertices, which is a dominating set of G,

Since G has no isolated vertices.

So γ ≤p-i. Hence that γ+i=p

CHAPTER 3

FACTOR DOMINATION

Factor Domination of graphs is concerned with a domination concept defined for a graph and specific factoring of that graph.

Definition: 3.1

A graph $H = (V, E)$ has a **t-factoring** into factors $G_1, G_2, \dots, G_t$ if each

graph $G_i = (V_i, E_i)$ has node set V_i =V and the collection $\{E_1, E_2, \dots, E_t\}$forms a partition of E.

Definition:3.2

Let $H = (V, E)$ be a graph having the t-factoring $G_1, G_2, \dots, G_t$

Then(i) $D_f \subseteq V$ is a **factor dominating set** for each G_i, 1$\leq i \leq t$ and

(ii)the **factor domination number** $\gamma_f(G_1, G_2, \dots, G_t)$ is the size of a smallest factor dominating set.

Observation: 3.3

Let H be a graph with factors $G_1, G_2, \dots, G_t$.

Then $\max_{1\leq i\leq t}\{\gamma_i\} \leq \gamma_f \leq \sum_{i=1}^{t} \gamma_i$

Observation:3.4

The decision problem of determining whether a graph and associated factoring have a factor domination set of size k or less is NP-complete.

Proof:

Restrict t=1.The problem reduces to Dominating set.

Note:3.5

General Factorings

Throughout this section we assume that the graph H is factored into $G_1, G_{2,\dots,} G_t$, and that the node set is $\{v_1, v_2, \dots, v_p\}$.

The next observation shows that the problem of finding γ_f can be reduced to determining the ordinary domination number of a specially constructed graph.

Observation:3.6

There is a graph H' ,constructible from H and its factors , such that $\gamma(H') = \gamma_f$

Proof :

Construct H' on p(t+1)nodes from disjoint copies of H, $G_1, G_2, \dots, G_t$.

Additional edges connect v_j of H to $v_j\ of\ G_i$ and to all nodes of G_i which are adjacent to v_j,1≤j≤ $p, 1 \leq i \leq t.$

It follows that some minimum dominating set of H' is contained in the copy of H and it is clear that the same nodes form a factor dominating set of H and its factorization ;

Thus $\gamma_f \leq \gamma(H')$.

The reverse inequality is obtained by observing that the nodes of any factor dominating set, when interpreted as nodes of H in H',form a dominating set of H'

Note:3.7

We now concentrate on finding bounds for γ_f

Theorem:3.8

Let $i_1, i_2,\ldots, i_t$be any permutation of 1,2,…,t and k be any integer such that

1$\leq k \leq t-1$.

Then $\max\{\gamma_f\big(G_{i_1}, G_{i_2}, \ldots, G_{i_k}\big), \gamma_f(G_{i_{k+1}}, G_{i_{k+2}}, \ldots, G_{i_t})\} \leq \gamma_f(G_1, G_2, \ldots, G_t) \leq$

$\gamma_f\big(G_{i_1}, G_{i_2}, \ldots, G_{i_k}\big) + \gamma_f(G_{i_{k+1}}, G_{i_{k+2}}, \ldots, G_{i_t})\}$

Note:3.9

The next two theorems present upper bounds for γ_f.

Here δ_i is the minimum degree of G_i and α_0 is the node covering number of H.

Theorem:3.10

$\gamma_f \leq$ p - min $_{1 \leq i \leq t}$ $\{\delta_i\}$.

Proof:

Let D_f be any set of $p - min_{1\leq i\leq t}$ $\{\delta_i\}$ nodes.

Clearly D_f is a factor dominating set.

Theorem: 3.11

Let I be the set of nodes in H which are isolated in at least one G_i.

Then $\gamma_f \leq \alpha_0$+$|I|$.

Proof:

Let X be a minimum node cover of H.

Suppose v is a non-isolated node of G_i with incident edge e.

Then e is also an edge of H so X dominates v in G_i.

It follows that X $\cup$ I is a factor dominating set.

Theorem: 3.12

$$If\ t \le \Delta, then\ \gamma_f \ge t, else\ \gamma_f = p.$$

Proof:

Let D_f be a minimum factor dominating set.

If $t \le \Delta$, any node v in H-D_f must have in H at least t edges to D_f so it can be dominated in each G_i.

Hence $|D_f| \ge t$.

If t $>$ Δ,no such node v can exist and H-D_f must be empty .

i.e., γ_f =p.

Theorem:3.13.

If t$\le$ Δ, then $\gamma_f \ge t + \gamma - 2$, else γ_f =p

Proof:

When $t > \Delta$ then shows γ_f =p.

Thus assume t$\le$ Δ and let D_f be a minimum factor dominating set.

If D_f =V(H),then γ_t =p≥ γ+Δ≥ γ+t> γ+t-2 and the result holds.

If D_f ≠V(H),let v∈ H-D_f.

v is adjacent to at least t nodes of D_f.

Let X⊆ D_f be a set of t-1 nodes contained in the neighborhood of v in H.

In H every node of H-(D_f ∪{v}) is adjacent to atleast one node of D_f-X.

Thus $(D_f - X)$ ∪{v} dominates H so $\gamma \leq |$ D_f-X| +1=γ_f-(t-1)+1

Theorem :3.14.

γ_f ≥pt/(Δ+t)

Proof:

If γ_f=p,the result holds.

Thus we may assume t≤ Δ and $\gamma_f < p$.

D_f be a minimum factor dominating set.

Each node of H-$D_f \neq \emptyset$ has at least t edges to D_f for a total of at least (p-γ_f)t such edges.

Thus (p-γ_f)t≤ $\sum_{v\in D_f} \deg(v) \leq \Delta\gamma_f$.

Solving for $\gamma_f$yields the result.

Definition:3.15

The **Invariant** $\boldsymbol{\varepsilon_f}$ is the cardinality of the largest set of nodes X such that in each G_i,$1 \le i \le t$, there is a spanning forest in which X is independent and each node of X has degree one.

Theorem: 3.16

$$\gamma_f + \varepsilon_f = \text{p}.$$

Proof:

If γ_f=p, then ε_f=0 and the result holds.

Thus we may assume t$\le \Delta$ and γ_f <p.

Let D_f be a minimum factor dominating set.

In each G_i arbitrarily select for each node of G_i-D_f one edge between it and D_f.

The subgraph of G_i thus formed is a union of stars centered on the nodes of D_f and is a spanning forest of G_i.

In each of these spanning forests the nodes of $G_i - D_f$ are independent and have degree one it follows that $\varepsilon_f \ge p - \gamma_f$.

Now suppose X is a set of ε_f nodes satisfying the requirements of invariant.

Then the nodes of H-X form a factor dominating set and γ_f ≤p- ε_f.

Note:3.17

In this section we restrict attention to 2-factors of the complete graph K_p.

Thus $G_1 \equiv G$ and $G_2 \equiv \bar{G}$.To simplify notation we employ $\gamma \equiv \gamma(G)$ and $\bar{\gamma} \equiv \gamma(\bar{G})$.

The same convention will apply to other graphical invariants of G and $\bar{G}$

We begin by noting that observation 3.3 may be rephrased for this special case as $\max\{\gamma, \bar{\gamma}\} \leq \gamma_f \leq \gamma + \bar{\gamma}$.

Equility with $\max\{\gamma, \bar{\gamma}\}$ is achieved by several classes of graphs and we list a few with easily computed values.

Here K_p is the complete graph, C_p is the cycle ,W_p is the wheel ,and G^* is the complete r-partite graph $K_{n_1,n_2,\dots,n_r}$,

(i) $\gamma_f(K_p, \bar{K}_p)$=p

(ii) $\gamma_f(C_p,\bar{C}_p)=3 \begin{cases} 3 & \text{if } p = 3,5 \\ [P/3] & \text{otherwise} \end{cases}$

(iii) $\gamma_f(W_p,\bar{W}_p)=\begin{cases} 4 \text{ if } p = 4 \\ 3 \text{ if otherwise} \end{cases}$

(iv) $\gamma_f(G^*,\bar{G}^*)$=r

A more general class of graphs for which $\gamma_f = \max\{\gamma, \bar{\gamma}\}$ is given in the following.

Theorem : 3.18

If either G or $\bar{G}$ is disconnected , $\gamma_f = \max\{\gamma, \bar{\gamma}\}$.

Proof:

Assume G is disconnected.

Any dominating set of G must contain at least one node from each of its components and such a set clearly dominates $\bar{G}$.

Note:3.19

We will see other conditions which guarantee $\gamma_f = \max\{\gamma, \bar{\gamma}\}$,but we first give a result which shows there are graphs for which γ_f may have any of the values between max{ $\gamma, \bar{\gamma}$} and $\gamma + \bar{\gamma}$.

Theorem: 3.20

For any integers m,n and k such that 2≤m≤ $n \leq k \leq m+n$,there exist graphs G for which $\gamma = m, \bar{\gamma}$=n and γ_f=k.

Theorem: 3.21

If G and $\bar{G}$ are connected, then

(i)γ_f=max{ $\gamma, \bar{\gamma}$} for d+$\bar{d}$≥ 7,

(ii) γ_f ≤max{3, $\gamma, \bar{\gamma}$} for d+ $\bar{d}$=6,

(iii)γ_f ≤max{ $\gamma, \bar{\gamma}$}+2 for d+ $\bar{d}$=5,

(iv)γ_f ≤min {$\delta, \bar{\delta}$}+1 for d=$\bar{d}$=2.

Proof:

Without loss of generality assume d≥ $\bar{d} \geq 2$ and that nodes x and y have distance d in G.

Let X denote the set of nodes containing x and its neighbors in G and similarly define Y for y.

CHAPTER 4

PERFECT DOMINATION

Perfect domination is closely related to perfect codes and perfect codes have been used in Coding Theory.In this chapter we study the effect of removing a vertex from the graph on perfect domination.

Definition: 4.1

A Subset S of V(G) is said to be a **Perfect Dominating Set** if for each vertex v not in S, v is adjacent to exactly one vertex of S.

Example:4.2

Consider the path P_4 with four vertices v_1,v_2,v_3,v_4. The set S={ $v_2,v_3,$} is perfect dominating set in this graph. It may be noted that if G is a graph then V(G) is always a perfect dominating set of G.

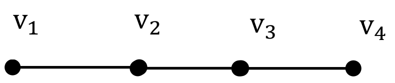


Fig 4.1

Definition: 4.3

A perfect dominating set S of the graph G is said to be **Minimal Perfect Dominating Set** if for each vertex v in S,S-{v} is not a perfect dominating set. It may be noted that it is not necessary that a proper subset of minimal perfect dominating set is not a perfect dominating set.

Example:

Consider the cycle graph $G=C_6$ with six vertices 1,2,3,4,5,6.Then obviously V(G) is a minimal perfect dominating set of G. However the set {1,4} is proper subset of V(G) and is a perfect dominating set in the graph G.

Definition: 4.5

A perfect dominating set with smallest cardinality is called **Minimum Perfect Dominating Set**. It is called γ_{pf} set of the graph G.

Definition :4.6

The cardinality of a minimum perfect dominating set is called the **Perfect Domination Number** of the graph G. It is denoted as $\gamma_{pf}(G)$.The perfect domination number of cycle C_6 is 2 and that of the path $P_3$is also 1.

Defintion: 4.7

Let S be a subset of V(G) and v$\in S$.

Then the **Perfect Private Neighbourhood** of v with respect to

$S = Ppf\{v,S\} = \{w \in V(G) - S: N(w) \cap S = \{v\}\}$ ∪{v,is adjacent to no vertex of S or at least vertices of S}

Theorem:4.8

A perfect dominating set S of G is minimal perfect dominating set if and only if for each vertex v is S. P_{pf}[v,s] is non- empty.

Proof:

Suppose S is minimal and v$\in S$.

Therefore there is a vertex w not in S-{v} such that either w is adjacent to no vertex of S-{v} or w is adjacent to at least two vertices of S-{v}.

If $w = v$,then this implies that v∈ P_{pf}[V,S].

If $w \neq v$,then it is impossible that w is adjacent to at least two vertices of S-{v} because S is a perfect dominating set.

Therefore w is not adjacent to any vertex of S-{v}.

Since S is a perfect dominating set w is adjacent to only v in S.

That is N(w)∩S={v}.

Thus w∈ P_{pf}[V,S].

Conversely suppose v∈S and P_{pf}[V,S] contain some vertex w of G.

If w=v, then w is either adjacent to atleast two vertices of s-{v} is not a perfect dominating set.

If w≠v,then N(w))∩S={v} implies that w is not adhacent to any vertex of S-{v}.

Thus ,in all cases S-{v} is not a perfect dominating set if v∈S.

Thus S is minimal

Note:4.9

A dominating set on path G=P_5 with five vertices v_1,v_2,v_3,v_4,v_5.

Note that S={v_2,v_5} is minimum and therefore minimal perfect dominating set.

$P_{pf}[v_2,S]=\{v_1,v_2,v_3\}$.

Now we define the following symbols

V^{+}_{pf}= {v∈ V(G): γ_{pf}(G)< γ_{pf}(G-V))}

V^{-}_{pf} = { v∈ V(G): γ_{pf}(G)> γ_{pf}(G-V))}

V^{0}_{pf} = { v∈ V(G): γ_{pf}(G)= γ_{pf}(G-V))}

Lemma :4.10

Let v∈ V(G) and suppose v is a pendent vertex and has a neighbour w of degree atleast two.If v∈ V_{pf} then γ_{pf}(G-V)= γ_{pf}(G)-1

Proof:

Let S_1 be a minimum perfect dominating set of G-{v}.

If w∈ S_1,then S_1 is a perfect dominating set of G with +|S_1|<γ_{pf}(G).

That is γ_{pf}(G)≤ |S_1|<γ_{pf}(G) this is a contradiction.

Therefore w ∉S_1.

Let S=S_1∪{w}.

Then S is a minimum perfect dominating set of G.

Therefore is γ_{pf}(G)= |S| = |S_1| + 1 =+γ_{pf}(G-V)+1.

Theorem :4.11

Let v be a vertex of G.

Then v∈V^{+}_{pf} if and only if the following conditions are satisfies.

(1)v belongs to everyγ_{pf} set ofG

(2)No subset S of G-{v} which is either disjoint from N[v] or intersects N[v] in at least two vertices and $|S| \le \gamma_{pf}(G)$can be a perfectly dominating set of G-{v}.

Proof:

(1)Suppose $v \in V^{+}_{pf}$.

Suppose S is a γ_{pf} set of G which does not contain v then S is a perfect dominating set of G-{v}.

Therefore $\gamma_{pf}(G-v) \le |S| = \gamma_{pf}(G)$.

Thus $v \notin V^{+}_{pf}$.This is a contradiction .

Thus, v must belong to every γ_{pf}set of G.

(2)If there is set S which satisfies the condition stated in (2).

Then S is a perfect dominating set of G-{v} and therefore $\gamma_{pf}(G-v) \le \gamma_{pf}(G)$.

This is a contradiction.

Conversely assume that (1) and (2) hold.

Suppose $v \in V^{0}_{pf}$.

Let S be a minimum perfect dominating set of G-{v}.

Then $|S| = \gamma_{pf}(G)$.

Suppose v is not adjacent to any vertex of S.

Then S is disjoint from N[v], $|S| = \gamma_{pf}(G)$ and S is a perfectly dominating set of G-{v}.This violates (2).

Suppose v is adjacent to exactly one vertex of S then S is a minimum perfect dominating set of G not containing v which violates (1).

Suppose v is adjacent to at least two vertices of S.

Then S∩N[v] in atleast two vertices and S is a perfectly dominating set of G-{v} with

$|S| = \gamma_{pf}(G)$ which again violate (2).

Thus $v \in V^0_{pf}$ implies (1) or (2)violated.

Suppose $v \in V^-_{pf}$.Let S_1 be a minimum perfectly dominating set of G-{v}.Then $|S_1|<\gamma_{pf}(G)$.

If v is not adjacent to any vertex of S_1 then as above (2) is violated.

If v is adjacent to exactly one vertex of S_1 then S_1 is a perfectly dominating set of G with $|S_1|<\gamma_{pf}(G)$- which is a contradiction.

If v is adjacent to at least two vertices of S_1 then $S_1 \cap N[v]$ in at least two vertices, $|S_1| \leq \gamma_{pf}(G)$ and S_1 is a perfect dominating set of G-{v}-which again violates (2).

Thus $v \in V_{pf}$ implies that (2) is violated.

Thus v does not belongs to V^0_{pf} or V_{pf}.

Hence $v \in V^+_{pf}$

Theorem :4.12

Let v be a pendent vertex which has the neighbor w of degree at least two then

$v \in V_{pf}$ if and only if there is γ_{pf}set S containing w and not containing v such that $P_{pf}[w,S]=\{v\}$

Proof:

Suppose$v \in V^{-}_{pf}$.

Let S_1 be a minimum perfect dominating set of G-{v}.

Then ,w does not belong to S_1.

Let $S=S_1 \cup \{w\}$.Then S is γ_{pf}containing w.

Since S_1 is a perfect dominating set of G-{v}, w is adjacent to some vertex of S_1.

Therefore w $\notin P_{pf}[w,s]$.

If x is any vertex different from v such that x is adjacent to w then x is also adjacent to some vertex of S_1 because S_1 is a perfect dominating set of G-{v}.

Thus x $\notin P_{pf}[w,s]$.Further v is adjacent to only w of S therefore $P_{pf}[w,s]=\{v\}$.

Conversely, Suppose there is a γ_{pf}set S containing w such that $P_{pf}[w,s]=\{v\}$.

Let S_1=S-{w}.Let x be any vertex of G-{v} which is not in S-{v}.

Since x $\notin P_{pf}[w,s]$, x must be adjacent to some unique vertex S_1.

Thus, S_1 is a minimum perfect dominating set of G-{v} with , $|S_1| \leq \gamma_{pf}$(G).

Thus,$v \in V^{-}_{pf}$.

Example 4.12:

Consider the path G=P_4 with vertices 1,2,3,4.Then γ_{pf}(G)=2.

Let v=1 and w=2. Now γ_{pf}(G-1)=1.

Thus 1∈ γ_{pf}also S=(2,3) is γ_{pf}set of G, containing w=2 and P_{pf}[2,S]={1}.

Theorem:4.14

Let S_1 and S_2 be two disjoint perfect dominating sets of G. Then $|S_1|=|S_2|$

Proof:

For every vertex x in S_1 there is a unique vertex v(x) in S_2 which is adjacent to x.

Also for every vertex y in S_2 there is a unique vertex u(y) in S_1 which is adjacent to y.

It may be noted that these functions are inverses of each other.

Therefore $|S_1|=|S_2|$.

Corollary:4.15

If in a graph G there are perfect dominating sets S_1 and S_2 such that $|S_1| \neq |S_2|$.$S_1 \cap S_2 \neq \varphi$

Corollary:4.16

Let G be a graph with n vertices.

If there is a perfect dominating set S with $|S_1|$<n/2 or ≥n/2.

Then V(G)-S is not a perfect dominating set.

CHAPTER 5

EQUITABLE DOMINATION

In Equitable Dominatioon, the characterization of the equitable dominating graphs which are either connected or complete are obtained.

Definition:5.1

Let G=(V, E) be a graph. A subset D of V is said to be a **Equitable Dominating Set** of if for every v ∈V -D there exists a vertex u ∈D such that uv∈E(G) and

|d (u)-d (v)| ≤1.

Definition:5.2

The minimum cardinality of such a dominating setD is called the **Equitable Domination Number** of G and is denoted by γ^e(G) .

Definition:5.3

An equitable dominating set is said to be **Minimal Equitable Dominating Set** if no proper subset of D is an equitable dominating set.

Definition 5.4:

A vertex u ∈V is said to be **Degree Equitable** with a vertex v ∈V if and

|d (u)-d (v)| ≤1.

Definition:5.5

A vertex u ∈V is said to be an **Equitable Isolate** if|d (u)-d (v)| ≥ 2 ∀v ∈V

Definition: 5.6

A minimal equitable dominating set of maximum cardinality is called $\mathbf{\Gamma^e}$ **Set** and its cardinality is denoted by Γ^e(G) .

Definition :5.7

Let u ∈V . The **Equitable Neighborhood** of u denoted by N^e(u) is defined as by N^e(u)={v ∈V /v ∈N(u), d (u)-d (v)| ≤ 1}.

Definition: 5.8

A subset S of V is called an **Equitable Independent Set**, if for any u ∈ S, v ∉ N^e(u) for all v∈ $S - \{u\}$.

Definition:5.9

The maximum cardinality of S is called **Equitable Independence Number** of Gand is denoted β^e(G) .

Definition: 5.10

The maximum order of a partition of V into equitable dominating sets is called **Equitable Domatic Number** of G and is denoted by $d^e(G)$

Definition :5.11

The **Equitable Dominating Graph ED (G)** of a graph G is a graph with V(ED(G))=V(G)∪D(G) where D(G) is the set of all minimal equitable dominating sets of G and u,v ∈ $V(ED(G))$are adjacent to each other if u ∈V (G) and v is a minimal equitable dominating set of G containing u .

Example :5.12

An example of the equitable dominating graph ED(G)of a graph G is given below:

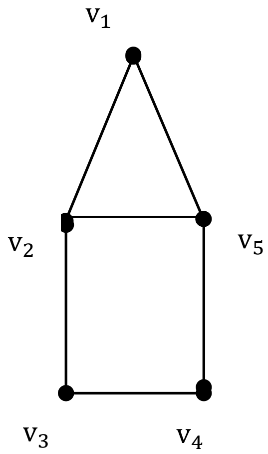


Fig 5.1

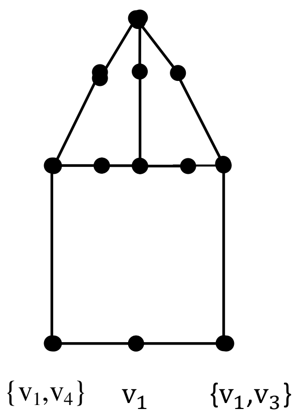


Fig 5.2

Theorem :5.13

Let G be a graph without equitable isolated vertices. If D is a minimal equitable dominating set, then V-D is an equitable dominating set.

Theorem :5.14

A graph is Eulerian if and only every of vertex of G is of even degree.

Theorem :5.15

For any graph G with $p \geq 2$ and without equitable isolated vertices, the equitable dominating graph ED(G) of G is connected if and only if $\Delta(G) < p-1$.

Proof:

Let $\Delta(G) < p-1$. Let D_1 and D_2 be two minimal equitable dominating sets of G

We consider the following cases:-

Case i):

Suppose there exists two vertices $u \in D_1$ and $v \in D_2$ such that u and v are not adjacent to each other.

Then, there exists a maximal equitable independent set D_3 containing u and v.

Since every maximal equitable independent set is a minimal equitable dominating set, D_3 is a minimal equitable dominating set joining D_1 and D_2 .

Hence there is a path in ED (G) joining the vertices of V (G) together with the minimal equitable dominating sets of G.

Thus, ED (G) is connected.

Case ii):

Suppose for any two vertices $u \in D_1$ and $v \in D_2$ there exists a vertex w does not belong to $D_1 \cup D_2$ such that w is adjacent to neither u not v.

Then, there exists two maximal equitable independent sets D_3 and D_4 containing u,w and w,v respectively.

Thus, the vertices u,v,w and the minimal equitable dominating sets D_1, D_2, D_3, D_4 are connected by the path D_1- u –D_3- w-D_4- v –D_2 .

Thus, ED(G) is connected.

Conversely, suppose that ED(G) is connected.

Let us assume that Δ (G)=p-1. and let {u}be a vertex of degree p -1.

Then, {u} is a minimal equitable dominating set of G and V-D has a minimal equitable dominating set say D' .

This implies that ED (G) has at least two components, a contradiction.

Hence, Δ (G)<p-1.

Hence the result.

Remark :5.16

In ED (G) , any two vertices u and v of V (G) are connected by a path of length at most four.

Theorem :5.17

For any graph with Δ (G)<p-1 and without equitable isolated vertices,

diam (ED(G)) $\leq$ 5.

Proof:

As Δ (G)<p-1 , G is connected.

Let ED(G)=V$\cup Y$ where Y is the set of all minimal equitable dominating sets of G.

Let u ,v $\in$V$\cup$ Y.

Then, diam(ED(G))$\leq$ 4 if u, v$\in$ V , or u ,v$\in$ Y , .

On the other hand, if u$\in$ V and v $\in$Y then v =D is a minimal equitable dominating set of G .

If u$\in D$, then d(u,v)$\leq$ 4;Otherwise, there exists a vertex w$\in D$ such that d(u,v)$+d(u,w)+d(w,v) \leq 4+1=5$ proves the result.

Theorem 5.18

For any graph without equitable isolated vertices, ED (G) is a complete bipartite graph if and only if $\overline{K}$ p .

Proof:

Suppose that ED (G) is not a complete bipartite graph with G≈$\overline{K}$p

. As G≈$\overline{K}$p the minimal equitable dominating set of G is V (G) , every isolated vertex in ED (G) is adjacent to the vertex V (G).

This implies that ED (G) is $K_{1,p}$, which is a contradiction.

Thus, ED(G) is complete bipartite graph.

Conversely, suppose that ED(G) is complete bipartite graph and G $\neq \overline{K}_p$.

Thus G contains a nontrivial subgraph G_1 .

Then, for some vertex u $\in G_1$,there exists a minimal equitable dominating sets D and $\overline{D}$ with u ∈ D and u∉ D , which is a contradiction to the fact that G is complete bipartite graph with u $\in G_1$.

Hence G≈ $\overline{K}$p ,

This complete the proof.

Theorem: 5.19

For any graph G without equitable isolated vertices, $d^e(G) \leq \beta^e$(ED(G)).

Further, the equality holds if and only if V (G) can be partitioned into union of disjoint minimal equitable dominating sets of cardinality one.

Proof:

Let S be the maximum order of equitable domatic partition of V (G).

If every equitable dominating set is minimal and S consists of all minimal equitable dominating sets of G , then S is a maximum equitable independent sets of ED(G) .

Hence . $d^e(G) = \beta^e$(ED(G)).

Otherwise, let D be a maximum equitable independent set with D ∉ S .

Hence, D is a minimal equitable dominating set of G.

Let u∈ D . Then, there are two following cases:

Case i):

If u∈D' where D'∈ S .

Then, clearly S ∪ {u} is a equitable independent set in ED(G) .

Hence the result holds.

Case ii): If u ∉ D' , where D'∈ S.

Then, there exists a vertex w ∈V(G) such that S ∪ {u,w} is an equitable independent set.

Hence the result.

Clearly, the equality condition follows as every component of ED (G) is K_2 as V(G) is the union of disjoint minimal equitable dominating sets of cardinality one.

This completes the proof.

Corollary :5.20

For any graph G , $|V(ED(G)| \geq d^e(G)$

Theorem: 5.21

For any graph G without equitable isolated vertices $p+d^e \leq p' \leq p(\beta^e(G)+1)$,where p ' is the number of vertices of ED(G) .

Further the lower bound is attained if and only if every minimal equitable dominating set of G is independent .

The upper bound is attained if and only if every maximum equitable independent set is of cardinality one.

Proof:

The graph ED(G) has the vertex set V (G)∪D (G).

It has at least $d^e(G)$ number of minimal equitable dominating sets, hence the lower bound follows.

Clearly upper bound follows as every maximal equitable independent set is a minimal equitable dominating set .

Every vertex is present in at most(p-1) minimal equitable dominating sets. Further, suppose that $p+ d^e(G) =p'$.

As there are $d^e(G)$ number of minimal equitable dominating sets

Each vertex is present in exactly one of the minimal equitable dominating set .

Hence these minimal equitable dominating sets are independent.

Also, suppose that every maximum equitable independent set is of cardinality one .

These are minimal equitable dominating sets of G and are independent and as every vertex is present in at most (p-1)minimal equitable dominating set, the equality holds.

This implies the necessary condition.

Converse of the result trivially holds.

Theorem :5.22

For any graph G without equitable isolated vertices $\left(\frac{p+d^e(G)}{2}\right) \leq q' \leq p(p-1),$, where q' is the number of edges of ED (G) .

Further, the lower bound is attained if and only if every minimal equitable dominating set is independent and the upper bound is attained if and only if G is

(p-2)- regular.

Proof:

Suppose the lower bound is attained.

As every vertex must be in exactly one of the dominating set, Cleary every minimal equitable dominating set is independent.

As every vertex is in at most (p-1) minimal equitable dominating set, upper bound follows. Suppose the upper bound is attained.

Then, each vertex is in exactly (p-1) minimal equitable dominating sets and hence G is (p-2)- regular.

This completes the proof.

Theorem: 5.23

For any graph G with p ≥3 ,d^e(ED(G)=1 if and only if G =$\overline{K}_p$,where $\overline{K}_p$ is the complement of K_p or ED (G) has an equitable isolated vertex.

Proof:

Suppose that d^e(ED(G)=1.

Then, ED (G) has a vertex D with D=V(G).

Thus ED(G)is $K_{1,p}$ and hence G= $\overline{K}_p$.

Otherwise, suppose assume that ED(G)has no equitable isolated vertex and V(ED(G))=p' . Thenγ^e(ED(G) $\leq \frac{p'}{2}$.

If D is an equitable dominating set, then V-D is an equitable dominating set and hence d^e(ED(G) ≥ 2 , a contradiction.

Hence ED (G) has an equitable isolated vertex.

The converse is obvious.

Theorem :5.24

If a graph G is connected, (p-1)- regular and without equitable isolated vertices then,

γ^e(ED(G)$= p$

Proof:

As G is connected and $\Delta(G)$ = p-1, ED(G) is disconnected.

Also, we know that every vertex is present in at most (p-1) minimal equitable dominating sets. Thus, ED(G) is a disconnected graph with each of the component being K_2 , there are p number of components.

Hence γ^e(ED(G)$= p$.

Theorem :5.25

For any graph G of order p $\geq$2, without equitable isolated vertices and $\Delta(G) <$ p-1, the equitable dominating graph ED (G) of a graph G is a tree if and only if G = $\overline{K}_p$.

Proof:

As G is a graph of order p$\geq$ 2, without equitable isolated vertices and $\Delta(G) <$ p-1 ED(G) is connected.

Suppose assume that ED (G) of G is a tree.

Then, clearly G has no cycle.

On the contrary assume that G$\neq$ $\overline{K}_p$ Then, d^e(ED(G))$\neq 1$.

Hence there exists at least two minimal equitable dominating sets containing where u and v are any two vertices in G .

If u and v are in the same minimal equitable dominating set D then, u-D-v-u is a cycle in ED(G) , a contradiction.

On the other hand, if u and v are in different minimal equitable dominating set.

Then, there exists vertices u_1 , v_1 and the minimal equitable dominating sets D_1, D_2 and D_3 such

that $uu_1 \in D_1, u_1v_1 \in D_2$ and $v_1v \in D_3$.

Thus , u and v are connected by two paths in ED(G) , a contradiction.

Hence $G = \overline{K}_p$

Conversely, suppose that $G = \overline{K}_p$ and $\Delta(G) < p-1$.

Then, ED(G) is connected.

Also, d^e(ED(G))= 1. i.e., there exists a minimal equitable dominating set D with

D =V(G).

Thus, ED (G) is connected, $K_{1,p}$ has no cycle.

Hence ED(G) is a tree.

This completes the proof.

Theorem :5.26

For any graph G , ED(G) is either connected or has at most one component that is not K_2

Proof:

We consider the following cases:-

Case i):

If $\Delta(G) < p-1$, then, ED (G) is connected.

Case ii):

If $\delta(G) = \Delta(G)$ =p-1, then G=K_p .

Hence each of the vertex v$\in V(G)$ is a minimal equitable dominating set of G and hence each of the component of ED (G) is K_2 .

Case iii):

If $\delta(G) < \Delta(G)$ =p-1 .

Let $v_1, v_2, v_3, \ldots v_n$ be n vertices of G of degree p-1.

Let H=G/{ $v_1, v_2, v_3, \ldots v_n$ } then $\Delta(H) < V(H) - 1$.

Hence , ED (H) is connected.

Since ED(G)=$\Omega(V(ED(H) \cup V(G_1) \cup V(G_2) \cup V(G_n))$are the graphs joining v_1, v_2, $v_3, \ldots v_n$ with $\{v_1\}, \{v_2\}, \{v_3\}, \ldots, \{v_n\}$ respectively.

Then, exactly one of the component of ED(G) is not K_2 .

Hence the proof

Theorem: 5.27

If G is a r- regular graph with $\Gamma^e(G) = 2$ and every vertex is in exactly even number of minimal equitable dominating sets then ED (G) is Eulerian.

Proof:

Let G is a r regular graph.

Since each of the vertex of G is in even number of minimal equitable dominating sets, each ofthen contributes even number to the degree of the vertex in ED(G)

And as $\Gamma(G) = 2$ each of the minimal equitable dominating set of G is a vertex of degree two in ED(G) .

Thus, ED (G) is Eulerian.

Theorem: 5.28

Let G be a graph with $\Delta(G) < p - 1$ and $\Gamma^e(G) = 2$.

If every vertex is present in exactly two minimal equitable dominating sets then, E(G) is Hamiltonian.

Proof:

As , $\Delta(G) < p - 1$, G is connected .

Also, since every vertex is present in exactly two minimal equitable dominating sets,

γ^e(G)= $\Gamma^e(G)$and also deg(u) = deg(D)=2 in ED (G) , where D is a minimal equitable dominating set in G .

Thus, ED (G) is connected and 2-regular.

Hence ED (G) is Hamiltonian.

A STUDY ON CUBIC WEAK BI-IDEALS OF NEAR-RINGS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

S. CHRISTY LIZY

Reg. No: 19SPMT06

Under the guidance of

Dr. R. MARIA IRUDHAYA ASPIN CHITRA M.Sc., M.Phil., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON CUBIC WEAK BI-IDEALS OF NEAR-RINGS"** is submitted to **St. Mary's college (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work is done during the year **2020-2021** by

S. CHRISTY LIZY (Reg. No: 19SPMT06)

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON CUBIC WEAK BI-IDEALS OF NEAR-RINGS "** submitted for the degree of Master of Science is my work carried out under the guidance of **Dr. R. Maria Irudhaya Aspin Chitra M.Sc., M.Phil., Ph.D.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

S. Christy Lizy.

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER-I

PRELIMINARIES

In this chapter I give all the basic definitions and results which are used in the subsequent chapters.

Definition : 1.1

By a near-ring, we shall mean an algebraic system $(N, +, .)$, where

a) $(N, +)$ forms a group.

b) $(N, .)$ forms a semi-group.

c) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in N$

((i.e) right and left distributive laws hold)

Definition : 1.2

For a near-ring, the zero-symmetric part of N denoted by $N_0$is defined by $N_0 = \{n \in N | n0 = 0\}$ and the constant part of N denoted by $N = \{n \in N | n0 = n\}$. It is well known that N_0 and N_c are subnear-rings of N.

Definition : 1.3

A subgroup M of N with $MM \subseteq M$ is called a sub near-ring of N.

Definition : 1.4

A non-empty subset I of a near-ring N is a subnear-ring of R if and only if $x - y \in I$ and $xy \in I$ for all $x, y \in I$.

Definition : 1.5

A normal subgroup I of N is called a right ideal if $IN \subseteq I$ and denoted by $I \lhd_r N$. It is called a left ideal if $n(s+i) - ns \in I$ for all $n, s \in N$ and $i \in I$ and denoted by $I \lhd_l N$. If such a normal subgroup I is both left and right ideal in N, then it is called an ideal in N and denoted by $I \lhd N$.

Definition : 1.6

Let U be an initial universal set and E be the set of parameters. Let A be a subset of E. Let $P(U)$ denote the power set of U. A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \to P(U)$.

For each $x \in A, F(x)$ is the set of x- approximate elements of the soft set (F, A).

Definition : 1.7

For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$. It is denoted by $(F, A) \subseteq (G, B)$. (F, A) is said to be soft super set of (G, B), if (G, B) is a soft subset of (F, A).

Definition : 1.8

For a soft set (F, A), the set $Supp(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called the support of the soft set (F, A). The null soft set is a soft set with an empty support, and a soft set (F, A) is non-null if $Supp(F, A) \neq \emptyset$.

Definition : 1.9

Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$. It is denoted by $(F, A) = (G, B)$.

Definition : 1.10

Let N be a soft set relation on (F, A), then

(i) N is reflexive if $F(a) \times F(a) \in N \; \forall \; a \in A$.

(ii) N is symmetric if $F(a) \times F(b) \in N \Longrightarrow F(b) \times F(a) \in N$

$\forall \; (a, b) \in A \times A$.

(iii) N is transitive if $F(a) \times F(b) \in N$ and $F(b) \times F(c) \in N$

$\Rightarrow F(a) \times F(c) \in N \; \forall \; a, b, c \in A$.

(iv) N is equivalence if it is reflexive, symmetric and transitive.

(v) N is an identity if $a \neq b, F(a) \times F(b) \in N$ but

$F(a) \times F(b) \notin N \; \forall \; a, b \in A$.

$(i, e.,)$ $F(a) \times F(b) \in N \Longrightarrow ab \; \forall \; a, b \in A$

Definition : 1.11

Let $(F, A), (G, B)$ and (H, C) be three soft sets over a common universe. Let R be a soft set relation from (F, A) to (G, B) and S be a soft set relation from (G, B) to (H, C). Then, a new soft set relation from (F, A) to (H, C) called the composition of R and S denoted by $S \; o \; R$ is defined as follows :

If $F(a) \in (F, A)$ and $H(c) \in (H, C)$, then

$F(a) \times H(c) \in S \; o \; R \Leftrightarrow F(a) \times G(b) \in R$ and $G(b) \times H(c) \in S$, for some $G(b) \in (G,B)$.

Definition : 1.12

Let (F,A) and (G,B) be two non-empty soft sets over U. Then a soft set relation f from (F,A) to (G,B) written $f:(F,A) \to (G,B)$ is called a soft set function if every element in the domain of f has a unique element in the range of f. If $F(a) f\, G(b)$, i.e., $F(a) \times G(b) \in (b) \in f$, then we write $f\big(F(a)\big) = G(b)$.

Definition : 1.13

A function f from (F,A) to (G,B) is called

(i) Injective (one-to-one) if $F(a) \neq F(b) \Longrightarrow f(F(a)) \neq f(F(b))$

(ii) Surjective (onto) if range $f = (G,B)$

(iii) Bijective (one-to-one and onto) if f is both injective and surjective.

Definition : 1.14

The identity soft set function I on a soft set (F,A) is defined by

$I:(F,A) \to (F,A)$ such that $I\big(F(a)\big) = F(a)\ \forall\ F(a) \in (F,A)$.

Definition : 1.15

If (F,A) and (G,B) are two soft sets over a common universe U then "(F,A) AND (G,B)" is a soft set denoted by $(F,A) \wedge (G,B)$ and is defined by

$(F,A) \wedge (G,B) = (H, A \times B)$ where $H(x,y) = F(x) \cap G(y)$ for all $(x,y) \in A \times B$.

Definition : 1.16

Let (F, A) and (G, B) be two soft sets in a soft class (U, E) with $A \cap B \neq \emptyset$. The interesection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C) where $C = A \cap B$, and $H(c) = F(c) \cap G(c)$ for all $c \in C$. We write $(F, A) \cap (G, B) = (H, C)$.

Definition : 1.17

Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then the restricted intersection of (F, A) and (G, B) is defined as $(F, A) \cap_R (G, B) = (H, C)$ where $C = A \cap B$ and for all $c \in C, H(c) = F(c) \cap G(c)$.

Definition : 1.18

The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is defined as $(F, A) \cap_\varepsilon (G, B) = (H, C)$, where $C = A \cup B$ and

for all $c \in C$.

$$H(c) = \begin{cases} F(c) & if\ c \in A \setminus B \\ G(c) & if\ c \in B \setminus A \\ F(c)\ \cap G(c) & if\ c \in A \cap B \end{cases}$$

Definition : 1.19

The union or extended union of two soft sets (F, A) and (G, B) over a common universe U is defined as $(F, A) \cup_\varepsilon (G, B) = (H, C)$, where $C = A \cup B$ and for all

$c \in C$.

$$H(c) = \begin{cases} F(c) & if\ c \in A \setminus B \\ G(c) & if\ c \in B \setminus A \\ F(c)\ \cup G(c) & if\ c \in A \cap B \end{cases}$$

Definition : 1.20

The restricted union of two soft sets (F, A) and (G, B) over a common universe U is defined as $(F, A) \cup_R (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C, H(c) = F(c) \cup G(c)$.

Definition : 1.21

Let $\{(F_i, A_i) | i \in I\}$ be a non-empty family of soft sets over a common universe U. The extended intersection of these soft sets is defined to be the soft set (G, B) such that $B = \cup_{i \in I} A_i$ for all $x \in B$, $G(x) = \cap_{i \in I(x)} F_i(x)$ for all $x \in B$ where

$I(x) = \{i \in I | x \in A_i\}$. In this case we write $\cap_\varepsilon (F_i, A_i) = (G, B)$.

Definition : 1.22

Let $\{(F_i, A_i) | i \in I\}$ be a non-empty family of soft sets over a common universe U. The restricted intersection of these soft sets is defined to be the soft set (G, B) such that $B = \cap_{i \in I} A_i$ for all $x \in B, G(x) = \cap_{i \in I(x)} F_i(x)$ for all $x \in B$ where

$I(x) = \{i \in I | x \in A_i\}$. In this case we write $\cap_R (F_i, A_i) = (G, B)$.

Definition : 1.23

Let $\{(F_i, A_i) | i \in I\}$ be a non-empty family of soft sets over a common universe U. The extended union of these soft sets is defined to be the soft set (G, B) such that $B = \cup_{i \in I} A_i$ for all $x \in B$, $G(x) = \cup_{i \in I(x)} F_i(x)$ for all $x \in B$ where

$I(x) = \{i \in I | x \in A_i\}$. In this case we write $\cup_\varepsilon (F_i, A_i) = (G, B)$.

Definition : 1.24

Let $\{(F_i, A_i)|i \in I\}$ be a non-empty family of soft sets over a common universe U. The restricted union of these soft sets is defined to be the soft set (G, B) such that $B = \cap_{i\in I} A_i$ for all $x \in B$, $G(x) = \cup_{i\in I(x)} F_i(x)$ for all $x \in B$ where

$I(x) = \{i \in I|x \in A_i\}$. In this case we write $\cup_R (F_i, A_i) = (G, B)$.

Definition : 1.25

Let $\{(F_i, A_i)|i \in I\}$ be a non-empty family of soft sets over a common universe U. The AND soft set $\wedge_{i\in I}(F_i, A_i)$ of these soft sets is defined to be the soft set (H, B) such that $B = \Pi_{i\in I} A_i$ and $H(x) = \cap_{i\in I(x)} F_i(x)$ for all $x = (x_i)_{i\in I} \in B$.

Definition : 1.26

A fuzzy set μ in U and $\mu: U \to [0,1]$. The set of all fuzzy sets of U is denoted by $F(U)$.

Definition : 1.27

Let μ be a fuzzy set in U and $t \in [0,1]$. Then the crisp set

$\mu_t = \{x \in U|\ \mu(x) > t\}$ is called a level subset of μ.

Definition : 1.28

The support of a fuzzy set μ, denoted by $Supp(\mu)$ is defined as

$Supp(\mu) = \{x \in U|\ \mu(x) > 0\}$.

Definition : 1.29

Let $(G, +)$ be a group and μ be a fuzy set in G. Then μ is said to be a fuzzy subgroup if :

a) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$ for all $x, y \in G$.

b) $\mu(-x) \geq \mu(x)$ for all $x \in G$.

Definition : 1.30

Let $(G, +)$ be a group and μ be a fuzzy set in G. Then μ is said to be a normal fuzzy subgroup if :

a) μ is a fuzzy subgroup of G.

b) $\mu(x) = \mu(y + x - y)$ for all $x, y \in G$.

Definition : 1.31

A fuzzy set μ of a near-ing N is said to be a fuzzy subnear –ring of N if for all $x, y \in G$.

a) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in G$

b) $\mu(-x) \geq \mu(x)$ for all $x \in G$

c) $\mu(xy) \geq\ min\{\mu(x), \mu(y)\}$

Definition : 1.32

Let μ be a normal empty fuzzy set in a near-ring N. Then μ is a fuzzy ideal of N if

a) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$ for all $x, y \in N$

b) $\mu(-x) \geq \mu(x)$ for all $x \in N$

c) $\mu(x) = \mu(y + x - y)$ for all $x, y \in N$

d) $\mu(xy \geq \mu(y)$ for all $x, y \in N$

e) $\mu\{(x + i)y - xy\} \geq \mu(i)$ for all $x, y \in N$

If μ satisfies (a), (b), (c) and (d) then it is called a fuzzy left ideal of N and if it satisfies (a), (b), (c) and (e) then it is called a fuzzy right ideal of N.

CHAPTER – II

SOFT NEAR RINGS

In this chapter, I collect the definition of soft near-rings and some important definitions and some theorems.

Definition : 2.1

Let (F, A) be a non-null soft set over a near-ring N. Then (F, A) is called a soft near-ring over N if $F(x)$ is a subnear-ring of N for all $x \in Supp(F, A)$.

Example : 2.2

Consider the additive group $(Z_6, +)$. Under a multiplication defined by following table, $(Z_6, +, .)$ is a (right) near-ring.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Let (F, A) be a soft set over Z_6, where $A = Z_6$ and $F: A \to P(Z_6)$ is a set-valued function defined by $F(x) = \{y \in Z_6 | \ xy \in \{0,3\}\}$ for all $x \in A$.

Then $F(0) = F(3) = Z_6$ and $F(1) = F(2) = F(4) = F(5) = \{0,3\}$ are subnear-rings of Z_6. Hence (F, A) is a soft near-ring over Z_6.

Example : 2.3

Consider the additive group $(Z_6, +)$. Under a multiplication defined by following table, $(Z_6, +, .)$ is a (right) near-ring.

Let (G, A) be a soft set over Z_6 , where $G: A \to P(Z_6)$ is defined by

$G(x) = \{y \in Z_6 | xy \in \{1,2,3\}\}$ for all $x \in A$.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
	0	4	2	0	4	2
5	3	5	1	3	5	1

Then $G(1) = \{0,1,3,4\}$ (Since $\{0,1,3,4\} \not\subseteq \{1,2,3\}$) is not a subnear-ring of Z_6 and hence (G, A) is not a soft near-ring over Z_6.

Theorem : 2.4

Let $(F, A), (G, B)$ and (K, A) be soft near-ring over N. Then

a) If it is non-null, then the soft set $(F, A) \tilde{\wedge} (G, B)$ is a soft near-ring over N.

b) If it is non-null, then the bi-intersection $(F, A) \cap_R (K, A)$ is a soft near-ring over N.

c) If A and B are disjoint, then $(F, A) \sqcup_\varepsilon (G, B)$ is a soft near-ring over N.

Proof :

Given $(F, A), (G, B)$ and (K, A) be soft near-ring over N.

a) To prove : If it is non-null, then the soft set $(F, A) \tilde{\wedge} (G, B)$ is a soft near-ring over N

By the definition of AND.

Given $(F, A) \tilde{\wedge} \ (G, B) = (H, A \times B)$ where

$H(x, y) = F(x) \cap G(y) \ \forall \ (x, y) \in A \times B.$

By hypothesis, $(H, A \times B)$, is non-null soft set over N.

If $(x, y) \in Supp(H, A \times B)$, then $H(x, y) = F(x) \cap G(y) \neq \phi$

$\Longrightarrow F(x) \neq \phi \ and \ G(y) \neq \phi$

Since $(F, A) and \ (G, B)$ are soft near-ring of N, $F(x)$ and $G(y)$ are

subnear-ring of N.

Hence $H(x, y)$ is a subnear-ring $\forall \ x, y \in Supp(H, A \times B)$

Therefore $(H, A \times B)$ is a soft near-ring over N.

b) To prove : If it is non-null, then the bi-intersection $(F, A) \ \cap_R \ (K, A)$ is a soft

near-ring ove N.

By the definition of Restricted Intersection,

Let $(F, A) \cap_R (K, A) = (H, A)$ where $H(x) = F(x) \cap K(x) \forall \ x \in A$

By hypothesis, (H, A) is a non-null soft set over N.

If $x \in Supp(H, A)$ then $H(x) = F(x) \cap K(x) \neq \phi$

$\Longrightarrow F(x) \neq \phi \ and \ K(x) \neq \phi$

Since $(F, A) and \ (K, A)$ are soft near-rings of N, $F(x)$ and $K(x)$ are subnear-rings of N.

Hence $H(x)$ is a subnear-ring of $N \ \forall \ x \in Supp(H, A)$

Therefore (H, A) is a soft near-ring over N.

c) To prove : If A and B are disjoint, then $(F, A) \sqcup_{\varepsilon} (G, B)$ is a soft near-ring over N.

By the definition of Extended Union,

Let $(F, A) \sqcup_{\varepsilon} (G, B) = (H, A \cup B)$

$$\text{Where } H(x) = \begin{cases} F(x) & if \ x \in A \setminus B \\ G(x) & if \ x \in B \setminus A \\ F(x) \cap G(x) & if \ x \in A \cap B \end{cases} \qquad \forall \ x \in A \cup B$$

If $A \cap B = \phi$, it follows that either $x \in \ A \setminus B \ \ or \ x \in \ B \setminus A \ \ \forall \ x \in A \cup B$

$If \ x \in \ A \setminus B$ then $H(x) = F(x)$ is a subnear-ring of N and

$If \ x \in \ B \setminus A$ then $H(x) = G(x)$ is a subnear-ring of N

Therefore $(H, A \cup B)$ is a soft near-ring over N.

Definition : 2.5

Let (F, A) and (G, B) be two soft near-rings over N_1 and N_2 respectively. The product of soft near-rings (F, A) and (G, B) is defined as

$(F, A) \times (G, B) = (U, A \times B)$, where $U(x, y) = F(x) \times G(y) \ \ \forall \ x, y \in A \times B$.

Theorem : 2.6

Let (F, A) and (G, B) be two soft near-rings over N_1 and N_2 respectively. If it is non-null, then the product $(F, A) \times (G, B)$ is a soft near-ring over $N_1 \times N_2$.

Proof :

Given (F, A) and (G, B) be two soft near-rings over N_1 and N_2

To prove : If it is non-null, then the product $(F, A) \times (G, B)$ is a soft near-ring over $N_1 \times N_2$.

By the definition of product of soft near-rings,

Let $(F, A) \times (G, B) = (U, A \times B)$ where $U(x, y) = F(x) \times G(y)$

$\forall\, x, y \in A \times B$

By hypothesis, $(U, A \times B)$ is non-null soft set on $N_1 \times N_2$.

If $(x, y) \in Supp(U, A \times B)$ then $U(x, y) = F(x) \times G(y) \neq \emptyset$.

$\Rightarrow F(x) \neq \emptyset$ and $G(x) \neq \emptyset$

Since (F, A) and (G, B) are soft near-rings of over $N_1 \times N_2$.

Therefore $F(x)$ is a subnear-ring of N_1 .and $G(x)$ is a subnear-rings of N_2.

It follows that $U(x, y)$ is a subnear-ring of $N_1 \times N_2\ \forall\, x, y \in Supp(U, A \times B)$

Therefore $(U, A \times B)$ is a soft near-ring over $N_1 \times N_2$.

Note : 2.7

For a near-ring N, we can obtain at least two soft near-rings over N using N_0 and N_c from Definition 1.2. We give these soft near-rings by the following:

Note : 2.8

Let N be a near-ring, $A = N_0$ and let $F_0 : A \to P(N)$ be a set-valued function defined by $F_0(x) = \{y \in A | yx \in N_0\}$ for all $x \in A$. Then (F_0, N_0) is a soft near-ring over N. To see this, we need to show the following.

1) $a - b \in F_0(x)$
2) $ab \in F_0(x)$

for all $x \in Supp(F_0, N_0)$ and for all $a, b \in F_0(x)$.

To prove (1), we need to show that $a - b \in N_0$ and $(a - b)x \in N_0$.

Since $a, b \in F_0(x)$, then $a \in N_0$, $b \in N_0$, $ax \in N_0$ and $bx \in N_0$.

Since $(N_0, +)$ is a subgroup of $(N, +)$, then $a - b \in N_0$ and

$(a - b)x = ax = bx \in N_0$

Therefore (1) is satisfied.

To prove (2), we need to show that $ab \in N_0$ and $(ab)x \in N_0$

Since $a, b, ax, bx \in N_0$, then $(ab)0 = a(b0) = a0 = 0$ and

$\big((ab)x\big)0 = a(bx)0 = a0 = 0.$

Hence $ab \in N_0$ and $(ab)x \in N_0$

Therefore (2) is satisfied.

Therefore $F_0(x)$ is a subnear-ring of N for all $x \in Supp(F_0, N_0)$

(i.e,) (F_0, N_0) is a soft near-ring over N.

Note : 2.9

Let N be a near-ring, $B = N$ and let $F_c : B \to P(N)$ be a set – valued function defined by $F_c(x) = \{y \in B | yx \in N_c\}$ for all $x \in B$. Then (F_c, N) is a soft near-ring over N. In fact, for all $x \in Supp(F_c, N)$ and for all $a, b \in F_c(x)$.

To see this, we need to show the following:

1) $a - b \in F_0(x)$
2) $ab \in F_0(x)$

for all $x \in Supp(F_0, N_0)$ and for all $a, b \in F_0(x)$.

To prove (1), $((a-b)x)0 = (ax)0 - (bx)0 = ax - bx = (a-b)x$

Since $ax \in N_c$ and $bx \in N_c$. Then $(a-b)x = ax - bx \in N_c$.

(i.e) (1) is satisfied.

To prove (2), $\big((ab)x\big)0 = a\big((bx)0\big) = a(bx) = (ab)x,$ since $bx \in N_c$

Then $(ab)x \in N_c \Rightarrow ab \in F_c(x)$

(i.e) (2) is satisfied.

Therefore $F_c(x)$ is a subnear-ring of N, for all $x \in Supp(F_c, N)$ (i.e.) (F_c, N) is a soft near-ring over N.

Definition : 2.10

Let (F, A) be a soft near-ring over N. We have the following:

a) (F, A) is called trivial if $F(x) = \{0_N\} \ \forall \, x \in Supp(F, A)$

b) (F, A) is said to be whole if $F(x) = N \ \forall \, x \in Supp(F, A)$

Proposition : 2.11

Let (F, A) and (G, B) be soft near-rings over N, where $A \cap B \neq \emptyset$. Then,

a) If (F, A) and (G, B) are trivial soft near-rings over N, then $(F, A) \cap_R (G, B)$ is a trivial soft near-ring over N.

b) If (F, A) and $(G, B)$are whole soft near-rings over N, then $(F, A) \cap_R (G, B)$ is a whole soft near-ring over N.

c) If (F, A) is a trivial soft near-ring over N and (G, A) is a whole soft near-rings over N, then $(F, A) \cap_R (G, A)$ is a trivial soft near-ring over N.

Proof :

Given (F, A) and (G, B) be soft near-rings over N, where $A \cap B \neq \emptyset$

a) **To prove** : If (F, A) and (G, B) are trivial soft near-rings over N, then $(F, A) \cap_R (G, A)$ is a trivial soft near-ring over N.

Since (F, A) is trivial then $F(x) = \{0_N\} \ \forall \, x \in Supp\ (F, A)$

Also (G, B) is trivial then $G(x) = \{0_N\} \ \forall \, x \in Supp\ (G, B)$

By the definition of Restricted Intersection,

Let $(F, A) \cap_R (G, B) = (H, C)$ where $C = A \cap B$ and $\forall \, c \in C,\ H(c) = F(c) \cap G(c)$

Since by hypothesis, $A \cap B \neq \emptyset \;\Rightarrow\; C \neq \emptyset$

Therefore $H(x) = \{0_N\}\;\; \forall\, x \in Supp\ (H, C)$

$\Rightarrow (H, C)$ is trivial soft near-ring over N.

Therefore $(F, A) \cap_R (G, B)$ is trivial soft near-ring over N.

b) To prove : If (F, A) and (G, B) are whole soft near-rings over N, then $(F, A) \cap_R (G, B)$ is a whole soft near-ring over N.

Since (F, A) is whole then $F(x) = N\;\; \forall\, x \in Supp\ (F, A)$

Also (G, B) is whole then $G(x) = N\;\; \forall\, x \in Supp\ (G, B)$

By the definition of Restricted Intersection,

Let $(F, A) \cap_R (G, B) = (H, C)$ where $C = A \cap B$ and $\forall\, c \in C$,

$H(c) = F(c) \cap G(c)$.

Since by hypothesis, $A \cap B \neq \emptyset \;\Rightarrow\; C \neq \emptyset$

Therefore $H(x) = N\;\; \forall\, x \in Supp\ (H, C)$ and $C = A \cap B$

$\Rightarrow\; (H, C)$ is whole soft near-ring over N.

Therefore $(F, A) \cap_R (G, B)$ is whole soft near-ring over N.

c) To prove : If (F, A) is a trivial soft near-ring over N and (G, A) is a whole soft near-rings over N, then $(F, A) \cap_R (G, A)$ is a trivial soft near-ring over N.

Since (F, A) is trivial then $F(x) = \{0_N\}\;\; \forall\, x \in Supp(F, A)$

Also (G, A) is whole then $G(x) = N \quad \forall\, x \in Supp(G, A)$

By Theorem 2.4 (b), $(F, A) \cap_R (G, A)$ is a soft near-ring over N.

Therefore $(F, A) \cap_R (G, A) = \{0_N\} \cap_R N = \{0_N\}$

Therefore $(F, A) \cap_R (G, A)$ is a trivial soft near-ring over N.

Proposition : 2.12

Let (F, A) and (G, B) be two soft near-rings over N_1 and N_2 respectively. Then,

a) If (F, A) and (G, B) are trivial soft near-rings over N_1 and N_2 respectively, then the product $(F, A) \times (G, B)$ is a trivial soft near-ring over $N_1 \times N_2$.

b) If (F, A) and (G, B) are whole soft near-rings over N_1 and N_2 respectively, then the product $(F, A) \times (G, B)$ is a whole soft near-ring over $N_1 \times N_2$.

Proof :

Given (F, A) and (G, B) be two soft near-rings over N_1 *and* N_2 respectively.

a) To prove : If (F, A) and (G, B) are trivial soft near-rings over N_1 *and* N_2 respectively, then the product $(F, A) \times (G, B)$ is a trivial soft near-ring over $N_1 \times N_2$.

Since (F, A) and (G, B) is trivial soft near-rings,

$F(x) = \{0_N\}\ \forall\, x \in Supp(F, A)$ and $G(y) = \{0_N\}\ \forall\, y \in Supp(G, B)$

By the definition of Product of Soft Near-Rings,

$(F, A) \times (G, B) = (U, A \times B)$ where $U(x, y) = F(x) \times G(y)\ \forall\, x, y \in A \times B$

By Theorem 2.6,

(F, A) and (G, B) are trivial. So they are non-null.

$\Rightarrow (U, A \times B)$ is a soft near-ring over $N_1 \times N_2$.

Since (F, A) is trivial $\Rightarrow F(x) = \{0_N\} \ \ \forall\, x \in Supp(F, A)$

and (G, B) is trivial $\Rightarrow G(y) = \{0_N\} \ \ \forall\, x \in Supp(G, B)$

Therefore $U(x, y) = F(x) \times G(y)$

$\Rightarrow U(x, y) = \{0_N\} \times \{0_N\}$

$\Rightarrow U(x, y) = \{0_N\}$

Therefore $(U, A \times B)$ is trivial

Therefore $(F, A) \times (G, B)$ is a trivial soft near-ring over $N_1 \times N_2$.

b) To prove : If (F, A) and (G, B) are whole soft near-ring over N_1 and N_2 respectively, then the product $(F, A) \times (G, B)$ is a whole soft near-ring over $N_1 \times N_2$.

Since (F, A) and (G, B) are whole soft near-rings, $F(x) = N$

$\forall\, x \in Supp(F, A)$ and $G(y) = N \ \ \forall\, x \in Supp(G, B)$

By the definition of Product of Soft Near-Rings,

$(F, A) \times (G, B) = (U, A \times B)$ where $U(x, y) = F(x) \times G(y) \ \ \forall\, (x, y) \in A \times B$

By Theorem 2.6,

(F, A) and (G, B) are whole, So they are non-null.

$\Rightarrow (U, A \times B)$ is a soft near-ring over $N_1 \times N_2$.

Since (F, A) is trivial $\Rightarrow$ $F(x) = N \ \forall\, x \in Supp(F, A)$

and (G, B) is trivial $\Rightarrow$ $G(y) = N \ \forall\, y \in Supp(G, B)$

Therefore $U(x, y) = F(x) \times G(y)$.

$\Rightarrow U(x, y) = N \times N$

$\Rightarrow U(x, y) = N$

Therefore $(U, A \times B)$ is whole.

Therefore $(F, A) \times (G, B)$ is a whole soft near-ring over $N_1 \times N_2$.

Definition : 2.13

Let (F, A) and (G, B) be soft near-rings over N. Then the near-ring (F, A) is called a soft subnear-ring of (G, B) if it satisfies :

a) $A \subset B$
b) $F(x)$ is a subnear-ring of $G(x)$ for all $x \in Supp(F, A)$

Proposition : 2.14

Let (F, A) and (G, B) be soft near-rings over N. Then we have the following:

a) If $F(x) \subset G(x)$ for all $x \in A$, then (F, A) is a soft subnear-ring of (G, A)
b) $(F, A) \cap_R (G, A)$ is a soft subnear-ring of both (F, A) and (G, A) if it is non-null.
c) If $(G, B) \widetilde{\subseteq} (F, A)$, then (G, B) is a soft subnear-ring of (F, A).

Proof :

Let (F, A) and (G, B) be soft near-rings over N.

a) To prove : If $F(x) \subset G(x)$ for all $x \in A$, then (F, A) is a soft subnear-ring of(G, A).

If $F(x) \subset G(x)$ $\forall\, x \in A$, then it is obvious that $F(x)$is a subnear-ring of $G(x)$.

By the definition of soft subnear-ring, then the result is true.

b) To prove : $(F, A) \cap_R (G, A)$ is a soft subnear-ring of both (F, A) and (G, A) if it is non-null.

By (a), (F, A) is a soft subnear-ring of (G, A).

Now by Theorem 2.4 (b) " If it is non-null, then the bi-intersection $(F, A) \cap_R (K, A)$ is a soft near-ring over N. "

Hence $(F, A) \cap_R (G, A)$ is a soft subnear-ring of both (F, A) and (G, A).

c) To prove : If $(G, B) \tilde{\subset} (F, A)$, then (G, B) is a soft subnear-ring of (F, A)

Since $F(x)$ and $G(x)$ are subnear-ring of N for all $x \in Supp(F, A)$ and $x \in Supp(G, B)$ and $G(x), F(x)$ are identical approximations for all $x \in Supp(G, B)$ and $B \subseteq A$.

By part (a), (G, B) is a soft subnear-ring of (F, A).

CHAPTER – III

SOFT IDEALS AND IDEALISTIC SOFT NEAR RINGS

In this chapter, I collect the definition of soft ideals and idealistic soft near-rings and some important definitions and some theorems and propositions.

Definition : 3.1

Let (F, A) be a soft near-ring over $N. A$ non-null soft set (G, I) over N is called a soft left (respectiviely right) ideal of (F, A) denoted by $(G, I) \,\tilde{\triangleleft}_l\, (F, A)$ (respectively $(G, I) \,\tilde{\triangleleft}_r\, (F, A)$) if it satisfies :

a) $I \subset A$

b) $G(x) \triangleleft_l\ F(x)$ (respectively $G(x) \triangleleft_r\ F(x)\ \forall x \in Supp(G, I)$).

If (G, I) is both soft left and soft right ideal of (F, A), then it is said that (G, I) is a soft ideal of (F, A) and denoted by $(G, I) \,\tilde{\triangleleft}\, (F, A)$.

Example : 3.2

Consider the additive group $(Z_6, +)$. Under a multiplication defined by following table, $(Z_{6,} +, .)$ is a (right) near-ring.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Let (F, A) be a soft set over Z_6, where $A = Z_6$ and $F: A \to P(Z_6)$ is a set-valued function defined by $F(x) = \{y \in Z_6 \,|xy \in \{0,2,4\}\}$ for all $x \in A$.

Then $F(0) = F(2) = F(4) = Z_6$ and $F(1) = F(3) = F(5) = \emptyset$ are

subnear-rings of Z_6. Hence (F, A) is a soft near-ring over Z_6.

Now, Let $I = \{0,2,4\}$ and $G: I \to P(N)$ be a set-valued function defined by

$G(x) = \{y \in I \mid xy \in \{0,2,4\} \quad \forall x \in I.$

Then $G(0) = G(2) = G(4) = \{0,2,4\}$. It is easily seen that for all

$x \in Supp(G, I) = \{0,2,4\}, G(x) \lhd F(x)$ and hence $(G, I) \tilde{\lhd} (F, A)$.

Theorem : 3.3

Let (G_1, I_1) and (G_2, I_2) be soft left ideals (respectively soft right ideals, soft ideals) of a soft near-ring (F, A) over a near-ring N. Then the soft set $(G_1, I_1) \cap_R (G_2, I_2)$ is a soft left ideal (respectively soft right ideal, soft ideal) of (F, A) if it is non-null.

Proof :

We give the proof for soft left ideals; the same proof can be seen for soft right ideals and hence for soft ideals.

Assume that $(G_1, I_1) \tilde{\lhd}_l (F, A)$ and $(G_2, I_2) \tilde{\lhd}_l (F, A)$.

By the definition of Restricted Intersection,

$(G_1, I_1) \cap_R (G_2, I_2) = (G, I)$ where $I = I_1 \cap I_2$ and

$G(x) = G_1(x) \cap G_2(x) \; \forall x \in I.$

Since $I_1 \subset A$ *and* $I_2 \subset A$, is is clear that $I \subset A$.

Suppose that the soft set (G, I) is non-null.

If $x \in Supp(G, I)$, then $G(x) = G_1(x) \cap G_2(x) \neq \emptyset$.

Since $G_1(x) \lhd_l F(x),\ G_2(x) \lhd_l F(x)$, and the intersection of left ideals is a left ideal in near-rings, $G(x) \lhd_l F(x) \quad \forall x \in Supp(G, I)$.

Therefore $(G_1, I_1) \cap_R (G_2, I_2)\ \widetilde{\lhd}_l\ (F, A)$.

Theorem : 3.4

Let (G_1, I_1) and (G_2, I_2) be soft left ideals (respectively soft right ideals, soft ideals) of a soft near-ring (F, A) over a near-ring N. Then the soft set $(G_1, I_1) \sqcup_\varepsilon (G_2, I_2)$ is a soft left ideal (respectively soft right ideal, soft ideal) of (F, A) if I_1 and I_2 are disjoint.

Proof :

We give the proof for soft left ideals; the same proof can be seen for soft right ideals and hence for soft ideals.

Assume that $(G_1, I_1)\ \widetilde{\lhd}_l\ (F, A)$ and $(G_2, I_2)\ \widetilde{\lhd}_l\ (F, A)$.

By the definition of Extended Union,

$(G_1, I_1) \sqcup_\varepsilon (G_2, I_2) = (G, I)$ where $I = I_1 \cup I_2$ and for all $x \in I$

$$G(x) = \begin{cases} G_1(x) & if\ x \in I_1 \setminus I_2 \\ G_2(x) & if\ x \in I_2 \setminus I_1 \\ G_1(x) \cup G_2(x) & if\ x \in I_1 \cap I_2 \end{cases}$$

Since $I_1 \subset A$ and $I_2 \subset A$, it is clear that $I \subset A$.

If $I_1 \cap I_2 = \emptyset$, then for all $x \in Supp(G, I)$, we know that either $x \in I_1 \setminus I_2$ or $x \in I_2 \setminus I_1$

If $x \in I_1 \setminus I_{2,}$ then $\emptyset \neq G_1(x) = G(x) \lhd_l F(x)$ and

If $x \in I_2 \setminus I_{1,}$ then $\emptyset \neq G_2(x) = G(x) \lhd_l F(x)$ for all $x \in Supp(G, I)$.

Therefore $(G_1, I_1) \sqcup_\varepsilon (G_2, I_2) \widetilde{\lhd}_l (F, A)$.

Example : 3.5

Let (F, A) be the soft near-ring over the near-ring $N = (Z_6, +, .)$ and let $(G, I) \widetilde{\lhd} (F, A)$ be the ones given in Example 3.2

(where $F(x) = \{y \in A \mid xy \in \{0,2,4\}\}\ \forall x \in A$) and

$G(x) = \{y \in I | xy \in \{0,2,4\}\}\ \ \forall x \in I.\ Let\ K: A \to P(N)$ be a set-valued function defined by $K(x) = \{y \in A | xy = 0\}$ for all $x \in A$.

Then $K(0) = Z_6, K(1) = \emptyset, K(2) = \{0,3\},\ K(3) = \emptyset, K(4) = \{0,3\}\ and\ K(5) = \emptyset$.

Since $K(x) \lhd F(x)\ \ \forall x \in Supp(K, A)$, then $(K, A) \widetilde{\lhd} (F, A)$.

Now we consider the bi-intersection of the soft ideals (K, A) and (G, I).

Then $(K, A) \cap_R (G, I) = (H, I)$ where $A \cap I = I$ and

$H(x) = K(x) \cap G(x)\ \forall x \in I$.

Since *Supp*$(H, I) = \{0,2,4\}$, it is non-null.

For all $x \in I\ Supp(H, I)$, we see that $H(0) = \{0,2,4\} \lhd F(0) = Z_6$,

$H(2) = \{0\} \lhd F(2) = Z_6$, and $H(4) = \{0\} \lhd F(4) = Z_6$

Therefore $(K, A) \cap_R (G, I) \mathrel{\underline{\widetilde{\lhd}}} (F, A)$.

Now, we consider $(K, A) \sqcup_\varepsilon (G, I)$. Then $(K, A) \sqcup_\varepsilon (G, I) = (T, A)$ where $A \cup I = A$ and

$$T(x) = \begin{cases} K(x) & if\ x \in A \setminus I = \{1,3,5\} \\ G(x) & if\ x \in I \setminus A = \emptyset \\ K(x) \cup G(x) & if\ x \in A \cap I = \{0,2,4\} \end{cases} \qquad \forall x \in A$$

Then $Supp\ (T, A) = \{0,2,4\}$ and $T(0) = Z_6,\ T(2) = \{0,2,3,4\} = T(4)$.

Nevertheless, $T(2)$ is not an ideal of $F(2)$ and hence $(K, A) \sqcup_\varepsilon (G, I)$ is not a soft ideal of (F, A).

Namely, we see that the condition 'disjoint' cannot be removed from the Theorem 3.4

Definition : 3.6

Let (F, A) be a soft near-ring over N. If for all $x \in Supp(F, A)\ F(x) \lhd_l N$ (respectively $F(x) \lhd_r N,\ F(x) \lhd N$), then

(F, A) is called a left idealistic (respectively right idealistic, idealistic) soft near-ring over N.

Example : 3.7

Let the soft near-rings (F, A) and (K, A) be the ones given in Example 3.2 over the near-ring $N = (Z_6, +, .)$. Then for all $x \in Supp(F, A) = \{0,2,4\},\ F(x) \lhd N$, $(i.e.)$ (F, A) is an idealistic over N.

Since $Supp(K, A) = \{0,2,4\}$ and $K(0) = Z_6 \lhd N,\ K(2) = K(4) = \{0,3\} \lhd N$. Then (K, A) is also an idealistic soft near-ring over N.

Theorem : 3.8

Let (F, A) and (G, B) be idealistic soft near-rings over N. Then we have the following:

a) If it is non-null, $(F, A) \cap_R (G, B)$ is an idealistic soft near-rings over N.

b) If A and B are disjoint, then $(F, A) \sqcup_\varepsilon (G, B)$ is an idealistic soft near-ring over N.

c) If it is non-null, $(F, A) \tilde{\wedge} (G, B)$ is an idealistic soft near-ring over N.

Proof :

Let (F, A) and (G, B) be idealistic soft near-rings over N.

(a) To prove : If it is non-null, $(F, A) \cap_R (G, B)$ is an idealistic soft near-ring over N.

By the Theorem 2.4(b).

$(F, A) \cap_R (G, B)$ is a soft near-ring over N.

By hypothesis, (F, A) and (G, B) be idealistic soft near rings over N.

Then (($F, A) \cap_R (G, B)$ is an idealistic soft near-rings over N.

(b) To prove : If A and B are disjoint, then $(F, A) \sqcup_\varepsilon (G, B)$ is an idealistic soft near-ring over N.

By the Theorem 2.4(c),

$(F, A) \sqcup_\varepsilon (G, B)$ is a soft near-ring over N.

By hypothesis, (F, A) and (G, B) be idealistic soft near rings over N.

Therefore $(F, A) \sqcup_\varepsilon (G, B)$ is an idealistic soft near-rings over N.

(c)To prove: If it is non-null, $(F,A)\,\tilde{\wedge}\,(G,B)$ is an idealistic soft near-ring over N.

By the Theorem 2.4 (a),

$(F,A)\,\tilde{\wedge}\,(G,B)$ is a soft near-ring over N.

By hypothesis, (F,A) and (G,B) be idealistic soft near rings over N.

Therefore $(F,A)\,\tilde{\wedge}\,(G,B)$ is an idealistic soft near-rings over N.

Definition :3.9

A near-ring N is said to satisfy the condition (C) if $I \lhd J \lhd N$, then $I \lhd N$.

Example :3.10

Consider the additive group $(Z_6,+)$. Under a multiplication defined by following table, $(Z_6,+,.)$ is a (right) near-ring.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Let (F,A) be a soft set over Z_6, where $A = Z_6$, and $F: A \to P(Z_6)$ is a set-valued function defined by $F(x) = \{y \in Z_6 | xy \in \{0,2,4\}\}$ for all $x \in A$.

Then $F(0) = F(2) = F(4) = Z_6$ and $F(1) = F(3) = F(5) = \emptyset$ are subnear-ring of Z_6.

Hence (F, A) is a soft near-ring over Z_6

Then for all $x \in Supp(F, A) = \{0,2,4\}$, $F(x) \lhd N$.

Now, Let $I = \{0,2,4\}$ and $G: I \to P(N)$ be a set-valued function defined by

$G(x) = \{y \in I | xy \in \{0,2,4\}\}\ \forall x \in I$

Then $G(0) = G(2) = G(4) = \{0,2,4\}$. It is easily seen that for all

$x \in Supp(G, I) = \{0,2,4\}$,

$G(x) \lhd F(x)$ and hence $(G, I) \tilde{\lhd} (F, A)$.

Hence, $G(x) \lhd F(x)\ and\ F(x) \lhd N$. By condition (C), $G(x) \lhd N$.

Proposition :3.11

Let N be a near-ring which satisfies the condition (C) and let (F, A) be an idealistic soft near-ring over N. If (G, I) is a soft ideal of (F, A), then (G, I) is also an idealistic soft near-ring over N.

Proof :

If $(G, I) \tilde{\lhd} (F, A)$ then for all $x \in Supp(G, I)$, $G(x) \lhd F(x)$

Since(F, A) is an idealistic soft near-ring over N, then

for all $x \in Supp(F, A)$, $F(x) \lhd N$

So we have $G(x) \lhd F(x) \lhd N$ for all $x \in Supp(G, I)$.

Since N satisfics condition(C), $G(x) \lhd N$ for all $x \in Supp(G, I)$.

$G(x)$ is also a subnear-ring of N for all $x \in Supp(G, I)$, since every ideal of N is also a subnear-ring of N.

Therefore (G, I) is a soft near-ring over N.

Furthermore, (G, I) is an idealistic soft near-ring over N.

Example : 3.12

Let (F, A) be the soft near-ring over N and let $(G, I) \ \tilde{\lhd}\ (F, A)$ be the ones given in Example -3.2. It is seen that (G, I) is also an idealistic soft near-ring over N.

Definition :3.13

Let (F, A) and (G, B) be soft near-ring over two near-rings N_1 and N_2 respectively.

Let $f: N_1 \to N_2$ and $g: A \to B$ be two mappings. Then the pair (f, g) is called a soft mapping from (F, A) to (G, B). A soft mapping (f, g) is called soft homomorphism if it satisfies the conditions below:

a) f is a near-ring homomorphism
b) g is a mapping.
c) $f(F(x)) = G(g(x))$ for all $x \in A$.

If (f, g) is a soft homomorphism and f and g are both surjective, then we say the (F, A) is softly near-ring homomorphic to (G, B) under the soft homomorphism (f, g), which is denoted by $(F, A) \sim (G, B)$. Then (f, g) is called a soft near-ring homomorphism. Furthermore, if f is an isomorphism of near-rings and g is a bijective mapping, then (f, g) is said to be a soft near-ring isomorphism. In this case, we say that (F, A) is soft isomorphic to (G, B), which is denoted by $(F, A) \simeq (G, B)$.

Example :3.14

Consider the additive group $(Z_6, +)$. Under a multiplication defined by following table, $(Z_6, +, \ .)$ is a (right) near-ring.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Let (F, A) be a soft set over Z_6, where $A = Z_6$ and $F: A \to P(Z_6)$ is a set-valued function defined by $F(x) = \{y \in Z_6 | xy \in \{0,2,4\}\}$ for all $x \in A$.

Then $F(0) = F(2) = F(4) = Z_6\ and\ F(1) = F(3) = F(5) = \emptyset$ are

subnear-ring of Z_6.

Hence (F, A) is a soft near-ring over Z_6

Let $f : Z_6 \to \{0,2,4\}$ be the mapping defined by $f(x) = 4x$. Obviously, f is an epimorphism of near-rings.

Let $g : Z_6 \to \{0,2,4\}\ by\ g(x) = 2x$ for all $x \in Z_6$. Then one can easily say that g is surjective.

Let (F, Z_6) be a soft set over Z_6, where $F: Z_6 \to P(Z_6)$ is a function by

$F(x) = \{0\} \cup \{y \in Z_6 | 3x = y\}$ for all $x \in Z_6$. It can be easily illustrated that $F(x) = \{0,3\}$ is a subnear-ring of Z_6 for all $x \in Z_6$. Thus (F, Z_6) is a soft near-ring over Z_6.

Let $(G, \{0,2,4\})$ be a soft set over $\{0,2,4\}$, where $G: \{0,2,4\} \rightarrow P\{0,2,4\}$ is a funtion with $G(x) = \{y \in \{0,2,4\} | x0 = y\}$ $\forall\, x \in \{0,2,4\}$. Then one can show that $(G, \{0,2,4\})$ is a soft near-ring over {0,2,4}.

Furthermore, $f(F(x)) = f(\{0,3\}) = \{0\}$ and $G(g(0)) = G(0) = \{0\}$,

$G(g(1)) = G(2) = \{0\}, G(g(2)) = G(4) = \{0\}, G(g(3)) = G(0) = \{0\}$,

$G(g(4)) = G(2) = \{0\}, G(g(5)) = G(4) = \{0\}$, for all $x \in Z_6$.

So it is to say that $f(F(x)) = G(g(x))$ $\forall$ $x \in Z_6$.

Therefore (f, g) is a soft near-ring homomorphism and $(F, Z_6) \sim (G, \{0,2,4\})$.

Theorem : 3.15

Let $(F, A), (G, B)$ and (H, C) be soft near-rings over N_1, N_2 and N_3 respectively. Let the soft mapping (f, g) from (F, A) to (G, B) is a soft homomorphism from N_1 to N_2 and the soft mapping (f^*, g^*) from (G, B) to (H, C) a soft homomorphism from N_2 to N_3 . Then the soft mapping $(f^*\, o\, f,\ g^*\, o\, g)$ from (F, A) to (F, A) to (H, C) is a soft homomorphism from N_1 to N_3 .

Proof :

Let the soft mapping (f, g) from N_1 to N_2 be a soft homomorphism from (F, A) to (G, B).

Then there exists a near-ring homomorphism f such that $f: N_1 \rightarrow N_2$ and a mapping g such that $g : A \rightarrow B$ which satisfy $f(F(x)) = G(g(x))$ for all $x \in A$.

And let the soft mapping (f^*, g^*) from N_2 to N_3 be a soft homomorphism from (G, B) to (H, C).

Then there exists a near-ring homomorphism f^* such that $f^*: N_2 \to N_3$ and a mapping g^* such that $g^* : B \to C$ which satisfy $f^*\big(G(x)\big) = H\big(g^*(x)\big)$ for all $x \in B$.

We need to show that $(f^* \, o \, f^*) \big(F(x)\big) = H\big((g^* \, o \, g)(x)\big)$ for all $x \in A$.

Let $x \in A$, then

$$((f^* o \, f)\big(F(x)\big) = (f^* \Big(f\big(F(x)\big)\Big) = (f^* \Big(G\big(g(x)\big)\Big) = H\Big((g^* \big(g(x)\big)\Big)$$

$$= H((g^* \, o \, g)(x))$$

Therefore, the proof is completed.

Theorem : 3.16

The relation $\simeq$ is an equivalence relation on soft near-rings.

Theorem : 3.17

Let N_1 and N_2 be near-rings and $(F, A), (G, B)$ be soft sets over N_1 and N_2, respectively.

If (F, A) is a soft near-ring over N_1 and $(F, A) \simeq (G, B)$, then (G, B) is a soft near-ring over N_2.

Proof :

We need to show that $G(y)$ is a subnear-ring of N_2 for all $y \in Supp(G, B)$.

Since $(F, A) \simeq (G, B)$, there exists a near-ring epimorphism f from N_1 to N_2 and a bijective mapping g from A to B which satisfies $f\big(F(x)\big) = G\big(g(x)\big)$ for all $x \in A$.

Assume that (F, A) is a soft near-ring over N_1. Then $F(x)$ is a subnear-ring of N_1 for all $x \in Supp(F, A)$, therefore $f\big(F(x)\big)$ is a subnear-ring of N_2

for all $x \in Supp(F, A)$.

Since g is a bijective mapping, for all $y \in Supp(G, B) \subseteq B$, there exists an $x \in A$ such that $y = g(x)$.

Hence $G(y)$ is a subnear-ring of N_2 for all $y \in Supp(G, B)$ since

$f\big(F(x)\big) = G(y)$.

CHAPTER – IV

FUZZY SOFT NEAR RINGS

In this chapter, I collect the definition of fuzzy soft near-rings and idealistic fuzzy soft near-rings and some important definitions and some theorems.

Definition : 4.1

Let $(N, +, .)$ be a near-ring and E be the set of parameters and $A \subset E$.

Let F be a mapping given by $F: A \rightarrow [0,1]^N$ where $[0,1]^N$ is the collection of all fuzzy subsets of N. Then (F, A) is called fuzzy soft near-ring over N if and only if for each $a \in A$, the corresponding fuzzy subset $F_a = \left(= F(a)\right)$ of N is a fuzzy sub near-ring of N. (i.e.),

a) $F_a(x + y) \geq min\left(F_a(x), F_a(y)\right)$

b) $F_a(-x) \geq min\left(F_a(x)\right)$

c) $F_a(xy) \geq min\left(F_a(x), F_a(y)\right)$ for all $x, y \in N$

Example : 4.2

Let $N = \{a, b, c, d\}$ be a a non-empty set with two binary operations '+' and '.' defined as follows:

+	A	b	c	D
a	A	b	c	D
b	B	a	d	C
c	C	d	b	A
d	D	c	a	B

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then ($N, +, .$) is a near-ring.

Let (F, A) be a soft set over N, where $A = N$ and $F: A \to P(N)$ is a set-valued function defined by

$F(x) = \{y \in N | xy \in \{a, b\}\}$ for all $x \in A$.

Then $F(a) = F(b) = F(c) = N$ and $F(d) = \{a, b\}$ are all subnear-rings of N.

Hence (F, A) is a soft near-ring over N.

Let $A = \{e_1, e_2, e_3\}$ be the set of parameters.

Define a fuzzy soft set (F, A) on a near-ring N by

:

F	e_1	e_2	e_3
a	0.2	0.4	0.7
b	0.2	0.3	0.6
c	0.1	0.2	0.3
d	0.1	0.2	0.3

Then clearly the fuzzy soft set (F, A) is a fuzzy soft near-ring over a near-ring N.

Theorem : 4.3

Let (F, A) be a fuzy soft set over a near-ring N. Then (F, A) be a fuzzy soft near-ring over N if and only if for each $a \in A$ and $x, y \in N$ the following conditions hold:

a) $F_a(x - y) \geq min(F_a(x), F_a(y))$
b) $F_a(xy) \geq min(F_a(x), F_a(y))$

Proof :

Let (F, A) be a fuzzy soft near-ring over N.

Let $a \in A$ and $x, y \in N$. Then

(a) $F_a(x - y) = F_a(x + (-y)) \geq min(F_a(x), F_a(-y))$

$$\geq min(F_a(x), F_a(y))$$

Since (F, A) is a fuzzy soft near-ring over N, the second condition holds.

Conversely, let (F, A) be a fuzzy soft set over a near-ring N satisfying the given conditions.

Now consider $F_a(0) = F_a(x - x) \geq min(F_a(x), F_a(x))$

$$\geq F_a(x)$$

Thus $F_a(0) \geq F_a(x)$ for all $x \in N$

Also $F_a(-x) \geq F_a(x)$ for all $x \in N$

Therefore, $F_a(x-y) = F_a(x+(-y))$

$$\geq min\big(F_a(x), F_a(-y)\big)$$

$$\geq min\big(F_a(x), F_a(y)\big)$$

Hence the theorem.

Theorem: 4.4

Let (F, A) and (G, B) be two fuzzy soft near-ring over N. If $(F, A) \wedge (G, B)$ ia non-null, then it is a fuzzy soft near-ring over N.

Proof :

Let (F, A) and (G, B) be two fuzzy soft near-ring over N.

Let $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(a, b) = H_{a,b} = F_a \cap G_b$ for all $(a, b) \in A \times B$.

Since $(H, A \times B)$ is non-null, there exists a pair $(a, b) \in A \times B$ such that

$H_{a,b} = F_a \wedge G_b \neq 0_N$.

Since F_a isy a fuzzy sub near-ring of N for all $a \in A$ and G_b is a fuzzy sub near-ring of N for all $b \in B$ and since the intersection of two fuzzy sub near-ring of N is a subnear-ring of N.

Therefore, $H_{a,b} = F_a \cap G_b$ is a sub near-ring of N.

Hence $(F, A) \wedge (G, B) = (H, A \times B)$ is a fuzzy soft near-ring over N.

Theorem:4.5

Let (F, A) and (G, B) be two fuzzy soft near-rings over N. If $(F, A) \cap_R (G, B)$ is non-null, then it is a fuzzy soft near-ring over N.

Proof :

Let $(F, A) \cap_R (G, B) = (H, C)$ where $C = A \cap B$.

Let $c \in C$.

Since $(F, A) \cap_R (G, B) = (H,C)$, wher $C = A \cap B$ and $H_c = F_c \cap G_c$ for all $c \in C$.

Since (H, C) is non-null, there exists $c \in C$ such that $H_c(x) \neq \emptyset$ for some $x \in N$

Since intersection of two sub near-rings is a sub near-ring we see that $F_c \cap G_c$ is a fuzzy sub near-ring of N.

Hence $(F, A) \cap_R (G, B)$ is a fuzzy soft near-ring over N.

Theorem : 4.6

Let (F, A) and (G, B) be two fuzzy soft near-ring over N. Then the extended intersection $(F, A) \cap_\varepsilon (G, B)$ is a fuzzy soft near-ring over N.

Proof ;

Let (F, A) and (G, B) be two fuzzy soft near-rings over N.

Let $(F, A) \cap_\varepsilon (G, B) = (H, C)$ where $C = A \cup B$.

Then consider the following cases :

(i) If $c \in B \setminus A$, then $H_c = G_c$ for all $c \in C$.

Since G_c is fuzzy sub near-ring of N, H_c is a fuzzy sub near-ring of N.

(ii) If $c \in A \setminus B$ then $H_c = F_c$ for all $c \in C$.

Since F_c is fuzzy sub near-ring of N, H_c is a fuzzy sub near-ring of N.

(iii) If $c \in A \cap B$ then $H_c = F_c \cap G_c$ for all $c \in C$.

Since intersection of two fuzzy sub near-rings of N is a fuzzy sub near-ring of N we see that $F_c \cap G_c$ is a fuzzy sub near-ring of N.

Therefore H_c is a fuzzy sub near-ring of N.

Thus in any case, H_c is a fuzzy sub near-ring of N.

Hence, $(F, A) \cap_\varepsilon (G, B) = (H, C)$ is a fuzzy soft near-ring over N.

Theorem : 4.7

Let , (F, A) and (G, B) be two fuzzy soft near-rings over N. Then the extended union , $(F, A) \cup_\varepsilon (G, B)$ is a fuzzy soft near-ring over , N if , $A \cap B = \phi$

Proof :

Let (F, A) and (G, B) be two fuzzy soft near-rings over N.

Let $(F, A) \cup_\varepsilon (G, B) = (H, C)$ where $C = A \cup B$.

If $c \in C$, since $A \cap B = \emptyset$ then either $c \in A$ or $c \in B$.

If $c \in A$, then $H_c = F_c$

If $c \in B$ then $H_c = G_c$

Since F_c , G_c is a fuzzy sub near-ring of N,

Thus in both cases, H_c is a fuzzy sub near-ring of N.

Hence, $(F, A) \cup_{\varepsilon} (G, B) = (H, C)$ is a fuzzy soft near-ring over N.

Theorem : 4.8

Let $\{(F_i, A_i) | i \in I\}$ be a non-empty family of fuzzy soft near-rings over N. Then we have the following :

a) If $\wedge \{((F_i, A_i) | i \in I\}$ is non-null, then it is a fuzzy soft near-ring.
b) If $\cap \{(F_i, A_i) | i \in I\}$ is non-null, then it is a fuzzy soft near-ring.
c) If $\{A_i | i \in I\}$ are pairwise disjoint and $\cup \{(F_i, A_i) | i \in I\}$ is non-null, then it is a fuzzy soft near-ring.

Proof :

Let $\{(F_i, A_i) | i \in I\}$ be a non-empty family of fuzzy soft near-ring over N.

(i) Let $\wedge \{(F_i, A_i) | i \in I\} = (G, B)$ where $B = A_1 \times A_2 \times \dots \dots \times A_n$ and $G_b = \wedge \{F_i(b_i) | i \in I\}$ for all $b_i \in B$.

Suppose the fuzzy soft set (G, B) is non-null.

If $b \in Supp(G, B)$, then $G_b = \wedge\ F_i(b_i)$.

Since (F_i, A_i) is a fuzzy soft near-ring for all $i \in I$, then $F_i(b_i)$ is a fuzzy sub near-ring of N.

Hence (G, B) is a fuzzy sub near-ring of N for all $b \in Supp(G, B)$.

Hence if $\wedge \{(F_i, A_i) | i \in I\}$ is non-null, then it is a fuzzy soft near-ring.

(ii) Let $\cap \{(F_i, A_i) | i \in I\} = (G, B)$ where $B = \cap A_i$ and $G_b = \cap \{F_i(b_i) | i \in I\}$.

Suppose the fuzzy soft set (G, B) is non-null.

If $b \in Supp(G, B)$, then $G_b = \cap F_i(b_i)$ is non-null.

Since (F_i, A_i) is fuzzy soft near-ring for all $i \in I$, then $F_i(b_i)$ is a fuzzy sub near-ring of N for all $i \in I$

Hence (G, B) is a fuzzy sub near-ring of N for all $b \in Supp(G, B)$.

Thus $\cap \{(F_i, A_i)|i \in I\}$ is a fuzzy soft near-ring.

(iii) Let $\cup \{(F_i, A_i)|i \in I\} = (G, B)$ where $B = \cup A_i$ and $\{A_i|i \in I\}$ are pairwise disjoint and

$G_b = \cup \{F_i(b_i)|i \in I\}$.

Suppose the fuzzy soft set (G, B) is non-null.

If $b \in Supp(G, B)$, then $G_b = \cup F_i(b_i)$ is non-null.

Since (F_i, A_i) is a fuzzy soft near-ring for all $i \in I$, then $F_i(b_i)$ is a fuzzy sub near-ring of N for all $i \in I$.

Hence (G, B) is a fuzzy sub near-ring of N for all $b \in Supp(G, B)$.

Thus if $\{A_i|i \in I\}$ are pairwise disjoint and $\cup \{(F_i, A_i)|i \in I\}$ is non-null, then it is a fuzzy soft

near-ring.

Definition :4.9

let (F, A) be a fuzzy soft near-ring over N. The (F, A) is said to be an idealistic fuzzy soft near-ring if F_a is a fuzzy ideal of N for all $a \in Supp(F, A)$. $(i.e)$

a) $F_a(x+y) \geq min\{F_a(x), F_a(y)\}$ for $x, y \in N$

b) $F_a(-x) \geq F_a(x)$ for all $x \in N$

c) $F_a(x) = F_a(y+x-y)$ for all $x, y \in N$

d) $F_a(xy) \geq F_a(y)$ for all $x, y \in N$

e) $F_a\{(x+i)y - xy\} \geq F_a(i)$ for all $x, y, i \in N$

If F_a satisfies $(a), (b), (c)$and (d) then it is called left idealistic fuzzy soft near-ring of N and if it satisfics $(a), (b), (c)$ and (e) then it is called right idealistic fuzzy soft near-ring of N.

Theorem : 4.10

Let (F, A) be a fuzzy soft set over a near-ring N and $B \subseteq A$. If (F, A) is an idealistic fuzzy soft near-ring then (F, B) is an idealistic fuzzy soft near-ring over N, provided that it is non-null.

Proof :

Let (F, A) be a non-null fuzzy soft set and $B \subseteq A$.

Let $a \in Supp\ (F, B)$.

Then $a \in Supp(F, B)$ implies that $a \in Supp(F, A)$

Let (F, A) be an idealistic fuzzy soft near-ring over N.

Then F_a is a fuzzy ideal of N for all $a \in Supp(F, A)$

But since $B \subseteq A, a \in Supp(F, A)$ implies that $a \in Supp(F, B)$

Hence F_a is a fuzzy ideal of N for all $a \in Supp(F, B)$.

Therefore (F, B) is an idealistic fuzzy soft near-ring over N.

Theorem :4.11

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N. Then the restricted intersection of (F, A) and (G, B) is an idealistic fuzzy soft near-ring over N if it is non-null.

Proof :

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N.

Let $(F, A) \cap_R (G, B) = (H, C)$ where $C = A \cap B$ and for all $c \in C$,

$H(c) = F(c) \cap G(c)$.

Suppose, (H, C) is non-null.

Then there exists $a \in Supp(H, C)$ such that $H_a = F_a \cap G_a \neq 0_N$.

Since F_a and G_a are fuzzy ideals of N, it follows that H_a is fuzzy ideal of N for all $a \in Supp(H, C)$.

Hence, $(F, A) \cap_R (G, B) = (H, C)$ is an idealistic fuzzy soft near-ring over N.

Theorem: 4.12

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N. Then the extended intersection of (F, A) and (G, B) is an idealistic fuzzy soft near-ring over N, if it is non-null.

Proof :

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N

Let $c \in Supp(H, C)$.

If $c \in A \setminus B$ then the $H(c) = F(c) = F_c$ is fuzzy ideal of N.

Therefore $H(c) = H_c$ is a idealistic fuzzy soft near-ring over N.

Similarly, if $c \in B \setminus A$ then $H(c) = G(c) = G_c$ and since G_c is fuzzy ideal of N.

Therefore $H(c) = H_c$ is a idealistic fuzzy soft near-ring over N.

Again, if $c \in A \cap B$ and $H(c) = F(c) \cap G(c)$.

Since intersection of two fuzzy ideals of if R is a fuzzy ideal of N.

Therefore $H(c) = F(c) \cap G(c)$ is a fuzzy ideal of N.

Hence, $(H, C) = (F, A) \cap_{\varepsilon} (G, B)$ is an idealistic fuzzy soft near-ring over N, if it is non-null.

Theorem : 4.13

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N. If A and B are disjoint then the extended union $(F, A) \cup_{\varepsilon} (G, B)$ ia an idealistic fuzzy soft near-ring over N.

Proof :

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N.

Let A and B be disjoint.

If $A \cap B = \emptyset$ then for $a \in Supp(H, C)$ we have either $a \in A \setminus B$ or $a \in B \setminus A$.

If $a \in A \setminus B$ then $H_a = F_a$ is a fuzzy ideal of N and so H_a is an idealistic fuzzy soft near-ring.

If $a \in B \setminus A$ then $H_a = G_a$ is a fuzzy ideal of N and so H_a is an idealistic fuzzy soft near-ring.

Thus for all $a \in Supp(H, C), H_a$ is a fuzzy ideal of N.

Hence, the extended union $(F, A) \cup_{\varepsilon} (G, B)$ is an idealistic fuzzy soft near-ring over N.

Note : 4.14

If A and B are not disjoint then the theorem is not true since the union of two fuzzy ideals of a near-ring N is not a fuzzy ideal of a near-ring N.

Theorem:4.15

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N. Then $(F, A) \wedge (G, B) = (H, A \times B)$ is an idealistic fuzzy soft near-ring over N, if it is non-null.

Proof :

Let (F, A) and (G, B) be two idealistic fuzzy soft near-rings over a near-ring N.

Let $(F, A) \wedge (G, B) = (H, A \times B)$ where $H_{(a,b)} = F_a \wedge G_a$ for all $(a, b) \in A \times B$.

Suppose $(H, A \times B)$ is non-null fuzzy soft set.

If $(a, b) \in Supp(H, A \times B)$, then $H_{(a,b)} = F_a \wedge G_a \neq 0_N$.

Since (F, A) and (G, B) are two idealistic fuzzy soft near-rings over a near-ring N, we conclude that F_a and G_b are fuzzy ideals of N.

Since intersection of two fuzzy ideals of N is a fuzzy ideals of N.

Therefore $H_{(a,b)}$ is a fuzzy ideal of N for all $(a, b) \in Supp(H, A \times B)$.

Thus, $(F, A) \wedge (G, B) = (H, A \times B)$ is an idealistic fuzzy soft near-ring over N.

CHAPTER-V

FUZZY SOFT IDEALS

Definition : 5.1

Let $(R,+,.)$ be a ring and E be a parameter set and $A \subseteq E$. Let $\mathcal{J}$ be a mapping given by $\mathcal{J} : A \to P(R)$. Then $(\mathcal{J}, A)$ is called a soft left ideal over R if and only if for each $a \in A,\ \mathcal{J}(a)$ is a left ideal of R (i.e,)

(i) $x, y \in \mathcal{J}(a) \Rightarrow x - y \in \mathcal{J}(a)$

(ii) $x \in \mathcal{J}(a),\ r \in R \Rightarrow r.x \in \mathcal{J}(a)$

Definition : 5.2

Let $(R,+,.)$ be a ring and E be a parameter set and $A \subseteq E$. Let $\mathcal{J}$ be a mapping given by $\mathcal{J} : A \to P(R)$. Then $(\mathcal{J}, A)$ is called a soft right ideal over R if and only if for each $a \in A,\ \mathcal{J}(a)$ is right ideal of R (i.e,)

(i) $x, y \in \mathcal{J}(a) \Rightarrow x - y \in \mathcal{J}(a)$

(ii) $x \in \mathcal{J}(a), r \in R \Rightarrow r.x \in \mathcal{J}(a)$

Definition : 5.3

Let $(R,+,.)$ be a ring and E be a parameter set and $A \subseteq E$. Let $\mathcal{J}$ be a mapping given by $\mathcal{J} : A \to P(R)$. Then $(\mathcal{J}, A)$ is called a soft ideal over R if and only if for each $a \in A,\ \mathcal{J}(a)$ is a ideal of R i.e.

(i) $x, y \in \mathcal{J}(a) \Rightarrow x - y \in \mathcal{J}(a)$

(ii) $x \in \mathcal{J}(a),\ r \in R \Rightarrow r.x \in \mathcal{J}(a)$

Definition : 5.4

Let $(R,+,.)$ be a ring and E be a parameter set and $A \subseteq E$. Let $\mathcal{J}$ be a mapping given by $\mathcal{J} : A \to [0,1]^R$, where $[0,1]^R$ denotes the collection of all fuzzy subsets of R.

Then $(\mathcal{J}, A)$ is called fuzzy soft left ideal over R if and only if for each $a \in A$, the corresponding fuzzy subset $\mathcal{J}_a : R \to [0,1]$ is a fuzzy left ideal of R (i.e.)

(i) $\mathcal{J}_a(x - y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y)$

(ii) $\mathcal{J}_a(x.y) \geq \mathcal{J}_a(y),\ \forall x, y \in R.$

Definition : 5.5

Let $(R, +, .)$ be a ring and E be a parameter set and $A \subseteq E$. Let $\mathcal{J}$ be a mapping given by $\mathcal{J} : A \to [0,1]^R$, where $[0,1]^R$ denotes the collection of all fuzzy subsets of R. Then $(\mathcal{J}, A)$ is called fuzzy soft right ideal over R if and only if for each $a \in A$, the corresponding fuzzy subset $\mathcal{J}_a : R \to [0,1]$ is a fuzzy right ideal of R (i.e.)

(i) $\mathcal{J}_a(x - y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y)$

(ii) $\mathcal{J}_a(x.y) \geq \mathcal{J}_a(x),\ \forall x, y \in R.$

Definition : 5.6

Let $(R, +, .)$ be a ring and E be a parameter set and $A \subseteq E$. Let $\mathcal{J}$ be a mapping given by $\mathcal{J} : A \to [0,1]^R$, where $[0,1]^R$ denotes the collection of all fuzzy subsets of R. Then $(\mathcal{J}, A)$ is called fuzzy soft ideal over R if and only if for each $a \in A$,the corresponding fuzzy subset $\mathcal{J}_a : R \to [0,1]$ is a fuzzy right ideal of R (i.e.)

(i) $\mathcal{J}_a(x - y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y)$

(ii) $\mathcal{J}_a(x.y) \geq max\{\mathcal{J}_a(x), \mathcal{J}_a(y)\},\ \forall x, y \in R$

Theorem : 5.7

Let $(R, +, .)$ be a ring and E be a parameter set and $A \subseteq E$. Then $(\mathcal{J}, A)$ is fuzzy soft left (respectively right) ideal over R if and only if for each $a \in A$, the corresponding fuzzy subset $\mathcal{J}_a$ of R satisfy following conditions:

(i) $\mathcal{J}_a(x - y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y)$ for all $x, y \in R$

(ii) $\chi_R \circ \mathcal{J}_a \leq \mathcal{J}_a$(respectively $\mathcal{J}_a \circ \chi_R \leq \mathcal{J}_a$)

Where χ_R stands for the characteristic function of R.

Proof :

Suppose $(\mathcal{J}, A)$ is a fuzzy soft left ideal over R. Then for each $a \in A$, the corresponding fuzzy subset $\mathcal{J}_a$ of R satisfy two conditions

(1) $\mathcal{J}_a(x-y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y)$

(2) $\mathcal{J}_a(x.y) \geq \mathcal{J}_a(y),\ \forall x, y \in R.$

Let z be an element of R. Then

$$(\chi_R \circ \mathcal{J}_a)(z) = \sup_{z=x.y} \{min\, \{\chi_R(x), \mathcal{J}_a(y)\}\}$$

$$= \sup_{z=x.y} \{\min \{\mathcal{J}_a(y)\} \leq \mathcal{J}_a(x-y)$$

$$= \mathcal{J}_a(z)$$

If z can not be expressed as $z = x.y$ where $x, y \in R$, then $(\chi_R \circ \mathcal{J}_a)(z) = 0 \leq \mathcal{J}_a(z)$ holds.

Therefore $\chi_R \circ \mathcal{J}_a \leq \mathcal{J}_a$.

Conversely, let $(\mathcal{J}, A)$ be a fuzzy soft subset over R such that for each $a \in A$ the corresponding fuzzy subset $\mathcal{J}_a$ of R satisfy given two conditions.

(i) $\mathcal{J}_a(x-y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y)$ for all $x, y \in R$

(ii) $\chi_R \circ \mathcal{J}_a \leq \mathcal{J}_a$

Let $x, y \in R$ then we have $\mathcal{J}_a(x.y) \geq (\chi_R \circ \mathcal{J}_a)(x.y)$

$$= \sup_{x.y=p.q} \{min\, \{\chi_R(p), \mathcal{J}_a(q)\}\}$$

$$\geq min\{\chi_R(x), \mathcal{J}_a(y)\} = \mathcal{J}_a(y).$$

This shows that for each $a \in A$, $\mathcal{J}_a$ is a fuzzy left ideal of R.

So $(\mathcal{J}, A)$ is fuzzy soft left ideal over R. Similar proof for a fuzzy soft right ideal over R.

This completes the proof.

Theorem : 5.8

Let $(R,+,.)$ be a ring and E be a parameter set and $A \subseteq E$. Then $(\mathcal{J}, A)$ is a fuzzy soft left (respectively right) ideal over R iff for each $\mathcal{J}_a(a \in A\ )$ each level subset $(\mathcal{J}_a)_t, t \in lm(\mathcal{J}_a)$ is a left (respectively right) ideal of R where $\mathcal{J}_a$ is the fuzzy subset of R corresponding to $a \in A$.

Proof:

Suppose $(\mathcal{J}, A)$ is a fuzzy soft left ideal over R. Then for each $a \in A$ the corresponding fuzzy subset $\mathcal{J}_a$ is a fuzzy left ideal of R.

Now let $t \in lm(\mathcal{J}_a)$ and $x, y \in (\mathcal{J}_a)_t,\ r \in R$.

Since $\mathcal{J}_a$ is a fuzzy left ideal of R, then $\mathcal{J}_a(x-y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y) \geq t$ and $\mathcal{J}_a(r.x) \geq \mathcal{J}_a(x) \geq t$. This implies that $x - y \in (\mathcal{J}_a)_t$ and $r.x \in (\mathcal{J}_a)_t$.

So $(\mathcal{J}_a)_t$ is a left ideal of R for each $t \in lm(\mathcal{J}_a)$

Conversely, let $(\mathcal{J}_a)_t$ is a left ideal of R for each $t \in lm(\mathcal{J}_a)$ and corresponding to each $a \in A$ and also let $x, y \in R$.

Suppose $\mathcal{J}_a(x-y) < \mathcal{J}_a(x) * \mathcal{J}_a(y) = t_1$ (say).

This implies $x, y \in (\mathcal{J}_a)_{t_1}$ but $x - y \notin (\mathcal{J}_a)_{t_1}$. This contradicts to $(\mathcal{J}_a)_{t_1}$ is a left ideal of R.

So $\mathcal{J}_a(x-y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y) \dots\dots\dots\dots\dots\dots\dots\dots\dots$ (1)

Again suppose $\mathcal{J}_a(x.y) < \mathcal{J}_a(y) = t_2$ (say).

This implies $y \in (\mathcal{J}_a)_{t_2}$ but $x.y \notin (\mathcal{J}_a)_{t_2}$. This contradicts to $(\mathcal{J}_a)_{t_2}$ is a left ideal of R.

So $\mathcal{J}_a(x.y) \geq \mathcal{J}_a(y) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$ (2)

So (1) and (2) together implies $\mathcal{J}_a$ is a fuzzy left ideal of R for each $a \in A$.

Similar proof for fuzzy soft right ideal over R. This completes the proof.

Theorem : 5.9

Let $(R,+,.)$ be a ring and E be a parameter set and $A \subseteq E$. If (I,A) be a soft subset over R, where $(I,A) = \{(a, I(a): I(a)$ be the subset of R corresponding to $a \in A\}$ and we define a fuzzy subset $\mathcal{J}_a$ of R corresponding to $a \in A$ by

$$\mathcal{J}_a(x) \quad = \begin{cases} s \ \ if \ x \in I(a) \\ t \ \ otherwise \end{cases}$$

for all $x \in R$ and $s,t \in [0,1]$ with $s > t$.

Then $(\mathcal{J},A)$ is a fuzzy soft left (respectively right) ideal over R if and only if (I,A) is a soft left (respectively right) ideal over R, where $(\mathcal{J},A) = \{(a,\mathcal{J}_a) : a \in A\}$.

Proof:

At first left $(\mathcal{J},A)$ be a fuzzy soft left ideal over R. Then for each $a \in A$, $\mathcal{J}_a$ is a fuzzy left ideal of R.

Let $x,y \in I(a)$ and $r \in R$. Then $\mathcal{J}_a(x) = \mathcal{J}_a(y) = s$.

Hence $\mathcal{J}_a(x-y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(y) = s$. So $x - y \in I(a)$.

Again $\mathcal{J}_a(r.x) \geq \mathcal{J}_a(x) = s$. So $r.x \in I(a)$. Hence $I(a)$ is a left ideal of R for each $a \in A$.

So, (I,A) is a soft left ideal of over R.

Conversely, let (I,A) be a soft left ideal over R. Then for each $a \in A$, $I(a)$ is a left ideal of R. Let $x,y \in R$, then the following four cases arise for consideration:

Case (i) : $x,y \in I(a) \Rightarrow x - y \in I(a),\ x.y \in I(a)$ [as $I(a)$ is a left ideal of R]

$$\Rightarrow \mathcal{J}_a(x-y) = S = \mathcal{J}_a(x) = \mathcal{J}_a(y) = \mathcal{J}_a(x.y).$$

Case (ii): $x,y \in I(a),\ y \notin I(a) \Rightarrow \mathcal{J}_a(x) = s \ and \ \mathcal{J}_a(y) = t.$

Case (iii): $x,y \notin I(a), y \in I(a) \Rightarrow x.y \in I(a)$ [as $I(a)$ is a left ideal of R]

$$\Rightarrow \mathcal{J}_a(x.y) = S, = \mathcal{J}_a(x) = t,\ \mathcal{J}_a(y) = s.$$

Case (iv): $x,y \notin I(a) \Rightarrow \mathcal{J}_a(x) = \mathcal{J}_a(y) = t.$

For each case we have $\mathcal{J}_a(x-y) \geq \mathcal{J}_a(x) * \mathcal{J}_a(x)$ and $\mathcal{J}_a(x.y) \geq \mathcal{J}_a(y)$.

So $(\mathcal{J}, A)$ is a fuzzy soft left ideal over R.

This completes the proof.

A STUDY ON EXTENSION OF EDGE GRACEFUL LABELING

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

S.CRENA

Reg. No: 19SPMT07

Under the guidance of

Ms. J. JENIT AJITHA M.Sc., M.Phil.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON EXTENSION OF EDGE GRACEFUL LABELING"** submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **S.CRENA (Reg. No: 19SPMT07)**

Signature of the Guide **Signature of the Coordinator**

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous,
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON EXTENSION OF EDGE GRACEFUL LABELING"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. J. JENIT AJITHA M.Sc., M.Phil.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

S. Crena

Signature of the Student

Station: Thoothukudi

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

1. PRELEMINARIES

In this chapter we give basic information which will be used in the remainder of the project. In this section we give some basic definitions related to graph theory, which is a background in the Edge graceful labeling.

Definition: 1.1

A graph G = (V, E) is a finite non empty set V of objects called **vertices** together with a set E of unordered pairs of distinct vertices called **edges.**

Definition: 1.2

The Cardinality of the vertex set of graph G is called the **order of G** and is denoted by p. The Cardinality of its edge set is called the **Size of G** and is denoted by q. A graph with p vertices and q edges is called a **(p, q) - graph.**

Definition: 1.3

If e= (u, v) is an edge of G, we write e = uv and we say that u and v are adjacent vertices of G. If two vertices are adjacent, then they are said to be **neighbours**. Further, vertex u and edge e are said to be incident with each others, as are v and e. If two distinct edges f and g are incident with a common vertex, then f and g are said to be adjacent edges.

Definition: 1.4

An edge having the same vertex as both its end vertices is called as **self-loop or loop.**

Definition: 1.5

The number of edges incident on a vertex v with self-loops counted twice is called the **degree of the vertex v.** It is denoted by d(v).the maximum degree of a graph G is denoted by$\Delta(G)$ and the minimum degree of G is denoted by $\delta(G)$. A vertex with degree 0 is called an **isolated vertex.** A vertex with degree 1 is called **Pendant vertex.**

A graph is regular of degree r if every vertex of G has degree r. Such graphs are called **r-regular graphs.**

Definition: 1.6

A graph H is a **subgraph** of a graph G, if all the vertices and all the edges H are in G, and each edge of H has the same end vertices as in G.

Definition: 1.7

A **Spanning subgraph** of G is a subgraph H with$V(H) = V(G)$.

(i.e) Hand G having exactly the same vertex set.

Definition: 1.8

A graph in which any two distinct vertices are adjacent is called a **complete graph** and is denoted by K_p.

Definition: 1.9

Two graphs G and G′ are said to be **isomorphic,** if there is a one to one correspondence between their vertices and between their edges, such that the incidence relationship is preserved.

Definition: 1.10

A **Walk** is a finite alternating sequence of vertices and edges, beginning and ending with vertices such that each edge is incident with the vertices proceeding and succeeding it [No edge appear more than once].

Definition: 1.11

A walk which beginning and ends with the same vertex is called a **closed walk** which no vertex appears more than once is called a **path.**

Definition: 1.12

A walk in which no vertex (except the initial and final vertex) appears more than once is called **cycle .**A graph with exactly one cycle is called a **unicycle graph.**

Definition: 1.13

Any cycle with a pendent edge attached at each vertex is called a **Crown graph** and is denoted by $C_n^+ (n \geq 3)$.

Definition: 1.14

Armed crowns are obtained from cycle by attaching paths of equal lengths at each vertex of the cycle. We dente an armed crown by$C_n \ominus P_m$ where P_m is a path of length m-1, where C_n is a $n-$ cycle.

Definition: 1.15

A **Wheel** is a graph obtained from a cycle by adding a new vertex and edges Joining to all the vertices of a cycle, the new edges are called the spokes of the wheel. The wheel of n vertices is denoted by W_n.

Definition: 1.16

A labeled graph is a graph whose vertices are each assigned an element from a set of symbols (letters, usually, but this is unimportant). The important thing is to note is that the vertices can be distinguished one from another.

Definition: 1.17

Let G = (V, E, F) be a simple graph with order n and size m.

Let V = $\{v_1, v_2, \ldots, v_n\}$ and E = $\{e_1, e_2, \ldots, e_m\}$. Let each node v_i be labeled with distinct nonnegative integer x_i . A map f defined on the Cartesian product of the set of labels of vertices to the set of labels of edges is called if$f(x_i, x_j) = |x_i - x_j|$ such that each $|x_i - x_j|$ is distinct and varies from l to m and the group G is called graceful graph.

2. STRONG EDGE-GRACEFUL LABELING OF SOME SPECIALGRAPHS

Introduction:

In 1985, Lo[18] introduced edge-graceful graph which is dual notation of graceful labeling. Every edge-graceful graph is Anti magic Lo[18] found a necessary condition for a graph with p vertices and q edges to be edge graceful as

$$q(q+1) \equiv \frac{p(p-1)}{2} \ (mod\ p).$$

Strong edge-graceful labeling was studied in detail by Subbiah. In this chapter, we obtain strong-edge-graceful labeling of some special graphs.

Definition: 2.1

A graph G with p vertices and q edges is said to be an **Edge-graceful labeling** if there exists a bijection f from the edge set $\{1,2,\dots,q\}$ so that the induced mapping f^+ from the vertex set to the set $\{0,1,\dots,p-1\}$ given by,

$f^+(x) = \sum\{f(xy) : xy \in E(G)\}(mod\ p)$ is a bijection.

Remark:2.2 Lo's Necessary Condition [18]

If a graph G= (p,q) is edge-graceful then $q(q+1) \equiv \frac{p(p-1)}{2}\ (mod\ p)$.

Definition: 2.3

A (p,q) graph G is said to have a strong edge-graceful labeling if there exits an injection f from the edge set $\left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ so that the induced mapping f^+

from the vertex set $\{0,1,\ldots,2p-1\}$ defined by a

$f^{+}(x) = \sum\{f(xy): xy \in E(G)\}(mod\ 2p)$ are

distinct. A graph G is said to be **Strong edge-graceful labeling.** Here,[x] denotes the integer part of x.

Illustration: 2.4

Consider the graph G in figure 2.4.1

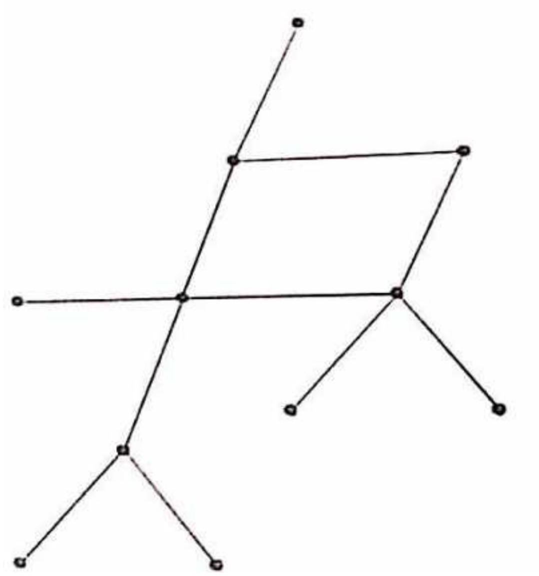

Figure 2.4.1: G

Here p=11, q=11. So, we define $E(G) \rightarrow \{1,2,\ldots,11\}$ and we obtain induced map $f^{+}: V(G) \rightarrow \{1,2,\ldots,10\}$.

The EGL of G is given in the figure 2.4.2

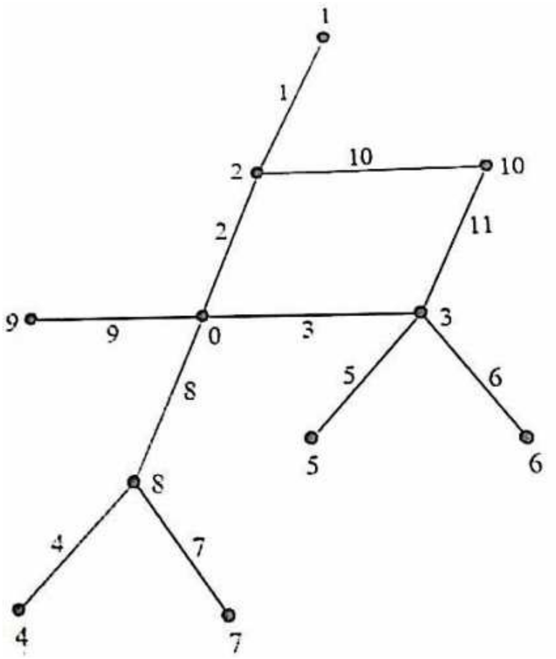

Figure 2.4.2 EGL of G

Illustration: 2.5

Consider the graph P_6. Here p=6 and q=5. Lo's condition is not satisfied. Hence,P_6 is not an Edge-graceful graph.

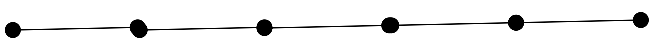

Figure 2.5.1: P_6

Illustration: 2.6

In Illustration 2.5 we have seen that P_6 is not EGC. But it is SEGG as given in the figure 2.6.1

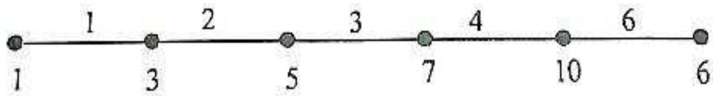

Figure 2.6.1 SEGL of: P_6

Remark: 2.7

An edge-graceful (p, q) graphs can be proved as strong edge-graceful graphs, where as the converse need not be true.

Definition: 2.8

A **Vertex Switching** $\boldsymbol{G_v}$ of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident at v and adding joining v to every other vertex which are not adjacent to v in G.

Theorem: 2.9

Switching of a pendent vertex in a path $P_n (n \geq 4)$ is a strong edge-graceful graph.

Proof:

Let $\{v_1, v_2, \dots, v_n\}$ be the vertices path P_n. Let G_v be the graph obtained by switching a pendent vertex in P_n. Without loss of generality, let the switched vertex be v_1. Let $\{e_1, e_2, \dots, e_{n-2}, e_{n-4,} \dots, e_{2n-4}\}$ be the edges of G_v.

We note that $|V(G_v)| = p = n$ and $|E\ (G_v)| = q = 2n - 4$

Case1: n ≡ $\boldsymbol{0, 1, 3\ (mod\ 4)}$

We first label the edges as follows:

Define $f: E(G_v) \rightarrow \left\{1,2, \dots, \left[\frac{3q}{2}\right]\right\}$ by

$$f(e_i) = 2n - i, \qquad 1 \leq i \leq n - 2$$

$$f(e_i) = 2n - i - 3, \qquad n - 1 \leq i \leq 2n - 4$$

Then the induced vertex labels are:

$$f^+(v_1) = \begin{cases} \frac{n+1}{2}, & n \equiv 0(mod\ 4) \\ n+1, & n \equiv 1(mod\ 4) \\ 1, & n \equiv 3(mod\ 4) \end{cases}$$

$f^+(v_i) = 2n - i + 1, \quad 2 \leq i \leq n-1$

$f^+(v_n) = 0$

Case2: n ≡ 2 ($\boldsymbol{mod}$ 4)

$f(e_i) = 2n - i, \qquad 1 \leq i \leq n-2$

$f(e_{n-1}) = n - 2, \qquad f(e_n) = 2n$

$f(e_i) = 2n - i - 3, \qquad n - 1 \leq i \leq 2n - 4$

Then the induced vertex labels are:

$f^+(v_1) = \frac{n+8}{2}$

$f^+(v_i) = 2n - i + 1, \quad 2 \leq i \leq n-1$

$f^+(v_{n-1}) = 5, \qquad f^+(v_n) = 0$

Clearly all the vertex labels are distinct. Hence, the above defined edge labeling function $f^+: V(G) \rightarrow$ {1,2, ... ,2p-1}. Hence f is a strong edge-graceful labelling.

Thus, switching of a pendent vertex in a path P_n, (n≥4) is a strong edge-graceful graph.

Illustration: 2.10

The SEGL of switching of a pendent vertex in the path P_7 are shown in the figure 2.10.1

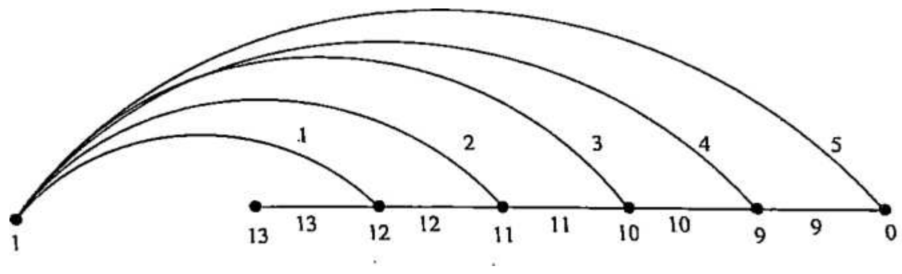

Figure 2.10.1 SEGL of switching of a pendent vertex in the path P_7

Theorem: 2.11

Switching of a vertex in a cycle C_n, ($n \geq 4$) is astrong edge-graceful graph.

Proof:

Let $v_1, v_2, \dots, v_n$ be the vertices of C_n. Let G' be the graph obtained by switching the vertex V_1.

Let $\{e_1, e_2, \dots, e_{n-2}, e_{n-4,} \dots, e_{2n-4}\}$ be the edges of G_v which are denoted as in figure.

We note that $|V(G')| = p = n$ and $|E(G')| = q = 2n - 5$

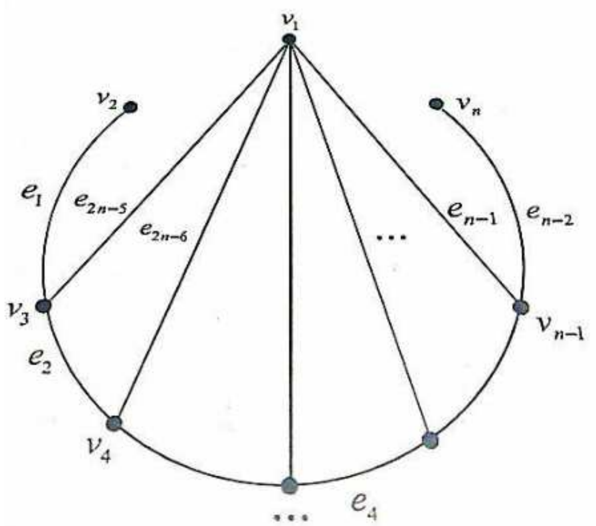

Figure 2.11.1: Ordinary labeling of switching of a vertex in a cycle C_n

Case1: n $\equiv \mathbf{0, 1, 3}$ $(\boldsymbol{mod}\ \mathbf{4}), (\boldsymbol{n} \geq \mathbf{7})$

We first label the edges as follows:

Define $f: E(G_v) \to \left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ by

$f(e_i) = 2n - 1, \qquad 1 \leq i \leq n - 3$

$f(e_i) = n - 3, \qquad n - 2$

$$f(e_{n-1}) = \begin{cases} 2n, & n \equiv 0 (mod\ 4) \\ 8, & n \equiv 1 (mod\ 4) \\ n - 2, & n \equiv 3 (mod\ 4) \end{cases}$$

$f(e_i) = 2n - i - 4, \qquad n \leq i \leq 2n - 5$

Then the induced vertex labels are:

$$f^+(v_1) = \begin{cases} \frac{n+12}{2}, & n \equiv 0 (mod\ 4) \\ 4, & n \equiv 1 (mod\ 4) \\ 6, & n \equiv 3 (mod\ 4) \end{cases}$$

$f^+(v_i) = 2n - i + 1, \quad 2 \leq i \leq n - 1$

$f^+(v_n) = n - 3$

Case2: n $\equiv \mathbf{2}$ $(\boldsymbol{mod}\ \mathbf{4}), (\boldsymbol{n} \geq \mathbf{10})$

$f(e_i) = 2n - i, \qquad 1 \leq i \leq n - 4$

$f(e_{n-1}) = n + 2, \qquad f(e_{n-1}) = n + 1$

$f(e_i) = 2n - i - 4, \qquad n - 1 \leq i \leq 2n - 5$

Then the induced vertex labels are:

$f^+(v_1) = \frac{n+6}{2}$

$f^+(v_i) = 2n - i + 1, \qquad 2 \leq i \leq n - 3$

$f^+(v_{n-2}) = n + 2 \quad ; \quad f^+(v_{n-1}) = n \quad ; f^+(v_n) = n + 1$

Clearly all the vertex labels are distinct. Hence, the above defined edge labeling function $f^{+}: V(G) \rightarrow \{1,2, \ldots ,2\text{p-}1\}$. Hence f is a strong edge-graceful labeling.

Thus, switching of a pendent vertex in a path C_n, ($n \geq 4$) is a strong edge-graceful graph.

Illustration: 2.12

The SEGL of switching of a pendent vertex in the path C_8 are shown in the figure 2.10.1

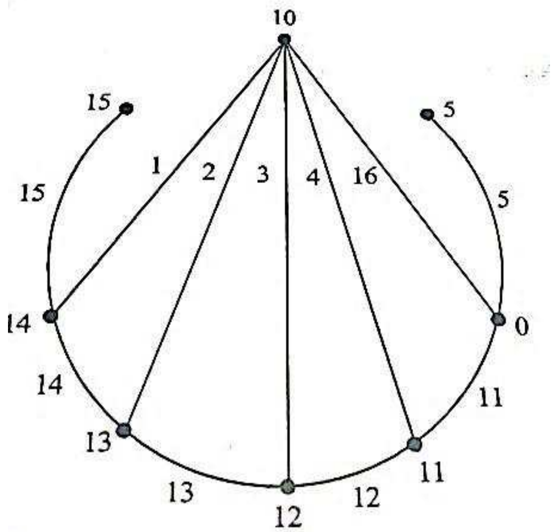

Figure 2.12.1 SEGL of switching of a vertex in C_8

Definition: 2.13

Let P_n denote the path on n vertices .Then the join of K_1 with P_n is defined as **fan** and is denoted by F_n (i.e) $F_n = K_1 + P_n$.

Theorem: 2.14

Switching of a vertex in a fan F_n, ($n \geq 3$) is astrong edge-graceful graph.

Let $\{v_1, v_2, \ldots, v_n\}$be the vertices of F_n. Let G_v be the graph obtained by switching the vertex V_1. Let $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1}, \ldots, e_{3n-5}\}$ be the edges of G_v

which are denoted as in figure.2.14.1

We denote that $|V(G')| = p = n+1$ and $|E(G')| = 3n-5$

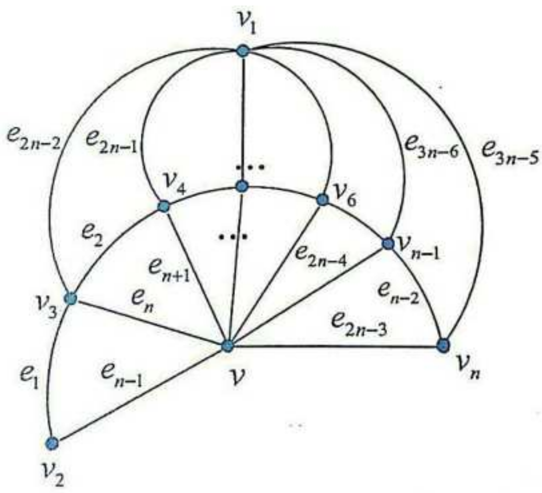

Figure 2.14.1: Ordinary labeling of switching of a vertex in a fan F_n

Case1: n ≡ 0 $(\boldsymbol{mod}\ 4)$

We first label the edges as follows:

Define $f: E(G_v) \to \left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ by

$f(e_i) = i, \qquad 1 \le i \le n-2$

$f(e_i) = 3n-i+1, \qquad n-1 \le i \le 2n-3$

$f(e_i) = i+6, \qquad 2n-2 \le i \le 3n-5$

Then the induced vertex label are:

$f^+(v_1) = n+1\ ; \qquad f^+(v_2) = 1$

$f(v_i) = 2n-2; \qquad 3 \le i \le 2n-3$

$f^+(v_n) = n-1\ ; \qquad f^+(v) = 2n-1$

Case2: n ≡ 1 $(\boldsymbol{mod}\ \mathbf{4})$

$f(e_i) = i, \qquad 1 \le i \le n-2$

$f(e_i) = 2n+3, \qquad i = n-1$

$f(e_i) = 2n-i+1, \qquad n \le i \le 2n-3$

$f(e_i) = i+6, \qquad 2n-2 \le i \le 3n-6$

$f(e_i) = 2(n+1), \qquad i = 3n-5$

Then the induced vertex labels are:

$f^+(v_1) = \frac{n+5}{2}$

$f^+(v_i) = 2i-2, \qquad 2 \le i \le n-1$

$f^+(v_{n-2}) = 0,$; $\qquad f^+(v) = \frac{3n-1}{2}$

Case3: n ≡ 1 $(\boldsymbol{mod}\ \mathbf{4})$

$f(e_i) = i, \qquad 1 \le i \le n-2$

$f(e_i) = 3n-i+1, \qquad n-1 \le i \le 2n-3$

$f(e_i) = i+5, \qquad 2n-2 \le i \le 3n-6$

$f(e_i) = n, \qquad i = 3n-5$

Then the induced vertex labels are:

$f^+(v_1) = n-6 \qquad f^+(v_2) = 1$

$f^+(v_i) = 2i-3, \qquad 3 \le i \le 2n-4$

$f^+(v_{n-2}) = n,$; $\qquad f^+(v) = n-2$

Case4: n ≡ 3$(\boldsymbol{mod}\ \mathbf{4}), (\boldsymbol{n} > \mathbf{3})$

$f(e_i) = i, \qquad 1 \le i \le n-2$

$f(e_i) = 3n - i, \qquad n - 1 \leq i \leq 2n - 4$

$f(e_i) = 2(n + 1), \qquad i = 3n - 5$

$f(e_i) = i + 5, \qquad 2n - 2 \leq i \leq 3n - 6$

Then the induced vertex label are:

$f^+(v_1) = \frac{3n+9}{2}$

$f^+(v_i) = 2i - 4, \qquad 2 \leq i \leq n$

$f^+(v) = \frac{n-5}{2}$

Clearly, all vertex labels are distinct. Hence, the above defined edge labelling function f^+:V$(G_v)\rightarrow \{0,1,2, \dots, 2p - 1\}$. Hence, f is a strong edge-graceful labeling.

Thus, switching of a vertex in a fan F_n,$(n \geq 4)$ is a strong edge- graceful graph.

Illustration: 2.16

The SEGL of switching of a vertex in the fanF_5 are shown in figure 2.15.1

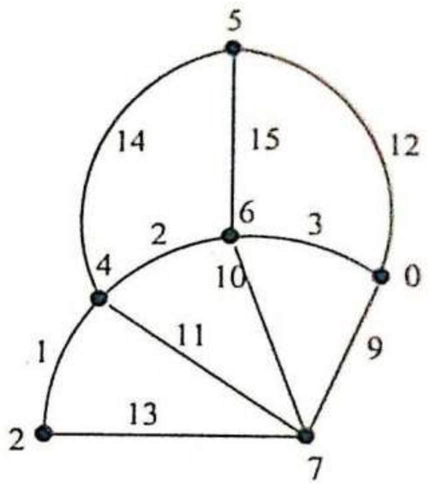

Figure 2.15.1: SEGL of switching of a vertex in F_5

Definition: 2.16

The **wheel** W_n is defined as $W_n = C_n + K_1$, Where C_n is the cycle of length n.

Theorem: 2.17

Switching of a rim vertex in a wheel W_n, (n$\geq$ 4) is a strong edge-graceful graph.

Proof:

Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of W_n. Let G_v be the graph obtained by switching the vertex v_1 . Let $\{e_1, e_2, \ldots, e_{n-2}, e_{n-4,} \ldots, e_{2n-4}\}$ be the edges of G_v which are denoted as in figure.2.17.1

We note that $|V(G')| = p = n + 1$ and $|E(G')| = q = 3n - 6$

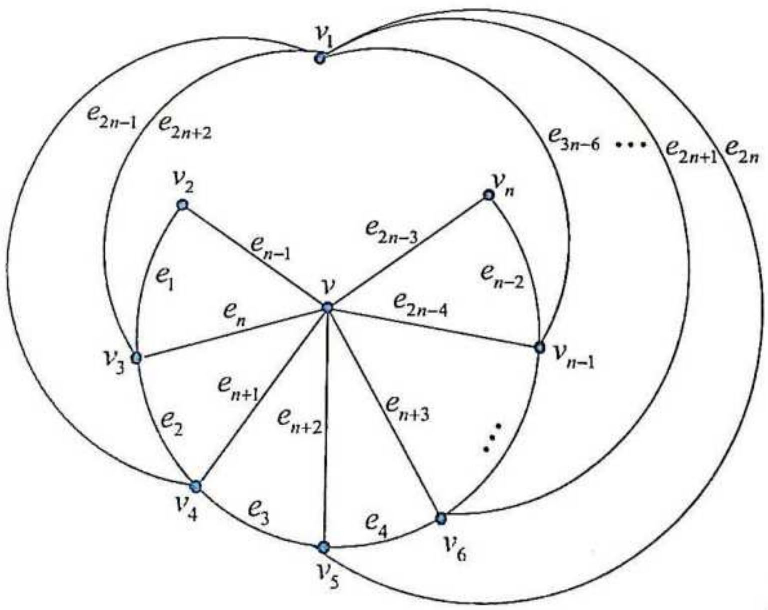

Figure2.17.1: Ordinary labeling of switching of a rim vertex W_n

Case1: n $\equiv \mathbf{0}$ $(\boldsymbol{mod}\ \mathbf{4}), (\boldsymbol{n} \geq \mathbf{8})$

We first label the edges as follows:

Define $f: E(G_v) \rightarrow \left\{1,2, \dots, \left[\frac{3q}{2}\right]\right\}$ by

$$f(e_i) = i, \qquad 1 \leq i \leq n-2$$

$$f(e_i) = 3n - i + 1, \quad n-1 \leq i \leq 2n-4$$

$$f(e_{2n-3}) = 3n,$$

$$f(e_i) = i + 5, \qquad 2n-2 \leq i \leq 3n-6$$

Then the induced vertex label are:

$$f^+(v_1) = \begin{cases} 1, & n = 4 \\ n+7, & n \geq 8 \end{cases}$$

$$f^+(v_i) = 2i - 4\ , \qquad 2 \leq i \leq n$$

$$f^+(v_n) = \begin{cases} 9, & n = 4 \\ n-5 & n \geq 8 \end{cases}$$

Case2: n $\equiv \mathbf{1}$ $(\boldsymbol{mod}\ \mathbf{4}), (\boldsymbol{n} \geq \mathbf{9})$

$$f(e_i) = i, \qquad 1 \leq i \leq n-2$$

$$f(e_{n-1}) = 2n + 3,$$

$$f(e_i) = 3n - i + 1, \quad n \leq i \leq 2n-3$$

$$f(e_i) = i + 6, \qquad 2n-2 \leq i \leq 3n-6$$

Then the induced vertex labels are:

$$f^+(v_1) = \frac{n+5}{2}$$

$$f^+(v_i) = 2i - 2, \qquad 2 \leq i \leq n-1$$

$$f^+(v_{n-2}) = 0, \quad ; \qquad f^+(v) = \frac{3n-1}{2}$$

Case3: n ≡ $\boldsymbol{2(mod\ 4), (n \geq 6)}$

$f(e_i) = i, \qquad 1 \leq i \leq n-2$

$f(e_i) = 3n - i + 1, \qquad n-1 \leq i \leq 2n-3$

$f(e_i) = i + 5, \qquad 2n-2 \leq i \leq 3n-6$

Then the induced vertex labels are:

$f^+(v_1) = 6$

$f^+(v_i) = 2i - 3, \qquad 2 \leq i \leq n-1$

$f^+(v_n) = 0, \quad ; \qquad f^+(v) = n-2$

Case4: n ≡ $\boldsymbol{3(mod\ 4), (n \geq 7)}$

$f(e_i) = i, \qquad 1 \leq i \leq n-2$

$f(e_i) = 3n - i, \qquad n-1 \leq i \leq 2n-4$

$f(e_{2n-3}) = 2n + 2,$

$f(e_i) = i + 5, \qquad 2n-2 \leq i \leq 3n-6$

$f(e_{3n-6}) = 3n$

Then the induced vertex label are:

$f^+(v_1) = \frac{n+15}{2}$

$f^+(v_i) = 2i - 4, \qquad 2 \leq i \leq n$

$f^+(v_{n-1}) = 2n - 5, \ f^+(v_n) = n - 2,$

$f^+(v) = \frac{n-5}{2}$

Clearly, all vertex labels are distinct. Hence the above defined edge labeling function induces the vertex labeling function $f^+\colon \mathrm{V}(G_v) \to \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling. Thus, Switching of a vertex in a wheel W_n is a strong edge-graceful graph for all n≥ 4.

We observe that switching of a rim vertex in wheel W_3 is disconnected graph with a unique vertex in one of the components.

Hence, it is not a strong edge-graceful graph.

Illustration: 2.18

The SEGL of switching of a vertex in the wheel W_{10} is shown in the figure 2.18.1

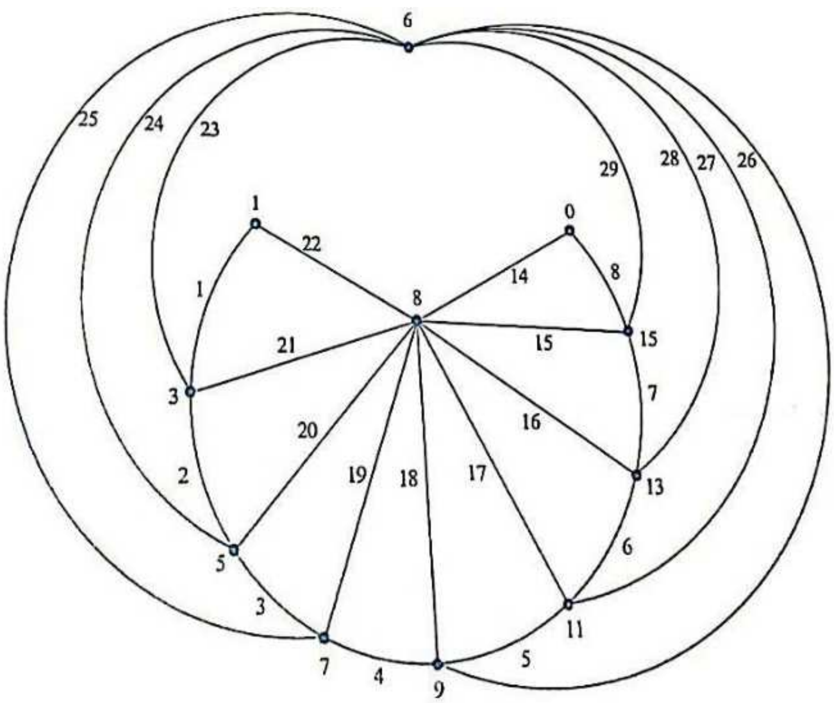

Figure 2.18.1: SEGL of W_{10}

3. STRONG EDGE-GRACEFUL LABELING IN THE CONTEXT OF SOME GRAPH OPERATIONS

Introduction:

In the previous chapter, we discussed strong edge-gracefull labelling of some special graphs. In this chapter, we provide strong edge-graceful labeling of the graphs on the context of some graph operations.

Definition: 3.1

An **Alternative triangular snake** $A(T_n)$ is agraph obtained from a path $u_{1,}u_2, u_3, \ldots, u_n$ by joining $u_i$and u_{i+1} (alternatively) to a new vertex v_i. That is, every alternative edge of a path is replaced by C_3. We observe than n is even.

Theorem: 3.2

An alternative triangular snake $A(T_n)$, (n $\geq$ 2) is a strong edge-graceful graph.

Proof:

Let $\left\{u_i, v_j/1 \leq i \leq n, 1 \leq j \leq \frac{n}{2}\right\}$ and $\left\{e_i, e_j^{'}/1 \leq i \leq n-1, 1 \leq j \leq n\right\}$ be the vertices and edges of $A(T_n)$ as shown in figure 3.2.1.

We note $|V\big(A(T_n)\big) = \frac{3n}{2}|$ and $|E\big(A(T_n)\big)| = 2n - 1$

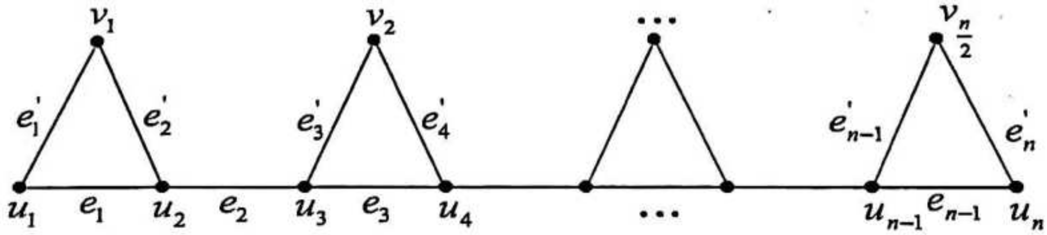

Figure 3.2.1: Ordinary labeling of $A(T_n)$

We first label the edges of $A(T_n)$ as follows:

Define $f: E\big(A(T_n)\big) \to \left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ by

$$f(e_i) = 3n - i - 1, \qquad 1 \le i \le n-2$$

$$f(e_{n-1}) = 1,$$

$$f(e_i{}') = i + 1, \qquad 1 \le i \le n-1$$

$$f(e_n{}') = 2n - 1$$

Then the induced vertex labels are:

$$f^+(u_1) = 0$$

$$f^+(u_i) = 3n - i, \qquad 2 \le i \le n, i \ne n-1$$

$$f^+(u_{n-1}) = 2$$

$$f^+(v_i) = 4i + 1, \qquad 1 \le i \le \frac{n-1}{2}$$

$$f^+\left(\frac{v_n}{2}\right) = 3n - 1$$

Clearly, all vertex labels are distinct. Hence the above defined edge labeling function induces the vertex labeling function $f^+: V\big(A(T_n)\big) \to \{0,1,2,\dots,2p-1\}$.

Hence, f is a strong edge-graceful labeling

Thus, an alternative triangular snake $A(T_n)$ is a strong edge-graceful graph for all n$\ge$ 2.

Illustration: 3.3

Strong edge-graceful labeling of $A(T_8)$ is shown in the figure 3.3.1

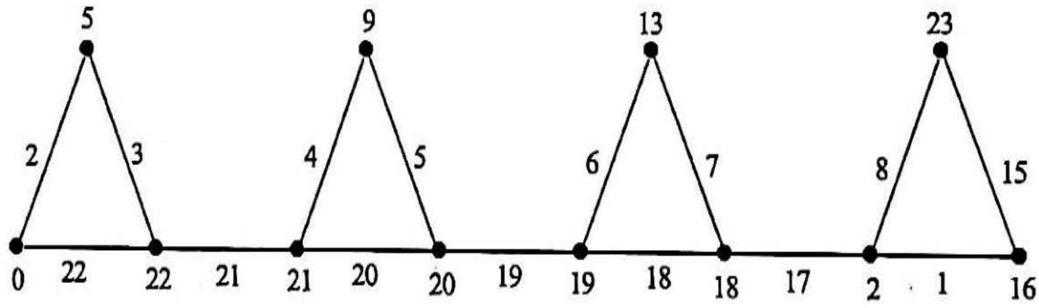

Figure 3.3.1: SEGL of $A(T_8)$

Definition: 3.4

A **Double triangular snake** $D(T_n)$ consists of two triangular snakes that have a common path.

Theorem: 3.5

The double triangular snake D (T_n), (n≥ 2) is astrong edge-graceful graph.

Proof:

Let $\{u_i, w_i, v_j / 1 \leq i \leq n, 1 \leq j \leq n+1\}$ and $\{a_i, b_i e_j / 1 \leq i \leq n-1, 1 \leq j \leq n\}$ be the vertices and edge of $D_2(T_n)$ as shown in the figure 3.5.1

We note that $|V(D(T_n))| = 3n + 1$ and $|E(D(T_n))| = 5n$

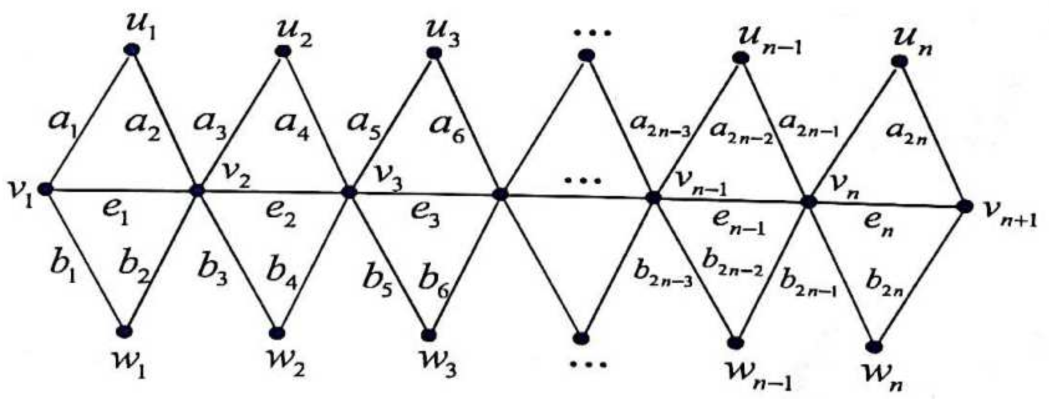

Figure3.5.1: Ordinary labeling of $D(T_n)$

Case1: n > 3

We first label the edges of $A(T_n)$ as follows:

Define $f: E\big(D_2(T_n)\big) \to \left\{1,2, \dots, \left[\frac{3q}{2}\right]\right\}$ by

$f(e_i) = i, \qquad 1 \le i \le n$

$f(a_1) = \begin{cases} 3n+2, & n\ odd \\ 3n+1, & n\ even \end{cases}$

$f(a_i) = \frac{4n-i+2}{2} \qquad 2 \le i \le 2n-2\ and\ i\ even$

$f(a_i) = \frac{8n-i+5}{2} \qquad 3 \le i \le 2n-1\ and\ i\ odd$

$f(a_{2n}) = 6n+3, \qquad f(b_1) = 6n+2,$

$f(a_i) = \frac{4n+i}{2} \qquad 2 \le i \le 2n-2\ and\ i\ even$

$f(a_i) = \frac{8n+i+1}{2} \qquad 3 \le i \le 2n-1\ and\ i\ odd$

Then the induced vertex label are:

$f^+(u_1) = \begin{cases} 5n+2, & n\ odd \\ 5n+1, & n\ even \end{cases}$

$f^+(u_i) = 6n+2i+4\ , \qquad 2 \le i \le n-1$

$f^{+}(u_i) = 3n + 4$

$$f^{+}(v_1) = \begin{cases} 3n+3, & n \text{ odd} \\ 3n+2, & n \text{ even} \end{cases}$$

$f^{+}(v_i) = 2i - 1, \qquad 2 \le i \le n$

$f^{+}(v_{n-1}) = 4n + 1, \qquad f^{+}(w_n) = 2n + 1$

$f^{+}(w_i) = 2i - 2, \qquad 2 \le i \le n$

Case2 : n= 3

Strong edge graceful labeling of $D(T_3)$ is shown in the figure 3.5.2

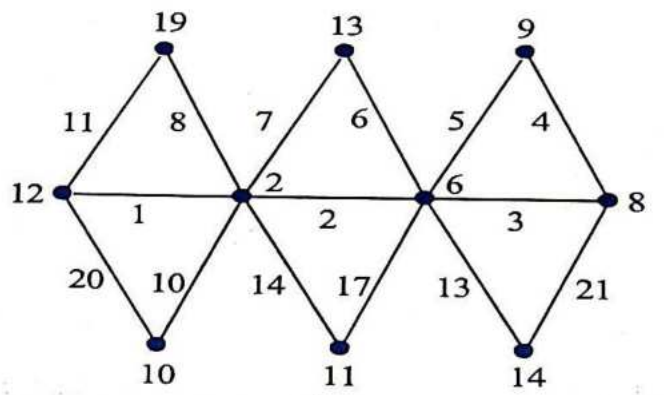

Figure3.5.2: SEGL of $D(T_3)$

Clearly, all vertex labels are distinct. Hence the above defined edge labeling function induces the vertex labeling function $f^{+}: V(D(T_n)) \to \{0,1,2,\ldots,2p-1\}$. Hence, f is a strong edge-graceful labeling. Thus, the double Triangular snake graph D(T_n) is a strong edge-graceful graph for all n≥2.

Illustration: 3.6

Strong edge-graceful labeling of D(T_6) are shown in the figure 3.6.1

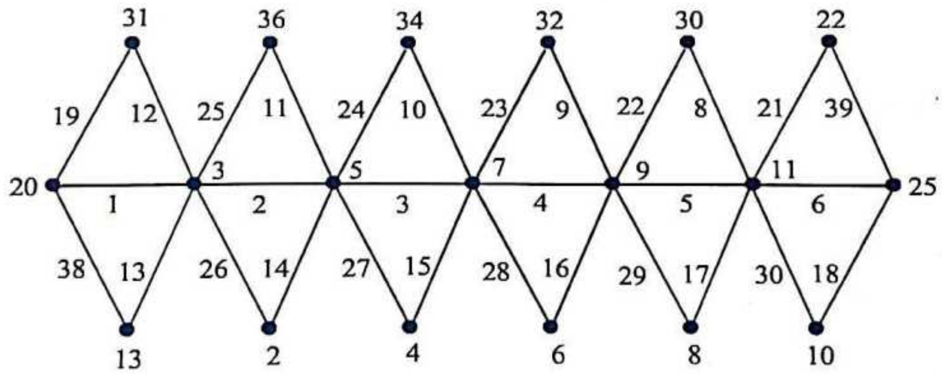

Figure 3.6.1: SEGL of $D(T_7)$

Definition: 3.7

The Cartesian product of G and H is a graph, denoted as GOH, whose vertex set is V(G) $\times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$.

Theorem: 3.8

The Book graph $K_{1,n} \times P_2$ is a strong edge-graceful graph for all n≥ 2.

Proof:

Let $\{u_1, u_2, v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$ be the vertices and $\{e, e_1, e_2, \dots, e_n, e'_1, e'_2, \dots, e'_n, e"_1, e"_2, \dots, e"_n\}$ be the edges of $K_{1,n} \times P_2$ as shown in

the figure 3.8.1. we note that $|V(K_{1,n} \times P_2)| = 2n + 2$ and $|E(K_{1,n} \times P_2)| = 3n + 1$.

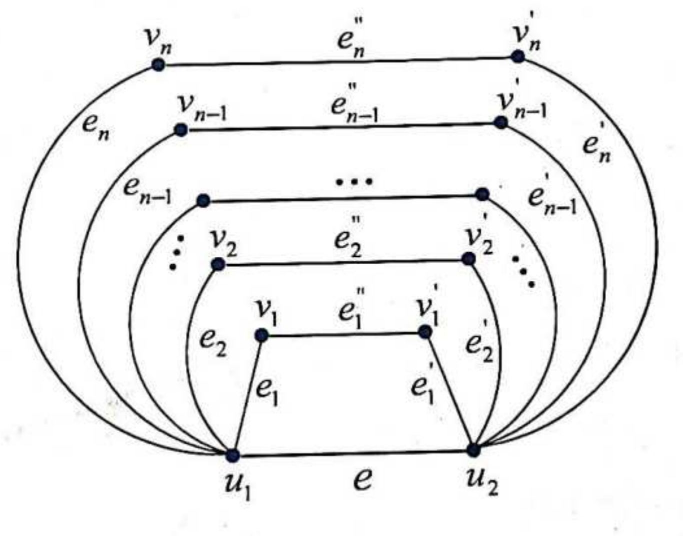

Figure 3.8.1 Ordinary labeling of $K_{1,n} \times P_2$

Case1: n $\equiv$ $0(mod\ 4)$

We first label the edges of as follows:

Define $f: E(K_{1,n} \times P_2) \to \left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ by

$f(e_i) = i, \qquad 1 \le i \le n$

$f(e'_i) = 2n + i, \qquad 1 \le i \le n$

$f(e"_i) = n + 1, \qquad 1 \le i \le n$

$f(e) = 3n + 1,$

Then the induced vertex label are:

$f^+(v_i) = n + 2i\ , \qquad 1 \le i \le n$

$f^+(v'_i) = 3n + 2i\ , \qquad 1 \le i \le \frac{n+2}{2}$

$f^+(v'_i) = 2i - n - 4\ , \quad \frac{n+4}{2} \le i \le n$

$$f^+(u_1) = \begin{cases} n+1, & n\ \equiv 0(mod\ 8) \\ n-i, & n\ \equiv 4(mod8) \end{cases}$$

$$f^{+}(u_2) = \begin{cases} n+1, & n \equiv 0(mod\ 8) \\ 3n+3, & n \equiv 4(mod 8) \end{cases}$$

Case2: n ≡ $\mathbf{1}(\boldsymbol{mod}\ \mathbf{4})$

$$f(e_i) = i, \qquad 1 \le i \le n$$

$$f(e'_i) = 2n + i, \qquad 1 \le i \le n$$

$$f(e"_i) = n + 1, \qquad 1 \le i \le n$$

$$f(e) = 3n + 2,$$

Then the induced vertex label are:

$$f^{+}(v_i) = n + 2i \ , \qquad 1 \le i \le n$$

$$f^{+}(v'_i) = 3n + 2i \ , \qquad 1 \le i \le \frac{n+3}{2}$$

$$f^{+}(v'_i) = 2i - n - 4 \ , \quad \frac{n+5}{2} \le i \le n$$

$$f^{+}(u_1) = \begin{cases} \frac{7n+5}{2}, & n \equiv 1(mod\ 8) \\ \frac{3n+9}{2}, & n \equiv 5(mod 8) \end{cases}$$

$$f^{+}(u_2) = \begin{cases} \frac{7n+9}{2}, & n \equiv 1(mod\ 8) \\ \frac{3n+5}{2}, & n \equiv 5(mod 8) \end{cases}$$

Case3: n ≡ $\mathbf{2}(\boldsymbol{mod}\ \mathbf{4})$

$$f(e_i) = 2i - 1, \qquad 1 \le i \le n$$

$$f(e'_i) = 2i, \qquad 1 \le i \le n$$

$$f(e"_i) = 2n + i, \qquad 1 \le i \le n$$

$$f(e) = \begin{cases} 3n+4 & n \equiv 2(mod\ 12) \\ 3n+2 & n \equiv 6(mod\ 12) \\ 3n+3 & n \equiv 10(mod\ 12) \end{cases}$$

Then the induced vertex label are:

Subcase1: n ≡ $\mathbf{6}(\boldsymbol{mod}\ \mathbf{12})$

$f^+(u_1) = 2$, $\qquad f^+(u_2) = n + 2$

$f^+(v_i) = 2n + 3i - 1, \quad 1 \le i \le \frac{2n+2}{3}$

$f^+(v_i) = 3i - 2n - 5\,, \quad \frac{2n+5}{3} \le i \le n$

$f^+(v'_i) = 2n + 3i, \qquad 1 \le i \le \frac{2n+2}{3}$

$f^+(v'_i) = 3i - 2n - 4, \quad \frac{2n+5}{3} \le i \le n$

Subcase2: n ≡ $\mathbf{6}(\boldsymbol{mod}\ \mathbf{12})$

$f^+(u_1) = 0$, $\qquad f^+(u_2) = n$

$f^+(v_i) = 2n + 3i - 1, \quad 1 \le i \le \frac{2n+2}{3}$

$f^+(v_i) = 3i - 2n - 5\,, \quad \frac{2n+6}{3} \le i \le n$

$f^+(v'_i) = 2n + 3i, \qquad 1 \le i \le \frac{2n+3}{3}$

$f^+(v'_i) = 3i - 2n - 4, \quad \frac{2n+6}{3} \le i \le n$

Subcase3: n ≡ $\mathbf{10}(\boldsymbol{mod}\ \mathbf{12})$

$f^+(u_1) = 1$, $\qquad f^+(u_2) = n + 1$

$f^+(v_i) = 2n + 3i - 1, \quad 1 \le i \le \frac{2n+4}{3}$

$f^+(v_i) = 3i - 2n - 5\,, \quad \frac{2n+7}{3} \le i \le n$

$f^+(v'_i) = 2n + 3i, \qquad 1 \le i \le \frac{2n+1}{3}$

$f^+(v'_i) = 3i - 2n - 4, \quad \frac{2n+4}{3} \le i \le n$

Case4: n ≡ $\mathbf{3}(\boldsymbol{mod}\ \mathbf{4})$

$f(e_i) = 1, \qquad 1 \le i \le n$

$f(e'_i) = 2n + i, \qquad 1 \le i \le n$

$f(e"_i) = n + 1, \qquad 1 \le i \le n$

$f(e) = 4n + 4,$

Then the induced vertex label are:

$f^+(v_i) = n + 2i \ , \qquad 1 \le i \le n$

$f^+(v'_i) = 3n + 2i \ , \qquad 1 \le i \le \frac{n+3}{2}$

$f^+(v'_i) = 2i - n - 4 \ , \ \frac{n+5}{2} \le i \le n$

$$f^+(u_1) = \begin{cases} \frac{3n+3}{2}, & n \equiv 3(mod\ 8) \\ \frac{7n+7}{2}, & n \equiv 7(mod 8) \end{cases}$$

$$f^+(u_2) = \begin{cases} \frac{3n+7}{2}, & n \equiv 3(mod\ 8) \\ \frac{7n+11}{2}, & n \equiv 7(mod 8) \end{cases}$$

Clearly, all vertex labels are distinct. Hence,the above defined edge labeling function induces the vertex labeling function $f^+: V(K_{1,n} \times P_2) \to \{0,1,2, \ldots ,2p - 1\}$.

Hence f is a strong edge-graceful labeling.

Thus, the book graph $K_{1n} \times P_2$ is a strong edge-graceful graph for all n≥ 2

Illustration: 3.9

The strong edge-graceful labeling of $K_{1,6} \times P_2$ is shown in the figure 3.9.1

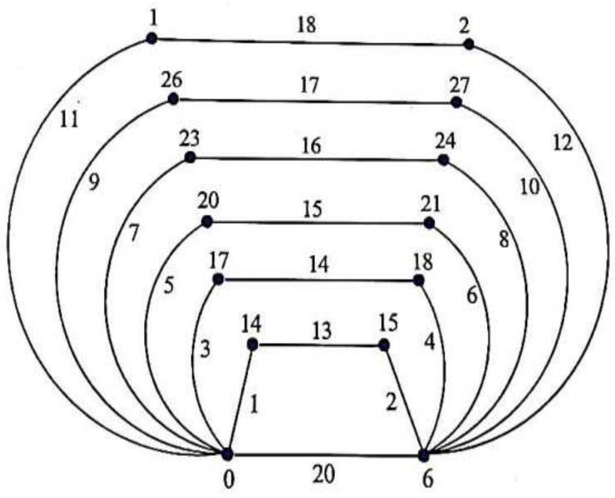

Figure 3.9.1 SEGL of $K_{1,6} \times P_2$

Theorem: 3.10

The ladder L_n=$P_n \times P_2$ is a strong edge graceful graph for all n≥ 2.

Proof:

Let $\{v_i/1 \leq i \leq 2n\}$ and $\{e_i/1 \leq i \leq 3n-2\}$ be the vertices and edges of L_n as shown in the figure 3.10.1. We note that $|V(L_n) = 2n|$ and $|E(L_n)| = 3n - 2$.

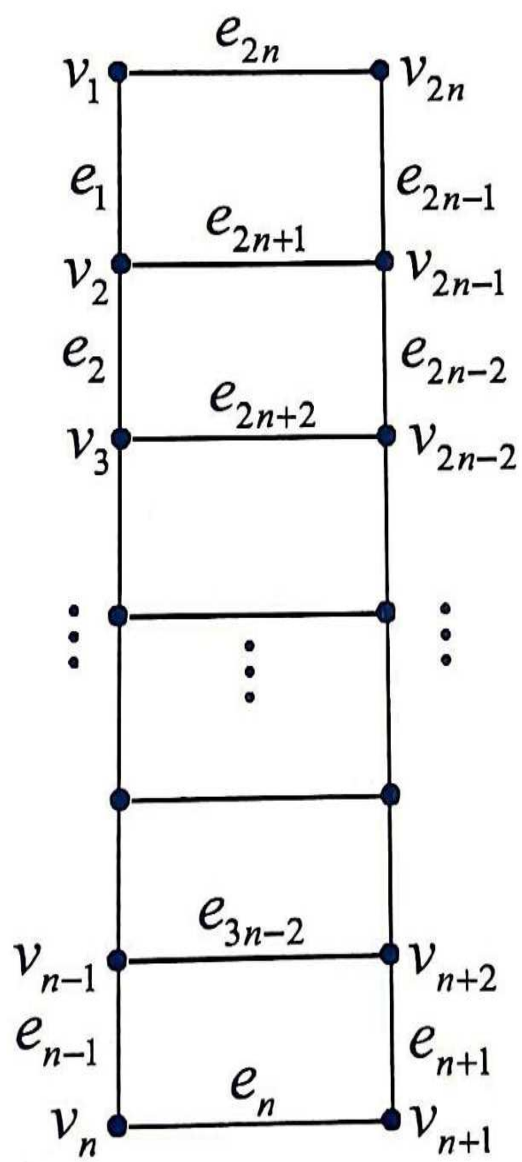

Figure 3.10.1: Ordinary labeling of L_n

We first label the edges of as follows:

Define $f: E(L_n) \rightarrow \left\{1,2,\ldots,\left[\frac{3q}{2}\right]\right\}$ by

$$f(e_i) = i, \qquad 1 \le i \le 3n-2, \qquad i \ne 2n$$

$$f(e_{2n}) = \begin{cases} 3n-1, & n \equiv 0,2,3(mod\ 6) \\ 3n, & n \equiv 1(mod\ 6) \\ 3n+1, & n \equiv 0,2,3(mod\ 6) \end{cases}$$

Then the induced vertex label are:

$$f^+(v_1)=\begin{cases}3n, & n\equiv 0,2,5(mod\ 6)\\ 3n+1, & n\equiv 1(mod 6)\\ 3n+2, & n\equiv 3,4(mod 6)\end{cases}$$

For $n \equiv 2,5\ (mod\ 6)$

$$f^+(v_i)=2n+3i-2\ ,\quad 2\le i\le \frac{2n-3}{3}$$

$$f^+(v_i)=3i-2n-2\ ,\quad \frac{2n+2}{3}\le i\le n-1$$

$$f^+(v_n)=2n-1\ ,$$

$$f^+(v_{n+1})=2n+1\ ,$$

$$f^+(v_i)=i-1\ ,\qquad n+2\le i\le 2n-1$$

$$f^+(v_{2n})=n-2$$

For $n \equiv 0,3\ (mod\ 6)$

$$f^+(v_i)=2n+3i-2\ ,\quad 2\le i\le \frac{2n}{3}$$

$$f^+(v_i)=3i-2n-2\ ,\quad \frac{2n+3}{3}\le i\le n-1$$

$$f^+(v_n)=2n-1$$

$$f^+(v_{n+1})=2n+1$$

$$f^+(v_i)=i-1\ ,\qquad n+2\le i\le 2n-1$$

$$f^+(u_2)=\begin{cases}n-2, & n\ \equiv 1(mod\ 6)\\ n, & n\ \equiv 4(mod\ 6)\end{cases}$$

For $n \equiv 1,4\ (mod\ 6)$

$$f^+(v_i)=2n+3i-2\ ,\quad 2\le i\le \frac{2n+1}{3}$$

$$f^+(v_i)=3i-2n-2\ ,\quad \frac{2n+4}{3}\le i\le n-1$$

$$f^+(v_n)=2n-1\ ,\qquad f^+(v_{n+1})=2n+1$$

$f^+(v_{n+1}) = 2n + 1$,

$f^+(v_i) = i - 1$, $\quad n + 2 \le i \le 2n - 1$

$$f^+(u_2) = \begin{cases} n-2, & n \equiv 1 (mod\ 6) \\ n, & n \equiv 4 (mod\ 6) \end{cases}$$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+: V(L_n) \rightarrow \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the ladder graph L_n is a strong edge-graceful graph for all n≥ 2.

Illustration: 3.11

The strong edge-graceful labeling of L_{13} are shown in the figure 3.11.1

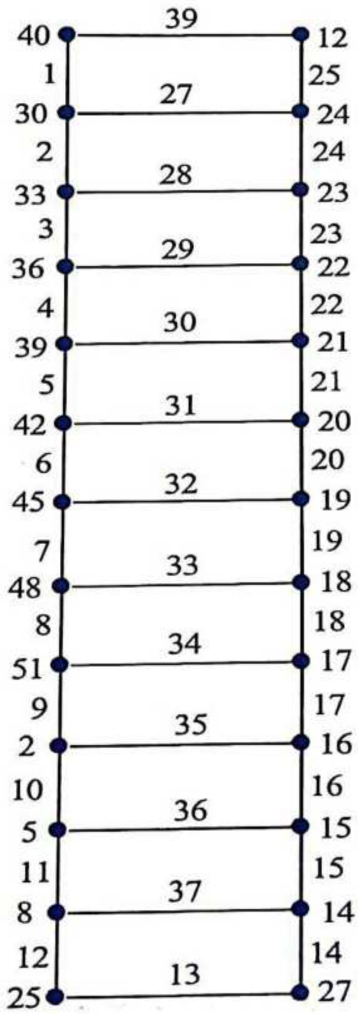

Figure 3.11.1: SEGL of L_{13}

4. STRONG EDGE GRCEFUL LABELING OF PATH AND CYCLE RELATED GRAPHS

Introduction:

In this chapter we discuss about strong edge graceful labeling of path and cycle related graphs.

Definition: 4.1

The **Middle graph** of the graph G is the graph whose vertex set V(G) ∪ E(G) in which two vertices are adjoint if and only if they are adjacent of edges of G or one is a vertex of G and other is an edge incident on it.

The middle graph is denoted by M(G).

Theorem: 4.2

The middle graph M(P_n), n≥ 3 is a strong edge graceful graph.

Proof:

Let $\{u_i, v_j / 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and

$\{e_i, e'_j / 1 \leq i \leq n-2, 1 \leq j \leq 2n-2\}$be the vertices and the edges of M (P_n) as shown in the figure 4.2.1.

We note that $|V(M(P_n))| = 2n - 1$and $|E(M(P_n))| = 3n - 4$.

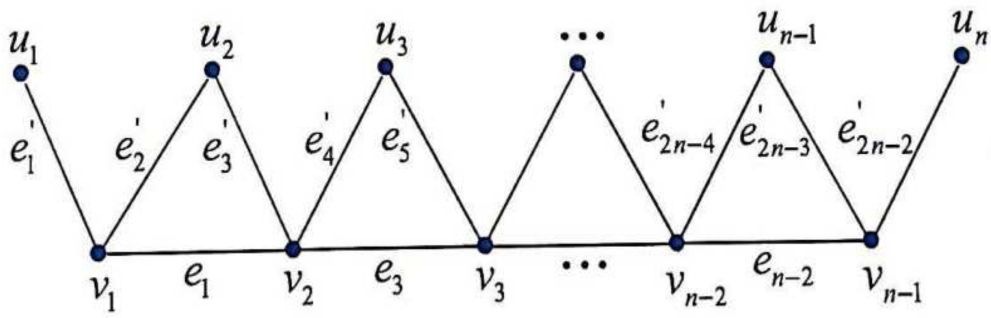

Figure 4.2.1: Ordinary labelling $M(P_n)$

We first label the edges of as follows:

Define $f: E\left(M(P_n)\right) \to \left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ by

$f(e_i) = i + 1, \qquad 1 \le i \le n+2$

$f(e'_i) = q + 3,$

$f(e'_i) = q - i + 2, \qquad 2 \le i \le 2n - 3$

$f(e'_{2n-2}) = 1,$

Then the induced vertex label are:

$f^+(u_1) = q + 3\ ,$

$f^+(u_i) = 2n - 4i + 1\ , \qquad 2 \le i \le \frac{n-1}{2}$

$f^+(u_i) = 6n - 4i - 1\ , \qquad \frac{n+1}{2} \le i \le n - 1$

$f^+(u_n) = 1\ , \qquad f^+(v_1) = 2n - 1$

$f^+(v_i) = 2n - 2i\ , \qquad 2 \le i \le n - 2$

$f(v_{n-1}) = 2n + 1$

Case2: n is even ($\boldsymbol{n \ge 6}$)

$f(e_i) = i + 1, \qquad 1 \le i \le n\text{-}2$

$f(e'_i) = q$

$f(e'_i) = q - i + 1, \qquad 2 \le i \le 2n-3$

$f(e'_{2n-2}) = 1$

Then the induced vertex label are:

$f^+(u_1) = q + 3$,

$f^+(u_i) = 2n - 4i + 1$, $\qquad 2 \le i \le \frac{n-1}{2}$

$f^+(u_i) = 6n - 4i - 1$, $\qquad \frac{n+1}{2} \le i \le n-1$

$f^+(u_n) = 1$, $\qquad f^+(v_1) = 2n - 5$

$f^+(v_i) = 2n - 2i - 2$, $\qquad 2 \le i \le n-2$

$f^+(v_{n-1}) = 2n$

Case3: $n = 3, 4$

Strong edge-graceful labeling of M(P_3) and M(P_4) are shown in the Figure 4.2.2 and figure 4.2.3 respectively.

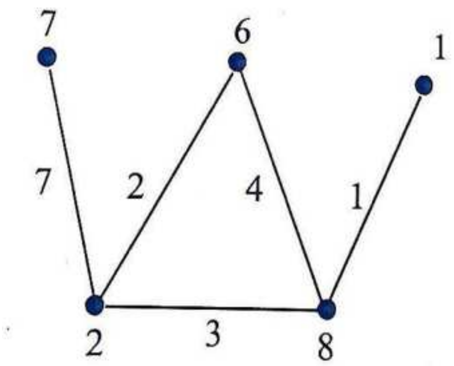

Figure 4.2.2: SEGL of $M(P_3)$

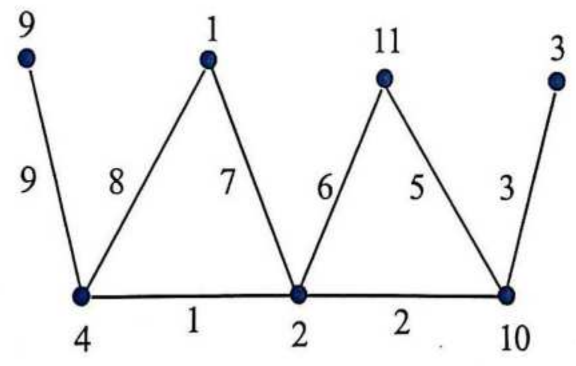

Figure 4.2.3: SEGL of $M(P_4)$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+: V(M(P_n)) \rightarrow \{0,1,2,\ldots,2p-1\}$.

Hence, f is a strong edge-graceful labeling

Thus, the middle graph M(P_n) is a strong edge-graceful graph for all n$\geq$ 3.

Illustration: 4.3

The SEGL of M(P_6) is shown in the figure 4.3.1.

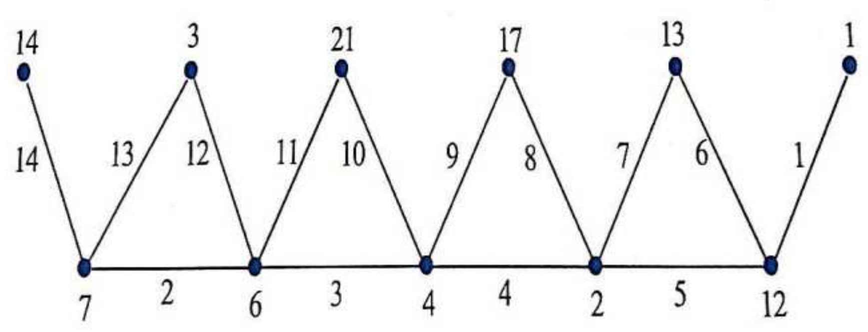

Figure 4.3.1: SEGL of $M(P_6)$

Definition: 4.4

Consider two copies of cycle $C_{n.}$ Then the mutual duplication of a pair of vertices v_k and v'_k respectively from each copy of cyle C_n produces a new graph G G such that N($v_{k)}$=N(v'_k).

Theorem:4.1

The mutual vertex duplication of a cycle $C_n(n \geq 4)$ is a strong edge-graceful graph.

Proof:

Let G$'$ denote the mutual vertex duplication of a cycle C_n. Let $\{u_i, v_i/1 \leq i \leq n\}$ be the vertices and $\{e_i, g_{i,}e'_j g'_j/1 \leq i \leq n, 1 \leq j \leq 2\}$ be the edges of the mutual vertex duplication of a life cycle C_n as shown in the figure 4.5.1.

We note that $|V(G')| = 2n$ and $|E(G')| = 2n + 4$.

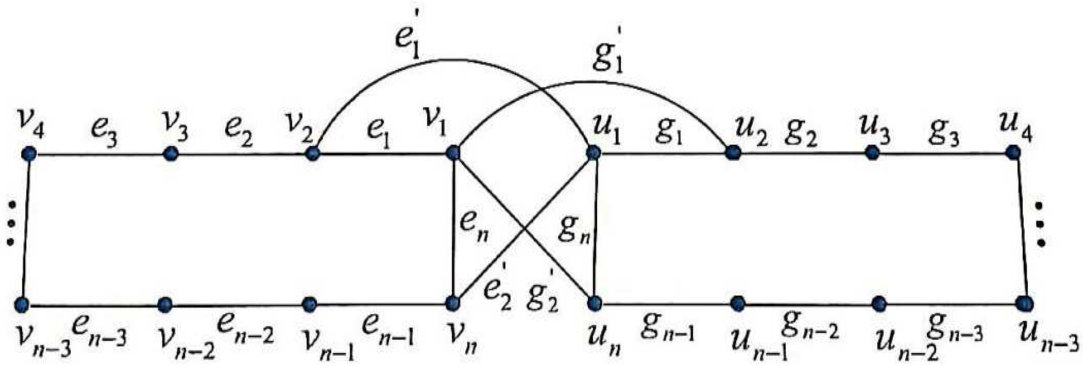

Figure 4.5.1: Ordinary labeling of mutual vertex duplication of a cycle C_n

Case1: n $\neq$ 4, 5, 7, 9

We first label the edges of as follows:

Define $f: E(G') \to \left\{1,2, \dots, \left[\frac{3q}{2}\right]\right\}$ by

$$f(e_i) = i, \qquad 1 \leq i \leq n$$

$f(g_i) = n + i, \qquad 1 \le i \le n$

$f(e'_1) = \begin{cases} 2n+3, & n\ odd \\ 2n+1, & n\ even \end{cases}$

$f(e'_2) = \begin{cases} 2n+3, & n\ odd \\ 2n+1, & n\ even \end{cases}$

$f(g'_1) = \begin{cases} 2n+7, & n\ odd \\ 2n+3, & n\ even \end{cases}$

$f(g'_2) = \begin{cases} 2n+5, & n\ odd \\ 2n+4, & n\ even \end{cases}$

Then the induced vertex label are:

$f^+(v_1) = \begin{cases} n+13, & n\ odd \\ n+8, & n\ even \end{cases}$

$f^+(v_2) = \begin{cases} 2n+6, & n\ odd \\ 2n+4, & n\ even \end{cases}$

$f^+(v_1) = 2i - 1, \qquad 3 \le i \le n-1$

$f^+(v_n) = \begin{cases} 0, & n\ odd \\ 1, & n\ even \end{cases}$

$f^+(u_1) = \begin{cases} 3n+5, & n\ odd \\ 3n+4, & n\ even \end{cases}$

$f^+(v_1) = \begin{cases} 10, & n\ odd \\ 6, & n\ even \end{cases}$

$f^+(u_i) = 2i + 2n - 1, \qquad 3 \le i \le n-1$

$f^+(u_n) = \begin{cases} 2n+4, & n\ odd \\ 2n+3, & n\ even \end{cases}$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+: V(G') \to \{0,1,2,\ldots,2p-1\}$.

Hence, f is a strong edge-graceful labeling.

Thus, the mutual vertex duplication of a cycle C_n(n≥ 4) is a strong edge-graceful graph.

Illustration: 4.6

The SEGL of mutual vertex duplicationC_{10} is given in the figure 4.6.1

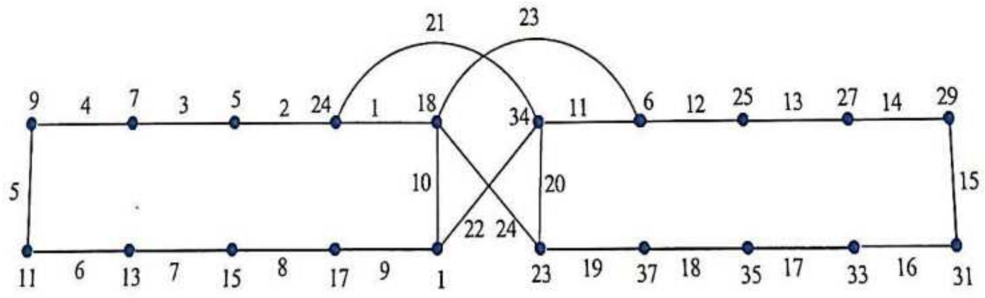

Figure 4.6.1: SEGL of mutual vertex duplication of C_{10}

Definition: 4.7

Let G be a graph with two or more vertices then the **Total graph $T(G)$** pf of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Theorem : 4.8

The total graph $T(P_n)(n \geq 3)$ is a strong edge-graceful graph.

Proof:

Let $\{u_i, v_j, /1 \leq i \leq n-1\}$ be the vertices and

$\{e_i, e'_j, e"_k, /1 \leq i \leq n-1, 1 \leq j \leq 2n-2, 1 \leq k \leq n-2\}$ be the edges of $T(P_n)$ as shown in the figure 4.8.1.

We note that $|V(T(P_n))| = 2n-1$ and $|E(T(P_n))| = 4n-5$.

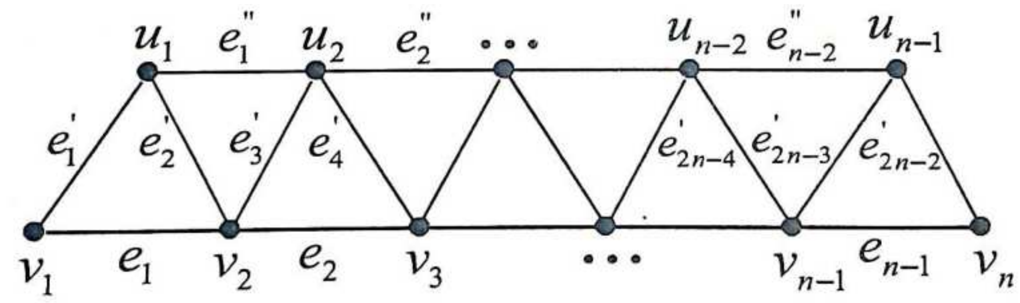

Figure 4.8.1: Ordinary labelling $T(P_n)$

Case1: n ≠ 3, 4

We first label the edges of as follows:

Define $f: E\big(T(P_n)\big) \rightarrow \left\{1,2, \dots, \left\lceil\frac{3q}{2}\right\rceil\right\}$ by

$f(e_1) = 4n - 4$

$f(e_i) = i, \qquad 2 \le i \le n-1$

$f(e'_i) = 1$

$f(e'_i) = 3n - i - 2, \qquad 2 \le i \le 2n-2 \; and \; i \neq 2n-3$

$$f(e'_{2n-2}) = \begin{cases} 2n+7, & n \; odd \\ 2n+3, & n \; even \end{cases}$$

$f(e'_i) = 3n + i - 3, \qquad 1 \le i \le n-2$

Then the induced vertex label are:

$f^+(v_1) = 4n - 3$

$f^+(v_1) = 2n - 7$

$f^+(v_i) = 2n - 2i\,, \qquad 3 \le i \le n-2$

$$f^+(v_{n-1}) = \begin{cases} 3n-2, & n \; odd \\ 3n-1, & n \; even \end{cases}$$

$f^+(v_n) = 2n - 1$

$f^{+}(u_i) = 2n - 3$

$f^{+}(u_i) = 4n - 2i - 6\,, \qquad 2 \leq i \leq n - 2$

$$f^{+}(u_{n-1}) = \begin{cases} n-4, & n\ odd \\ n-3, & n\ even \end{cases}$$

Case2: n = $\mathbf{3, 4}$

SEGL of $T(P_3)$ and $T(P_4)$ are shown in the figure 4.8.2 and figure 4.8.3 respectively.

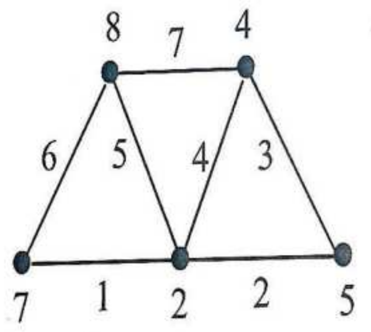

Figure 4.8.2: SEGL of $\boldsymbol{T(P_3)}$

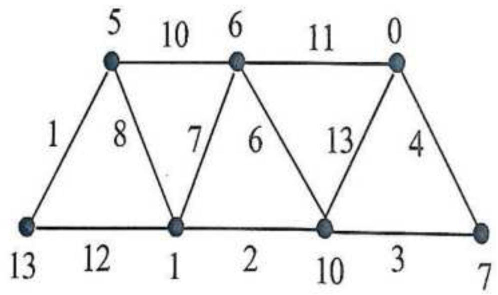

Figure 4.8.3: SEGL of $\boldsymbol{T(P_4)}$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^{+}: V(T(P_n)) \rightarrow \{0,1,2, \ldots, 2p-1\}$.

Hence, f is a strong edge-graceful labeling

Thus, the total graph T(P_n) is a strong edge-graceful graph for all n≥ 3.

Illustration:4.9

The SEGL of T(P_8) is shown in the figure 4.9.1.

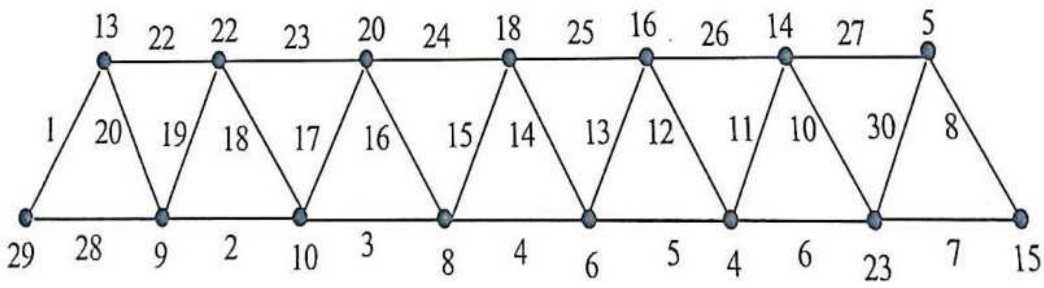

Figure 4.9.1: SEGL of $T(P_8)$

Definition: 4.10

For $p \geq 4$, a cycle (of order p) with one chord is a simple graph obtained from a p-cycle by adding a chord. Let p-cycle be $v_1, v_1, \ldots, v_p v_1$. Without loss of generality, we assume that the chord joins v_1 with any one v_j, where $3 \leq j \leq p-1$.This graph is denoted by $C_p(j)$.

For example $C_p(5)$ means a graph obtained from a p-cycle by adding a chord between the vertices v_1 and v_5. In this graph, $q = p + 1$.

Theorem: 4.11

The graph $C_n(j), (n \geq 4)$ is a strong edge-graceful graph.

Proof:

Let $\{v_i: 1 \leq i \leq n\}$ and $\{e_i, e: 1 \leq i \leq n\}$ be the vertices and the edges of $C_n(j)$ as shown in the figure 4.11.1.

Let $e = (v_1, v_j)$ be the chord. We note that $|V(C_n(j))| = n$ and

$|E(C_n(j))| = n+1$.

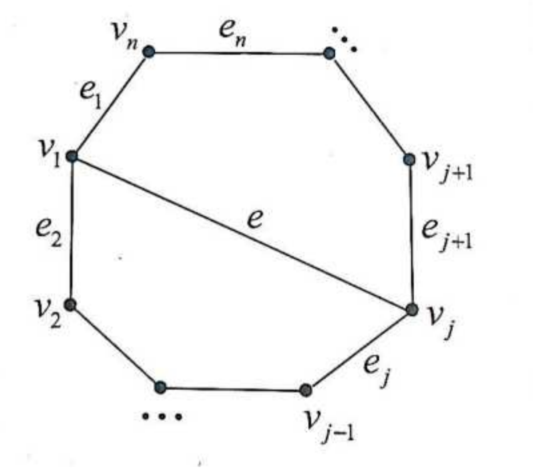

Figure 4.11.1 Ordinary labeling of $C_n(j)$

Case1: n $\boldsymbol{is\ odd\ and\ j \neq \frac{n-1}{2}, 3 \leq j \leq n-1}$

We first label the edges of as follows:

Define $f: E(T(P_n)) \to \left\{1,2, \dots, \left[\frac{3q}{2}\right]\right\}$ by

$$f(e_i) = i + 1 \qquad\qquad 1 \leq i \leq n$$

$$f(e) = 1,$$

Then the induced vertex label are:

$$f^+(v_1) = 6$$

$$f^+(v_i) = 2i + 3 \qquad\qquad 2 \leq i \leq n-3 \ and\ i \neq j$$

$$f^+(v_j) = 2j + 4, \qquad\qquad j \neq n-1, j \neq n-2$$

$$f^+(v_{n-2}) = \begin{cases} 0, & j = n-2 \\ 2n-1, & j \neq n-2 \end{cases}$$

$$f^+(v_{n-1}) = \begin{cases} 1, & j = n-1 \\ 2, & j \neq n-1 \end{cases}$$

$$f^+(v_n) = n+3$$

Case2: n $\boldsymbol{is\ odd\ and\ j \neq \frac{n-1}{2}, (n > 5)}$

$$f(e_i) = n - i + 2 \qquad 1 \leq i \leq n$$

$$f(e) = 1,$$

Then the induced vertex label are:

$$f^+(v_1) = 2$$

$$f^+(v_i) = 2n - 2i + 3 \qquad 2 \leq i \leq n-1 \ and \ i \neq \frac{n-1}{2}$$

$$f^+\left(v_{\frac{n-1}{2}}\right) = n+5\,, \qquad f^+(v_n) = n+3$$

Case3: n $\boldsymbol{is\ odd\ and\ j \neq \frac{n-1}{2}, (n > 5)}$

$$f(e_1) = \begin{cases} n+1, & j \neq \frac{n}{2}+1 \\ n+3, & j = \frac{n}{2}+1 \end{cases}$$

$$f(e_i) = i \qquad 2 \leq i \leq n$$

$$f(e) = 1,$$

Then the induced vertex label are:

$$f^+(v_1) = \begin{cases} n+4, & j \neq \frac{n}{2}+1 \\ n+6, & j = \frac{n}{2}+1 \end{cases}$$

$$f^+(v_i) = 2i+1, \qquad 2 \leq i \leq n-2 \ and \ i \neq j \ and \ j \neq \frac{n}{2}+1$$

$$f^+(v_j) = 2j+2 \qquad 2 \leq i \leq n-1 \ and \ j \neq n-1$$

$$f^+(v_{n-1}) = \begin{cases} 2n-1, & j \neq n-2 \\ 0, & j = n-1 \end{cases}$$

$$f^+(e_n) = \begin{cases} 1, & j \neq \frac{n}{2}+1 \\ 3, & j = \frac{n}{2}+1 \end{cases}$$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^{+}: V\big(C_n(j)\big) \rightarrow \{0,1,2,\ldots,2p-1\}$.

Hence, f is a strong edge-graceful labeling

Thus,T (C_n(j)) is a strong edge-graceful graph for all n≥ 4.

Illustration: 4.12

SEGL of $C_7(5)$ is shown in the figure 4.12.1

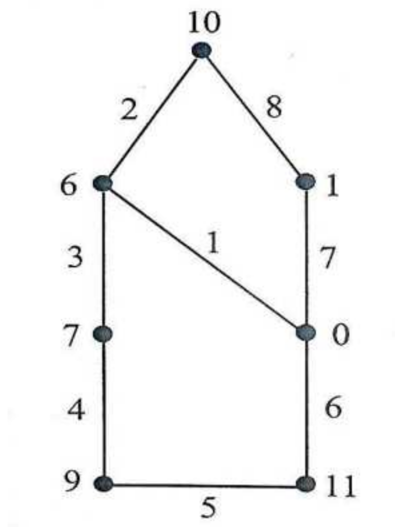

Figure 4.12.1: SEGL of $C_7(5)$

Definition: 4.13

Duplication of a vertex v_k of a graph G produces a new graph $VD(G)$ by adding a new vertex v'_k in such a way that N(v_k) = N (v'_k).

Theorem:4.14

V D (P_n), (n≥ 5) is a strong edge-graceful graph.

Proof:

Let $\{v', v_i/1 \le i \le n\}$ and$\{e_i, e'_j/1 \le i \le n-1, 1 \le j \le 2\}$ be the vertices and the edges of $VD(P_n)$ as shown in the figure 4.14.1.

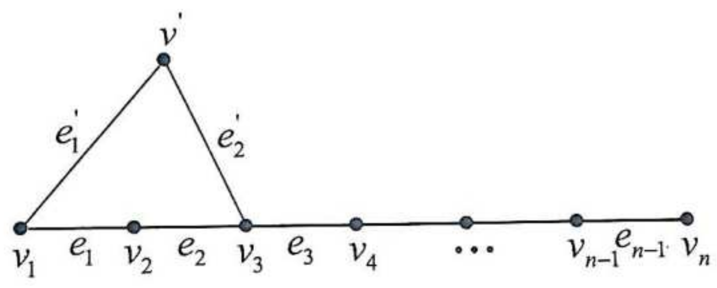

Figure 4.14.1 Ordinary labeling of $VD(P_n)$

We note that $|V(VD(P_n))| = n + 1$ and $|E(VD(P_n))| = n + 1$.

Case1: n ≥ 5 $and\ n \neq 8, 9$

We first label the edges of as follows:

Define $f: E(VD(P_n)) \rightarrow \left\{1,2, \dots, \left[\frac{3q}{2}\right]\right\}$ by

$$f(e_i) = i \qquad 1 \leq i \leq n$$

$$f(e_i) = i + 2, \qquad 3 \leq i \leq n - 2$$

$$f(e_{n-1}) = \begin{cases} n+1, & n\ odd \\ n+2, & n\ even \end{cases}$$

$$f(e'_i) = 5 - i, \qquad 1 \leq i \leq n - 2$$

Then the induced vertex label are:

$$f^+(v_i) = 7 - 2i \qquad 1 \leq i \leq 2$$

$$f^+(v_3) = 10$$

$$f^+(v_i) = 2i + 3\,, \qquad 4 \leq i \leq n - 2$$

$$f^+(v_{n-1}) = \begin{cases} 2n+1, & n\ odd \\ 0, & n\ even \end{cases}$$

$$f^+(v_n) = \begin{cases} n+1, & n\ odd \\ n+2, & n\ even \end{cases}$$

Hence, the above defined edge labeling function induces the vertex labeling function $f^{+}: V(T(P_n)) \rightarrow \{0,1,2, \dots ,2p-1\}$.

Hence, f is a strong edge-graceful labeling

Thus, $VD(P_n)$, (n≥ 5) is a strong edge-graceful graph.

Illustration: 4.15

The SEGL of $VD(P_6)$ is shown in the figure 4.15.1

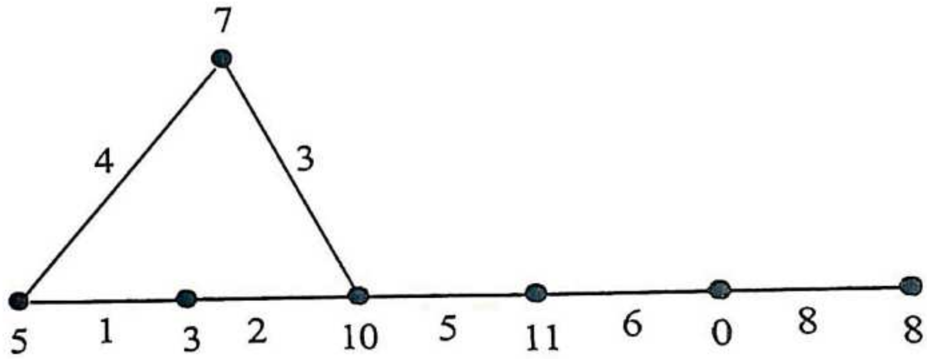

Figure 4.15.1: SEGL of $VD(P_6)$

A STUDY ON SUPER MEAN LABELING

A project submitted to

ST. MARY'S COLLEGE (AUTONOMOUS), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

T.DIYANA JUDE

Reg. No: 19SPMT08

Under the Guidance of

Ms. K. AMBIKA M.Sc., B.Ed., S.E.T.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April - 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON SUPER MEAN LABELING"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the **Degree of Master of Science in Mathematics** and is the work done during the year **2020 – 2021** by **T. DIYANA JUDE (Reg. No: 19SPMT08)**.

K. Sumathi.

Signature of the Guide

P. Anbarasi Anbarasi

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonom
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON SUPER MEAN LABELING"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. K. AMBIKA M.Sc., B.Ed., S.E.T.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

T. Diyana Jude
Signature of the Student

Station: Thoothukudi

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

1 . PRELIMINARIES

Definition: 1.1

A graph G(V,E) consists of a finite non-empty set V=V(G) of p points (called **vertices**) together with a prescribed set E=E(G) of q unordered pair of distinct vertices of V. Each pair e= $\{u,v\}$ of vertices in E is a line (called **edge**) of G.

We write e=uv for an edge and say that u and v are **adjacent vertices**; vertex u and edge e are **incident** with each other,as are v and e.

If two distinct edges e_1 and e_2 are incident with a common vertex then they are **adjacent edges.**

Definition: 1.2

A walk of a graph g is an alternating sequence of vertices and edges $v_0, e_1, v_1, \ldots\ldots v_{n-1}, e_n, v_n$ beginning and ending with vertices, in which each edge is incident with two vertices immediately preceding and following it. This walk joins v_0 and v_n, and may also be denoted $v_0\ v_1\ v_2 \ldots.\ v_n$;it is sometimes called a $v_0 - v_n$walk. It is **closed** if $v_0 = v_n$ and is open otherwise. It is **trail** if all the edges are distinct, and a **path** if all vertices and edges are distinct.

If the walk is closed, then it is a **cycle** provided its n vertices are distinct and n≥3. A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n.

The cycle C_3 is called a triangle.

Definition: 1.3

A vertex v in G is called an **isolated vertex** if deg v=0 and it is called an **end vertex** (or **pendant vertex**) if deg v=1. An edge incident to a pendent vertex is called a **pendant edge.**

Definition: 1.4

The **complete graph** K_p has every pair of its p vertices adjacent. Thus K_p has $\binom{p}{2}$ edges and is regular of degree p – 1. The graph $\overline{K_p}$ is totally disconnected and regular of degree 0 .

Definition: 1.5

A **bigraph** (**or bipartite graph**) G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2.

If G contains every edge joining V_1 and V_2, then G is a **complete bigraph**. If V_1 and V_2 have m and n vertices, We write $G = K_{m,n} = K(m, n)$.

Definitions: 1.6

The **square** G^2 of a graph G has V(G^2) = V(G) with u,v adjacent in G^2 whenever $d(u, v) \leq 2$ in G.

Definition: 1.7

The complete bipartite graph $K_{1,n}$ is called **a star graph** and it is denoted by S_m .

Definition: 1.8

The **union** of two graphs G_1 and G_2 is a graph $G_1 \cup G_2$ with

V(G_1 U G_2) = V(G_1)U V(G_2) and E(G_1 U G_2) = E(G_1)U E(G_2). The distinct union of m

copies of a graph G is denoted by mG.

Definition: 1.9

The **join** of two graphsG_1 and G_2 is a graph $G_1 + G_2$with V(G_1 U G_2)=V(G_1)U V(G_2)

and E(G_1 ∪ G_2)= E(G_1)∪ E(G_2) ∪ $\{uv: u \in V(G_1) and\ V(G_2)\}$.

Definition: 1.10

The graph C_n +v_1 v_3 is obtained from the cycle C_n : v_1 v_2 ….v_n v_1 by adding an edge

between the vertices v_1 and v_3.

Definition: 1.11

The **Corona** $G_1 \Theta G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1

(with p vertices) and p copies of G_2 and then joining the i^{th} vertex in the i^{th} copy of G_2.

Definition: 1.12

The **corona of a graph** G on p vertices $v_1, v_2, \dots, v_p$ is the graph obtained from G by adding p new vertices $u_1, u_2, \dots, u_p$ and the new edges $u_i v_i$ for $1 \leq i \leq p$, denoted by$G \odot K_{1.}$. The graph $P_n \odot K_1$ is called a **comb.**

Definition: 1.13

The **2-corona of G** is the graph obtained from G by identifying the center vertex of the star S_2 a each vertex of G, denoted by G ʘ S_2.

Definition: 1.14

The **balloon of a graph** $G, P_n(G)$ is the graph obtained from G by identifying an end vertex of P_n at a vertex of G.

Definition: 1.15

The **balloon of the triangular snake** $T_n(C_m)$ is the graph obtained from C_m by identifying an end vertex of the basic path in T_n at vertex of C_m

Definition: 1.16

The **H-graph** of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \dots, v_p$ and $u_1, u_2, \dots, u_p$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n}{2}}$ if n is even.

Definition: 1.17

Let$G(V,E)$ be a graph with p vertices and q edges. For every assignment $f: V(G) \rightarrow \{0,1,2,\dots,q\}$, an induced edge labeling $f^*: E(G) \rightarrow \{1,2,\dots,q\}$ is defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2}, & if\ f(u)\ and\ f(v)\ are\ of\ the\ same\ pary \\ \frac{f(u)+f(v)+1}{2}, & Otherwise \end{cases}$$

For every edge $uv \in E(G)$. If $f^*(E) = \{1,2,\dots,q\}$, then we say tthat f is a mean labeling

Of G. If a graph G admits a mean labeling then G is called a **mean graph**.

Example: 1.18

A mean labeling of the cube Q_3 is given in the Figure 1.1

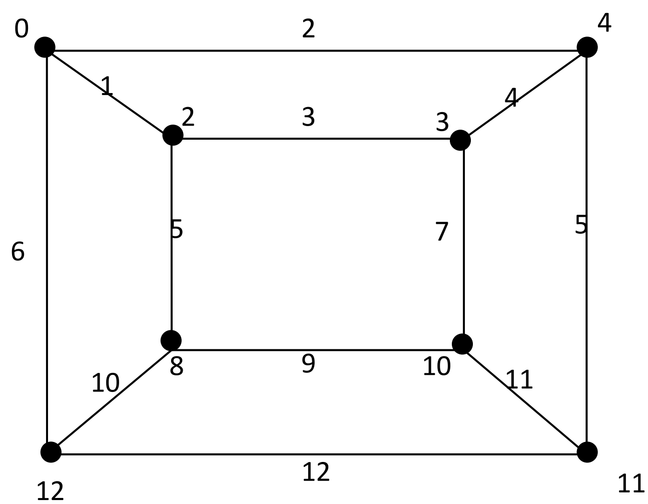


Figure 1.1

2. SUPER MEAN LABELING OF PAH AND H-GRAPH

RELATED GRAPHS

Definition: 2.1

Let G be a graph and : V(G) → $\{1,2,3, \ldots\ldots p+q\}$ be an injection.For each edge

e=uv, the induced edge labeling f^* is defined as follows:

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2}, & if\ f(u)+f(v)\ \ is\ even \\ \frac{f(u)+f(v)+1}{2}, & if\ f(u)+f(v)\ \ is\ odd \end{cases}$$

Then f is called a **super mean labeling** if

f(V(G)) ∪{f*(e) :e ∈ E(G)} = $\{1,2,3, \ldots\ldots p+q\}$

A graph that admits a super mean labeling is called a super mean graph. In this chapter,

the super meanness of the path P_n , comb $P_n \odot K_1$, the H-graph, the corona of a H-graph

2 corona of a H-graph are discussed.

Example: 2.2

The super mean labeling of the graph P_5^2 is shown in Figure 2.1

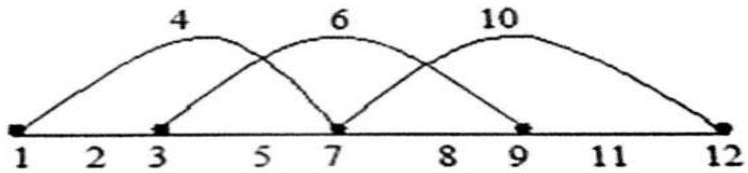


Figure 2.1

Theorem: 2.3

The path P_n is a super mean graph.

Proof:

Let $u_1, u_2, \dots, u_n$ be the vertices of the path P_n .

Define : $V(P_n) \rightarrow \{1,2,3, \dots, p+q\}$ as follows:

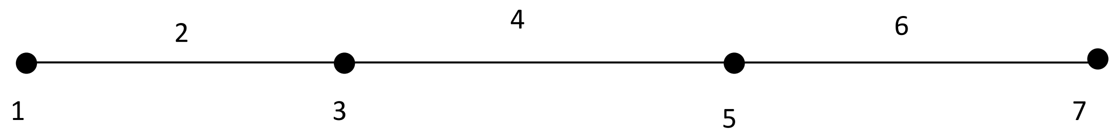
$$f\ (u_i)\ =\ 2i - 1 \qquad\qquad 1 \leq i \leq n$$

For the vertex labeling f , the induced edge labeling f * is defined as follows:

$$f^*\ (u_i u_{i+1})\ = 2i, \qquad\qquad 1 \leq i \leq n-1$$

Clearly, f is a mean labeling. Hence , the path P_n is a super mean graph.

For example, the super mean labeling of the path P_4 is shown in Figure 2.2



P_4

Figure 2.2

Theorem: 2.4

If the path P_n is a super mean , then the comb $P_n \odot K_1$ is a super mean graph.

Proof:

By theorem 2.2, there exists a super mean labeling f for P_n .

Let $u_1, u_2, \dots, u_n$ be the vertices of the P_n.

Let V ($P_n \odot K_1$) = V (P_n) ∪ { $u_1', u_2', \dots . u_n'$ } and

E ($P_n \odot K_1$) = E (P_n) ∪ { $u_i u_i' : 1 \leq i \leq n$ }

Define g : V ($P_n \odot K_1$) → $\{1,2,3, \dots, p+q\}$ as follows:

$$g(u_i) = f(u_i) + 2i, \qquad 1 \leq i \leq n$$

$$g(u_1') = f(u_1)$$

$$g(u_1') = f(u_i) + 2i - 3, \qquad 2 \leq i \leq n$$

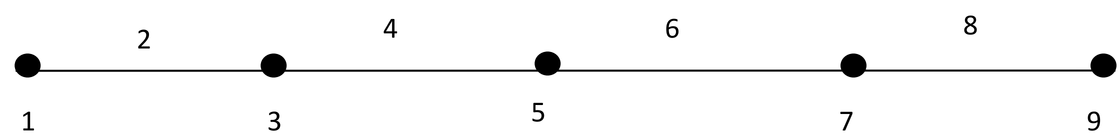
For the vertex labeling g the induced edge labeling g * is defined as follows:

$$g^*(u_iu_{i+1}) = f^*(u_iu_{i+1}) + 2i+1, \quad 1 \le i \le n-1$$

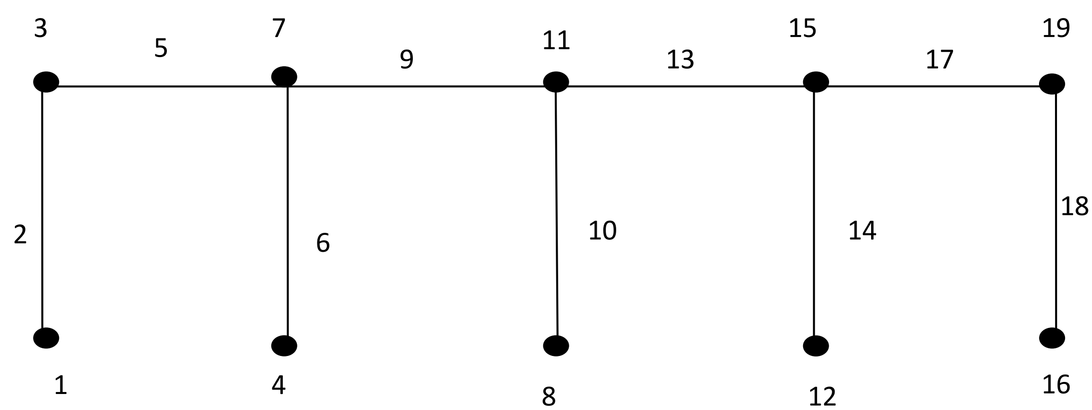
$$g^*(u_iu_i') = f(u_i) + 2i - 1, \quad 1 \le i \le n$$

Then, g is a super mean labeling and hence $P_n \odot K_1$ is a super mean graph.

For example, the super mean labeling of $P_5 \odot K_1$ for the path P_5 is shown in Figure 2.3



P_5



$P_5 \odot K_1$

Figure 2.3

Theorem:2.5

The H-graph G, is a super mean graph.

Proof:

Let $v_1, v_2, \ldots., v_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the H-graph G.

Define : V(G) → $\{1,2,3, \ldots, p+q\}$ as follows:

$f\ (v_i)\ =\ 2i - 1, \qquad 1 \leq i \leq n$

$f\ (u_i)\ =\ 2n + 2i - 1, \qquad 1 \leq i \leq n$

For the vertex labeling , the induced edge labeling $f\ ^*$ is defined as follows:

$f^*\ (v_i v_{i+1})\ = 2i, \qquad 1 \leq i \leq n-1$

$f^*\ (v_i v_{i+1})\ = 2i, \qquad 1 \leq i \leq n-1$

$f^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) = 2n$ if n is odd

$f^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) = 2n$ if n is even

Clearly, f is a super mean labeling. Hence , the H-graph G is a super mean graph.

For example, the super mean labeling of H-graph G_1 and G_2 are shown in Figure 2.4

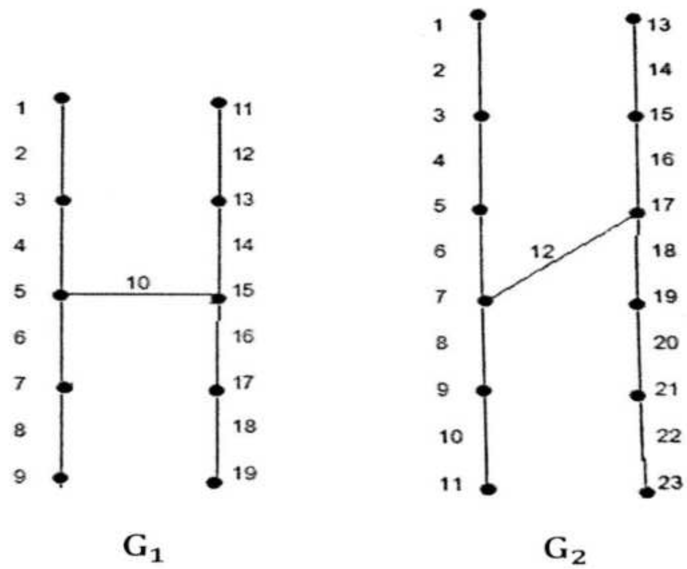


Figure 2.4

Theorem: 2.6

If a H – graph G is a super mean graph, G ⊙ K_1 is a super mean graph.

Proof:

By theorem 2.4, there exists a super mean labeling f for g.

Let $v_1, v_2, \ldots., v_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the G.

Let V (G ⊙ K_1) = V $(G) \cup \{v_1', v_2', \ldots., v_n'\} \cup \{ u_1', u_2', \ldots. u_n' \}$ and

E (G ⊙ K_1) = E (G) $\cup \{ v_i v_i', u_i u_i' : 1 \leq i \leq n \}$

Define g : V (G ⊙ K_1) → $\{1,2,3, \ldots, p+q\}$ as follows:

$$g(v_i) = f(v_i) + 2i, \qquad 1 \le i \le n$$

$$g(u_i) = f(u_i) + 2n + 2i, \qquad 1 \le i \le n$$

$$g(v_1') = f(v_1)$$

$$g(v_i') = f(v_i) + 2i - 3, \qquad 2 \le i \le n$$

$$g(u_i') = f(u_i) + 2n + 2i + 3 \quad 1 \le i \le n$$

For the vertex labeling g, the induced edge labeling g* is defined as follows:

$$g^*(v_i v_{i+1}) = f^*(v_i v_{i+1}) + 2i + 1, \qquad 1 \le i \le n-1$$

$$g^*(u_i u_{i+1}) = f^*(u_i u_{i+1}) + 2n + 2i + 1, 1 \le i \le n-1$$

$$g^*(v_i v_i') = f^*(v_i) + 2i - 1, \qquad 1 \le i \le n$$

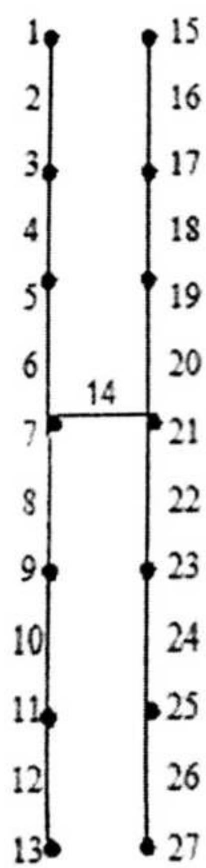
$$g^*(u_i u_i') = f^*(u_i) + 2n + 2i - 1, \qquad 1 \le i \le n$$

$$g^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) = 2 f^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) + 1 \qquad \text{if n is odd}$$

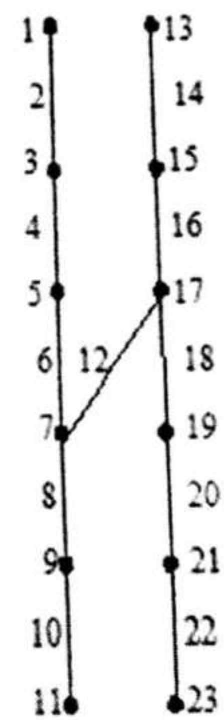
$$g^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) = 2 f^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) + 1 \qquad \text{if n is even}$$

Then, g is a super mean labeling and hence G ʘ K_1 is a super mean graph.

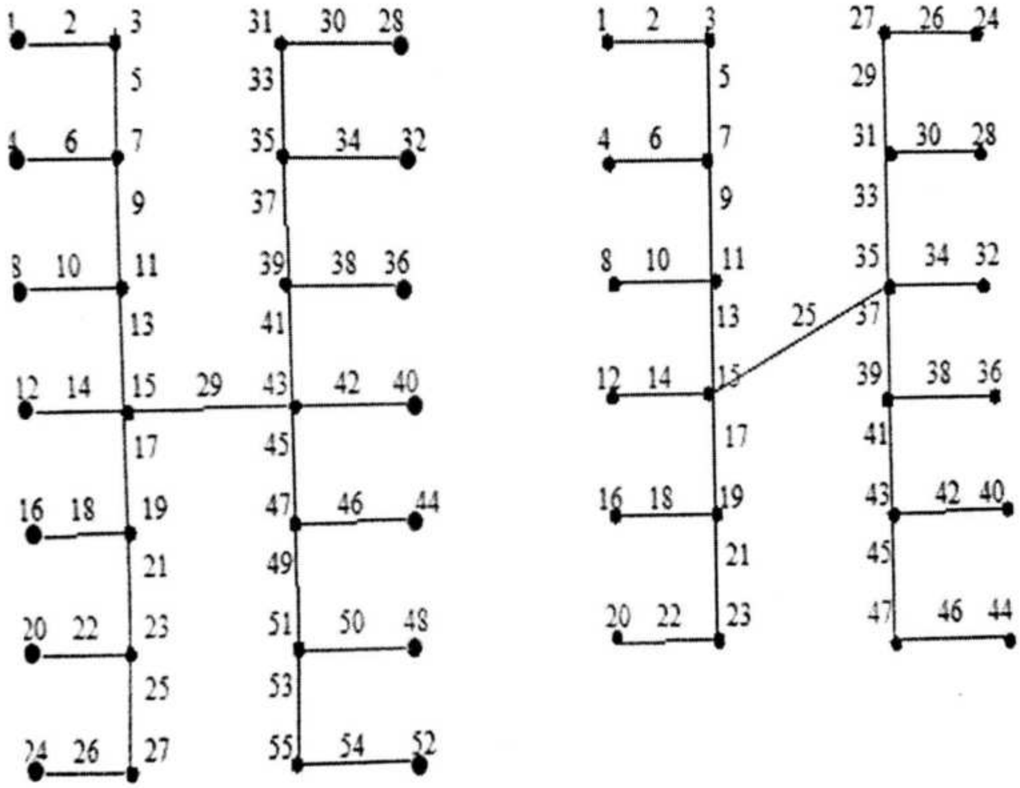
For example, the super mean labeling of $G_1 \odot K_1$ and $G_2 \odot K_1$ for the H-graphs G_1 and G_2 are shown in Figure 2.5



G_1



G_2



$G \odot K_1$ Figure 2.5 $G \odot K_2$

Theorem: 2.7

If a H – graph G is a super mean graph, then $G \odot S_2$ is a super mean graph.

Proof:

By theorem 2.3, there exists a super mean labeling f for G. Let $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the G.

Let $V(G)$ together with $v_1', v_2', \ldots, v_n', v_1'', v_2'', \ldots, v_n'', u_1', u_2', \ldots, u_n'$ and $u_1'', u_2'', \ldots, u_n''$ form the vertex set of $G \odot S_2$ and the edge set is $E(G)$ together with

$\{v_i v_i', v_i v_i'', u_i u_i'', u_i u_i'' : 1 \leq i \leq n\}$

Define g : V (G ⊙ K_1) → $\{1,2,3, \dots, p+q\}$ as follows:

$$g(v_i) = f(v_i) + 4 - 2, \quad 1 \leq i \leq n$$

$$g(v_i') = f(v_i) + 4i - 4, \quad 1 \leq i \leq n$$

$$g(v_i'') = f(v_i) + 4i, \quad 1 \leq i \leq n$$

$$g(u_i) = f(u_i) + 4n + 4i - 2, \quad 1 \leq i \leq n$$

$$g(u_i') = f(u_i) + 4n + 4i - 4, \quad 1 \leq i \leq n$$

$$g(u_i'') = f(u_i) + 4n + 4i, \quad 1 \leq i \leq n$$

For the vertex labeling g, the induced edge labeling g^* is defined as follows:

$$g^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) = 3 f^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) \quad \text{if n is odd}$$

$$g^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) = 3 f^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) \quad \text{if n is even}$$

$$g^*(v_i v_{i+1}) = f^*(v_i v_{i+1}) + 4i, \quad 1 \leq i \leq n-1$$

$$g^*(v_i v_i') = f(v_i) + 4i - 3, \quad 1 \leq i \leq n$$

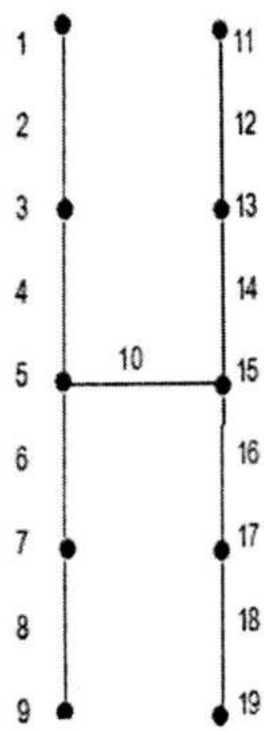
$$g^*(v_i v_i'') = f(v_i) + 4i - 1, \quad 1 \leq i \leq n$$

$$g^*(u_i u_{i+1}) = f^*(u_i u_{i+1}) + 4n + 4i, \quad 1 \leq i \leq n-1$$

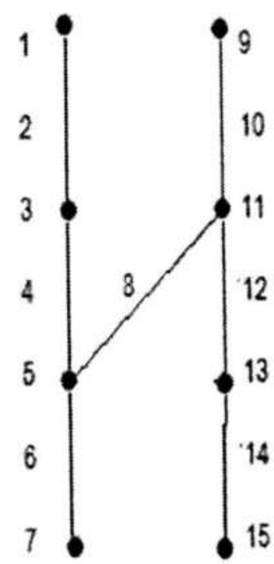
$$g^*(u_i u_i') = f(u_i) + 4n + 4i - 3, \quad 1 \leq i \leq n$$

$$g^*(u_i u_i'') = f(u_i) + 4n + 4i - 1, \quad 1 \leq i \leq n$$

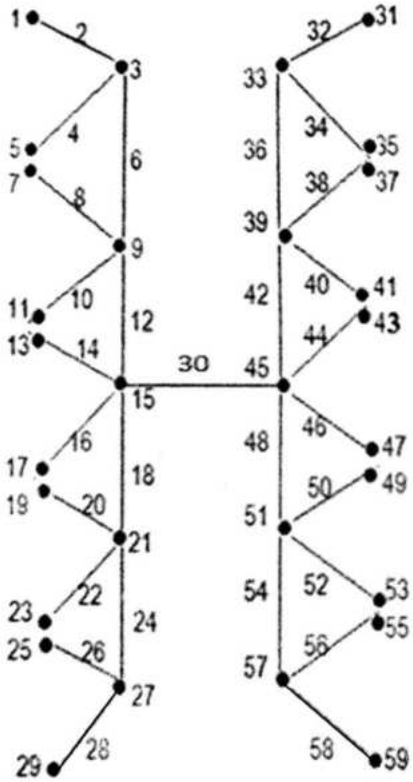
Then, g is a super mean labeling and hence G ʘ S_2 is a super mean graph. For example, the super mean labeling of $G_1 \odot S_2$ and $G_2 \odot S_2$ for the H - graphs G_1 and G_2 are in Figure 2.6



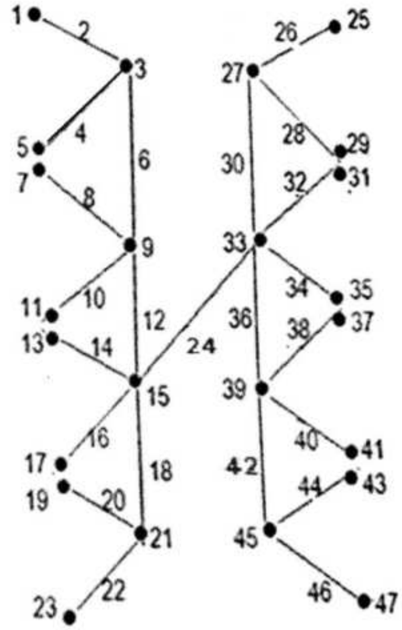
G_1



G_2



$G_1 \odot S_2$



$G_1 \odot S_2$

Figure 2.6

Definition: 2.8

A **triangular snake** T_n obtained from a path with vertices $v_1, v_2, \ldots, v_{n+1}$ by joining v_i and v_{i+1} to a new vertex w_i for $1 \leq i \leq n$, that is, every edge of a path is replaced by a triangle C_3.

Theorem:2.9

The triangular snake T_n Super Mean graph

Proof:

Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the vertices on the path of length n in T_n and let v_i, $1 \leq i \leq n$ be the vertices of T_n in which v_i is adjacent to u_i and u_{i+1}.

Define $f : V(T_n) \to \{1,2,3, \ldots, p+q\}$ as follows:

$$f(u_i) = 5\mathrm{i} - 4, \qquad 1 \leq i \leq n+1$$

$$f(v_i) = 5\mathrm{i} - 2, \qquad 1 \leq i \leq n$$

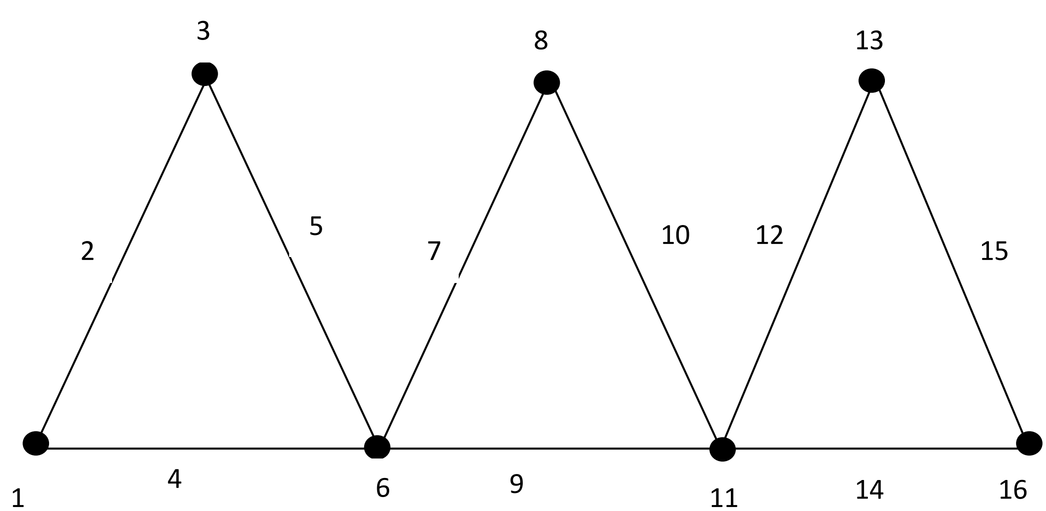
For the vertex labeling f,the induced edge labeling $f*$ is defined as follows:

$$f*(u_i v_i) = 5\mathrm{i} - 3, \qquad 1 \leq i \leq n$$

$$f*(u_i u_{i+1}) = 5\mathrm{i} - 1, \qquad 1 \leq i \leq n$$

$$f*(u_{i+1} v_i) = 5\mathrm{i}, \qquad 1 \leq i \leq n$$

Clearly, f is a super mean labeling. Hence T_n is a super mean graph. For example, the super mean labeling of T_3 is shown in figure2.7



T_3

Figure 2.7

Theorem: 2.10

The graph $T_n \odot K_1$ is a super mean graph, for $n \geq 1$.

Proof:

Let $u_1, u_2, \dots., u_n, u_{n+1}$ be the vertices on the path of length n in T_n and let v_i, $1 \leq i \leq n$ be the vertices of T_n in which v_i is adjacent to u_i and u_{i+1}. Let $v_i' v_i$ be the path attached at each v_i,$1 \leq i \leq n$ and $u_i' u_i$be the path attached at each u_i, $1 \leq i \leq n+1$

Define $f: V(T_n \odot K_1) \rightarrow \{12,3, \dots, p+q\}$ as follows:

$f(u_i)$ = 9i – 6, $1 \leq i \leq n+1$

$f(v_i)$ = 9i – 4, $1 \leq i \leq n$

$f(v_i')$ = 9i – 2, $1 \leq i \leq n$

$f(u_i')$ = 9i – 8, $1 \leq i \leq n+1$

For the vertex labeling f, the induced edge labeling $f*$ is defined as follows:

$f*(u_i u_{i+1})$ = 9i − 1, $1 \leq i \leq n$

$f*(u_i v_i)$ = 9i − 5, $1 \leq i \leq n$

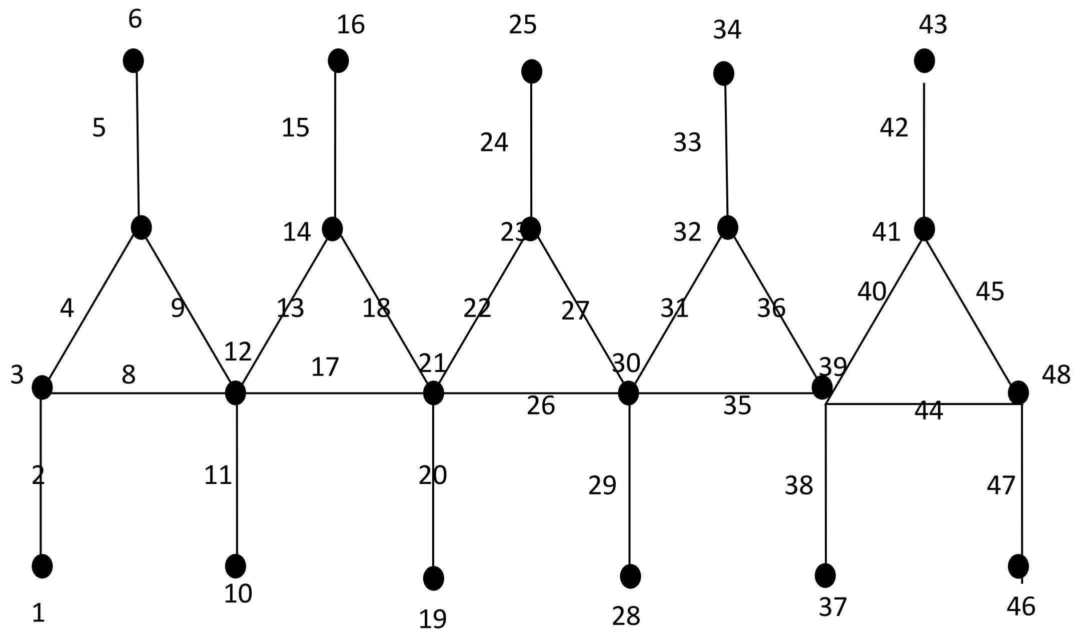
$f*(v_i v_{i+1})$ = 9i, $1 \leq i \leq n$

$f*(v_i v_i')$ = 9i −3, $1 \leq i \leq n$

$f*(u_i u_i')$ = 9i −7, $1 \leq i \leq n+1$

Thus, f is a super mean labeling and hence $T_n \odot K_1$ is a super mean graph, for $n \geq 1$.

For example, the super mean labeling of $T_5 \odot K_1$ is shown in Figure 2.8



$T_5 \odot K_1$

Figure 2.8

3. SUPER MEAN LABELING OF STAR AND CYCLE RELATED GRAPHS

Definition: 3.1

Let G be a graph and $f : V(G) \rightarrow \{1,2,3, \ldots, p+q\}$ be an injection.

For each edge $e = uv$, the induced edge labeling $f*$ is defined as follows:

$$f*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & if\ f(u)+f(v)\ \ is\ even \\ \frac{f(u)+f(v)+1}{2} & \end{cases}$$

Then f is called a **super meanlabeling**if f(V(G)) ∪{f*(e):e∈ E(G)} = $\{1,2,3, \ldots \ldots p+q\}$.

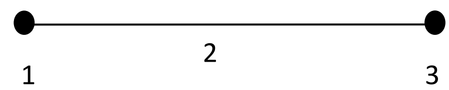
A graph that admits a super mean labeling is called a **super mean graph**. In chapter, the Super meanness of the star $K_{1,n}$, bistar $B_{m,n}$ for $m = n$ or $m = n+1$, C_{2n} for $n \geq 3$ and $n \neq 4$, C_{2n+1} for $n \geq 1$, union of super mean graphs, mC_n- snake for $m \geq 1$,$n \geq 3$ and $n \neq 4$, dragon $P_n(C_m)$ for $m \geq 3$, $C_n + v_1v_3$ for $n \geq 5$,and $C_m \times P_n$ for $m = 3,5$ are discussed.

Theorem: 3.2

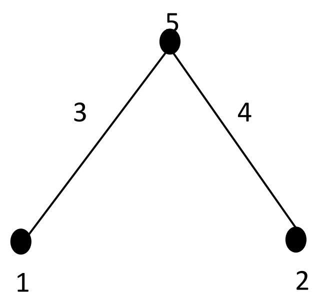
The star $K_{1,n}$ is a super mean graph, $n \leq 3$

Proof:

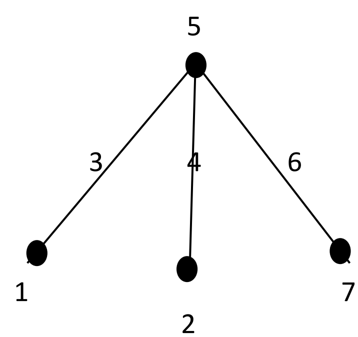
The super mean labeling of $K_{1,1}$, $K_{1,2}$, and $K_{1,3}$ are shown in figure 3.2



$K_{1,1}$



$K_{1,2}$



$K_{1,3}$

Figure 3.2

Remark: 3.3

$K_{1,n}$ is not a super mean graph for $n > 3$ and hence $K_{1,n}$ is a super mean graph except for $n > 3$.

Definition: 3.4

The **bistar** $B_{m,n}$ is the graph obtained from K_2 by identifying the center vertices $K_{1,m}$ and $K_{1,n}$ at the end vertices of K_2 respectively. $B_{m,m}$ is often denoted by $B(m)$.

Theorem: 3.5

The bistar $B_{m,n}$ is a super mean graph for $m = n$ or $m = n + 1$.

Proof:

Case (i): $m = n$

Let $V(B_{m,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ *and*

$E(B_{m,n}) = \{u, v, uu_i, vv_i : 1 \leq i \leq n\}$.

Define $f: V(B_{m,n}) \rightarrow \{1,2,3,,..,p + q\}$ as follows:

$f(u) = 1$

$f(u_i) = 4i - 1, \quad 1 \leq i \leq n$

$f(v) = 4n + 3$

$f(v_i) = 4i + 1, \quad 1 \leq i \leq n$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$f^*(uv) = 2n + 2,$

$f^*(uu_i) = 2i, \quad 1 \leq i \leq n$

$f^*(vv_i) = 2(n+i) + 2, \quad 1 \le i \le n$

Clearly, f is a super mean labeling.

Case (ii): $m = n + 1$

Let $V(B_{n+1,n}) = \{u, v, u_i, v_j : 1 \le i \le n+1, 1 \le j \le n\}$ and

$E(B_{n+1,n}) = \{uv, uu_i, vv_j : 1 \le i \le n+1, 1 \le j \le n\}$.

Define $f: V(B_{n+1,n}) \to \{1,2,3,,..,p+q\}$ as follows:

$f(u) = 3$

$f(u_i) = 4\text{i} - 3, \quad 1 \le i \le n$

$f(v) = 4\text{n} + 5$

$f(v_i) = 4\text{j} + 3, \quad 1 \le i \le n$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$f^*(uv) = 2\text{n} + 4,$

$f^*(uu_i) = 2\text{i}, \quad 1 \le i \le n$

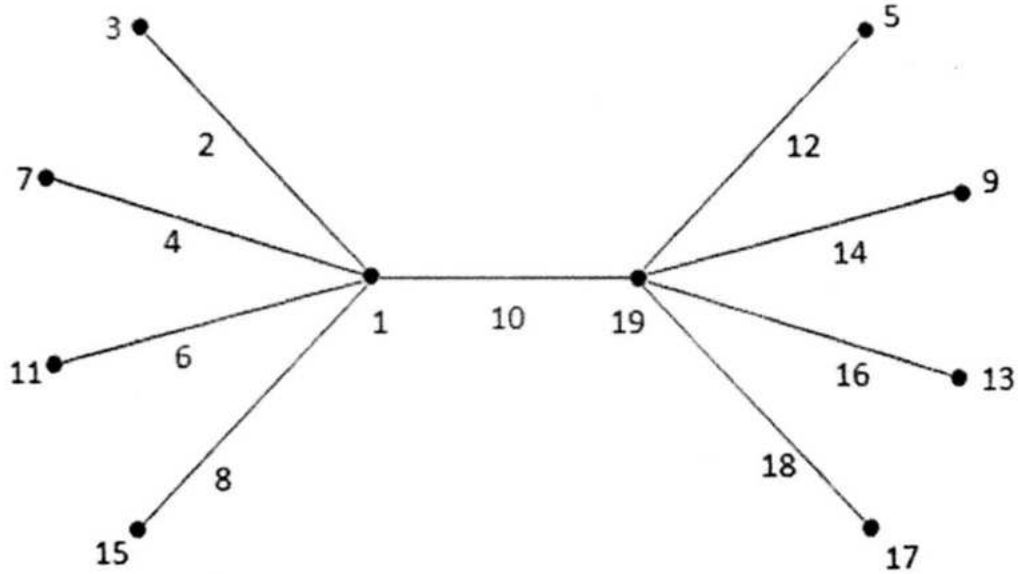
$f^*(vv_i) = 2(n+j) + 4, \quad 1 \le i \le n$

Clearly, f is a super mean labeling.

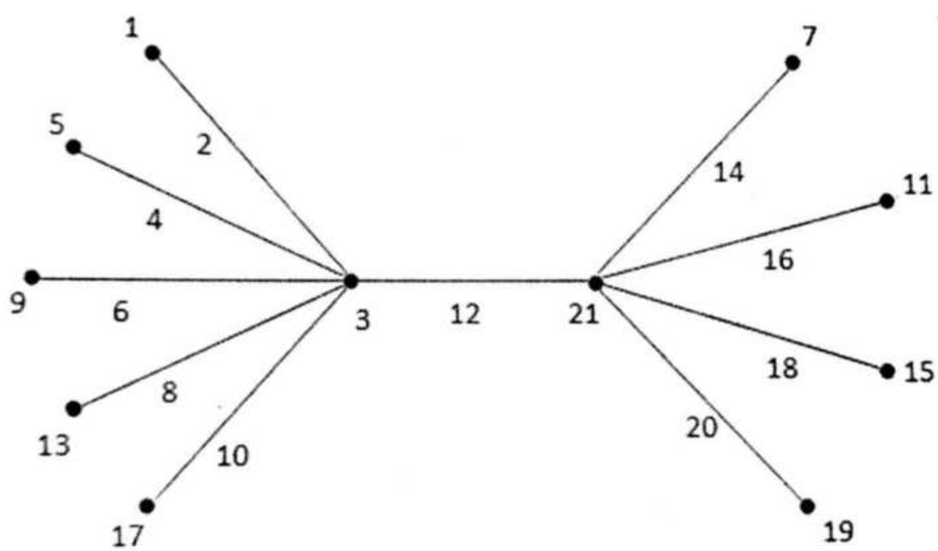
Hence, $B_{m,n}$ is a super mean graph for $m = n$ or $m = n + 1$.

For example, the super mean labeling of $B_{4,4}$ and $B_{5,4}$ are shown in Figure 3.3

$B_{4,5}$ Figure3.3



$B_{4,4}$



$B_{4,5}$ Figure3.3

Theorem: 3.6

Cycle C_{2n} is a super mean graph for $n \geq 3$.

Proof:

Let C_{2n} be a cycle with vertices $u_1, u_2, \dots, u_{2n}$ and edges $e_1, e_2, \dots, e_{2n}$.

Define $f: V(C_{2n}) \rightarrow \{1,2,3, \dots, p+q\}$ as follows:

$f(u_1) = 3$

$f(u_i) = 4i - 5, \quad 2 \leq i \leq n$

$f(u_{n+j}) = 4n - 3j + 3, \quad 1 \leq j \leq 2$

$f(u_{n+j+2}) = 4n - 4j - 2, \quad 1 \leq i \leq n-2$

For the vertex labeling f, the induced edge labeling $f*$ is defined as follows:

$f^*(e_1) = 2$

$f^*(e_i) = 4i - 3, \quad 2 \leq i \leq n-1$

$f^*(e_n) = 4n - 2,$

$f*(e_i) = 4i - 3,$

$f^*(e_{n+1}) = 4n - 1,$

$f^*(e_{n+j+1}) = 4n - 4j, \quad 1 \leq j \leq n+1$

Clearly, f is a super mean labeling and hence C_{2n} is a super mean graph. For example, the super mean labeling of C_{10} is shown in Figure 3.4

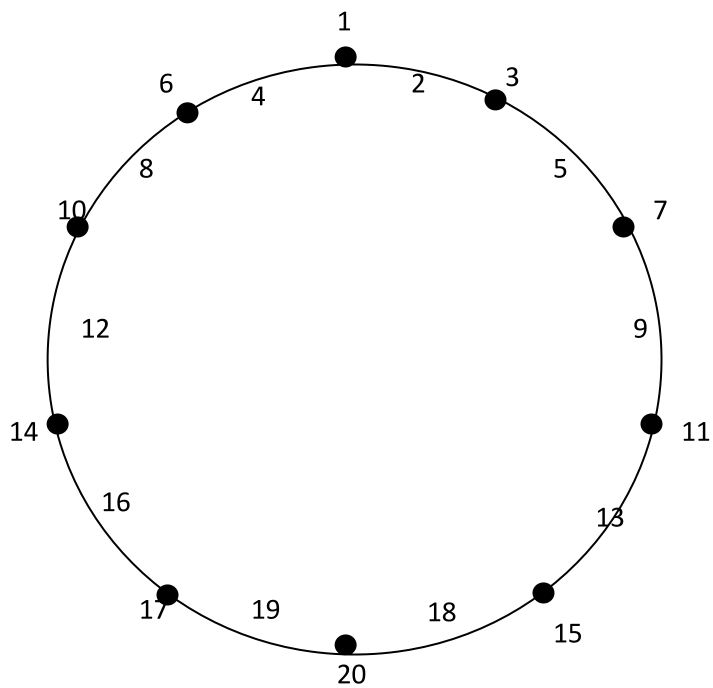


Figure 3.4

Theorem: 3.7

Cycle C_{2n+1} is a super mean graph for $n \geq 1$.

Proof:

Let C_{2n+1} be a cycle with vertices $u_1, u_2, \ldots, u_{2n+1}$ and edges $e_1, e_2, \ldots, e_{2n+1}$.

Define $f: V(C_{2n+1}) \rightarrow \{1,2,3, \ldots, p+q\}$ as follows:

$f(u_1) \quad = 1$

$f(u_i) \quad = 4i - 5, \qquad 2 \leq i \leq n+1$

$$f(u_{n+j+2}) \quad = 4n - 4j + 2, \qquad 0 \leq j \leq n-1$$

For the vertex labeling f, the induced edge labeling $f*$ is defined as follows:

$$f^*(e_1) = 2$$

$$f^*(e_i) = 4i - 3, \qquad 2 \leq i \leq n+1$$

$$f^*(e_{n+j+2}) = 4n - 4j, \qquad 0 \leq j \leq n-1$$

Clearly, f is a super mean labeling and hence C_{2n+1} is a super mean graph.

For example, the super mean labeling of C_9 is shown in Figure 3.5

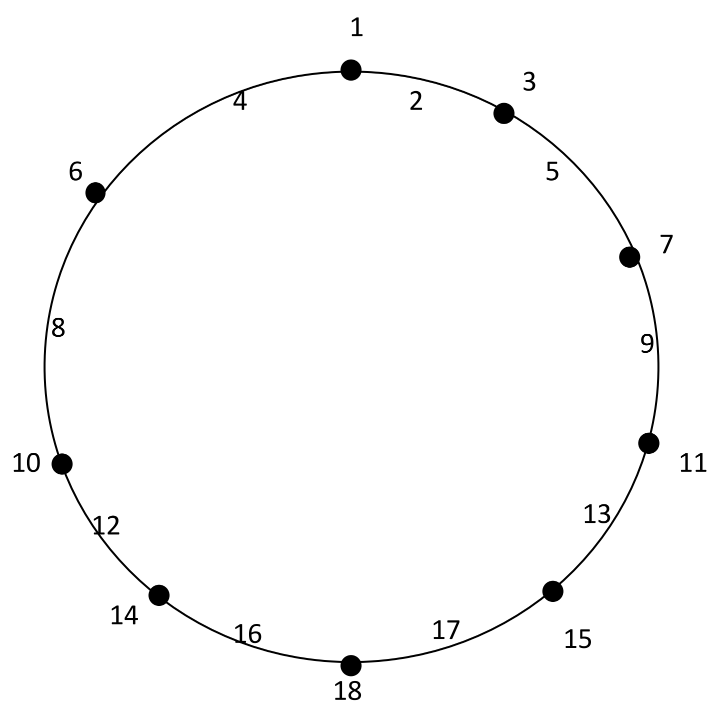


Figure 3.5

Remark: 3.8

C_4 is not a super mean graph and hence the cycle C_n is a super mean graph for $n \geq 3$ and $n \neq 4$.

Theorem: 3.9

If $G_1, G_2, G_3, \ldots, G_m$ are super mean graphs, then $G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_m$ is a super mean graph.

Proof:

If $G_1 = (p_1, q_1)$, $G_2 = (p_2, q_2)$, $G_3 = (p_3, q_3), \ldots, G_m = (p_m, q_m)$ are any m super mean graphs with super mean labeling $f_1, f_2, f_3, \ldots, f_m$ respectively, then $G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_m$ has $p_1 + p_2 + \cdots + p_m$ vertices and $q_1 + q_2 + \cdots + q_m$ edges.

Let $u_{1_i}(1 \leq i \leq p_1), u_{2_i}(1 \leq i \leq p_2), \ldots, u_{m_i}(1 \leq i \leq p_m)$ and $e_{1_i}(1 \leq i \leq q_1), e_{2_i}(1 \leq i \leq q_2), \ldots, e_{m_i}(1 \leq i \leq q_m)$ be the vertices and edges of the graphs of the graphs $G_1, G_2, G_3, \ldots, G_m$ respectively.

Define $g: V[G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_m] \to \{1,2, \ldots, p_1 + p_2 + \ldots + p_m + q_1 + q_2 + \cdots + q_m\}$ as follows:

$g(u_{1_i}) = f(u_{1_i})$

$g(u_{2_i}) = p_1 + q_1 + f(u_{2_i})$

$g(u_{3_i}) = p_1 + p_2 + q_1 + q_2 + f_3(u_{3_i})$

$$g(u_{4_i}) = p_1 + p_2 + p_3 + q_1 + q_2 + q_3 + f_4(u_{4_i})$$

… … … … … … … … … … …

… … … … … … … … … … …

$$g(u_{m_i}) = p_1 + p_2 + p_3 + \cdots + p_{m-1} + q_1 + q_2 + q_3 + \cdots + q_{m-1} + f_m(u_{m_i})$$

$$g * (e_{1_i}) = f_1 * (e_{1_i})$$

$$g^*(e_{2_i}) = p_1 + q_1 + f_2 * (e_{2_i})$$

$$g^*(e_{3_i}) = p_1 + p_2 + q_1 + q_2 + f_3 * (e_{3_i})$$

$$g^*(e_{4_i}) = p_1 + p_2 + p_3 + q_1 + q_2 + q_3 + f_4 * (e_{4_i})$$

… … … … … … … ….

… … … … … … … ….

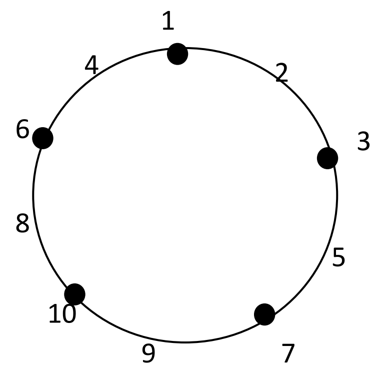
$$g^*(e_{m_i}) = p_1 + p_2 + p_3 + \cdots + p_{m-1} + q_1 + q_2 + q_3 + \cdots + q_{m-1} + f_m * (e_{m_i})$$

Then g is a super mean labeling.

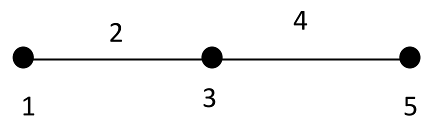
Hence, $G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_m$ is a super mean graph.

For example, the super mean labeling of G_1, G_2, G_3, G_4 and $G_1 \cup G_2 \cup G_3 \cup G_4$

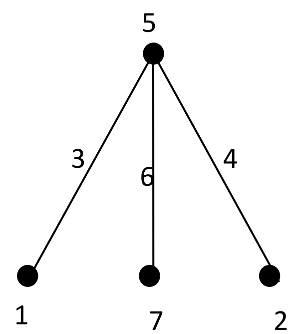
are shown in Figure 3.6



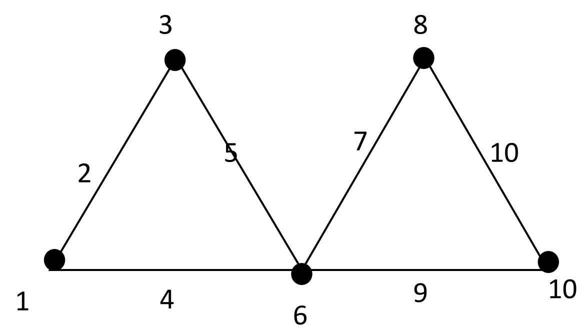
G_1



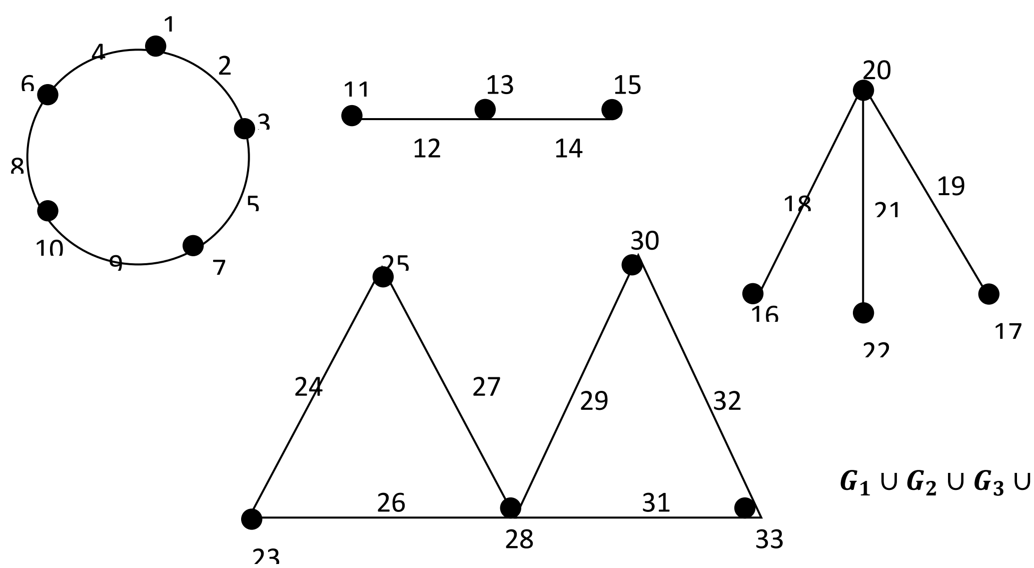
G_2



G_3



G_4



$\boldsymbol{G_1} \cup \boldsymbol{G_2} \cup \boldsymbol{G_3} \cup \boldsymbol{G_4}$

Remark: 3.10

If G is a super mean graph, then mG is also super mean graph, for all $m \geq 1$.

Definition: 3.11

A **cycle snake** mC_n is the graph obtained from m copies of C_n by identifying the vertex $v_{(k+2)_j}$ in the j[th] copy at a vertex $v_{1_{j+1}}$ in the (j+1)[th] copy if $n = 2k + 1$ and identifying the vertex $v_{(k+1)_j}$ in j[th] copy at a vertex $v_{1_{j+1}}$ in the (j+1)[th] copy if $n = 2k$.

Theorem: 3.12

The graph mC_n – snake, $m \geq 1, n \geq 3$ and $n \neq 4$ has a super mean labeling.

Proof:

We prove this result by induction on m.

Let $v_{1_j}, v_{2_j}, \dots, v_{n_j}$ be the vertices and $e_{1_j}, e_{2_j}, \dots, e_{n_j}$ be the edges of $mC_n -$ snake for $1 \leq j \leq m$.

Let f be a super mean labeling nof the cycle C_n.

When $m = 1$, by remark 3.4, C_n is a super mean graph $n \geq 3$ and $n \neq 4$.

Hence the result is true when $m = 1$.

Let $m = 2$

The $2C_n$- snake is the obtained from 2 copies of C_n by identifying the vertex $v_{(k+2)_1}$

in the first copy of C_n at a vertex v_{1_2} in the second copy of C_n, when $n = 2k + 1$

identifying the vertex $v_{(k+1)_1}$ in the first copy of C_n at a vertex v_{1_2} in the second copy of C_n, when $n = 2k$.

Define a super mean labeling g of $2C_n$ −snake as follows:

For $1 \leq i \leq n$,

$$g(v_{i_1}) = f(v_{i_1})$$

$$g(v_{i_2}) = f(v_{i_1}) + 2n - 1$$

$$g^*(e_{i_1}) = f^*(e_{i_1})$$

$$g^*(e_{i_2}) = f^*(e_{i_1}) + 2n - 1$$

Thus, $2C_n -$ snake is a super mean graph.

Assume that $mC_n -$ snake is a super mean graph for any $m \geq 1$. To complete the induction process, it is enough to prove that $(m + 1)C_n$- snake is a super mean graph.

Define a vertex labeling g of $(m + 1)C_n$- snake as follows:

$$g\left(v_{i_j}\right) = f(v_{i_1}) + (j - 1)(2n - 1), \quad 1 \leq i \leq n, 2 \leq j \leq m$$

$$g(v_{i_{m+1}}) = f(v_{i_1}) + m(2n - 1), \qquad 1 \leq i \leq n.$$

For this vertex labeling g, the induced edge labeling g^* is defined as follows:

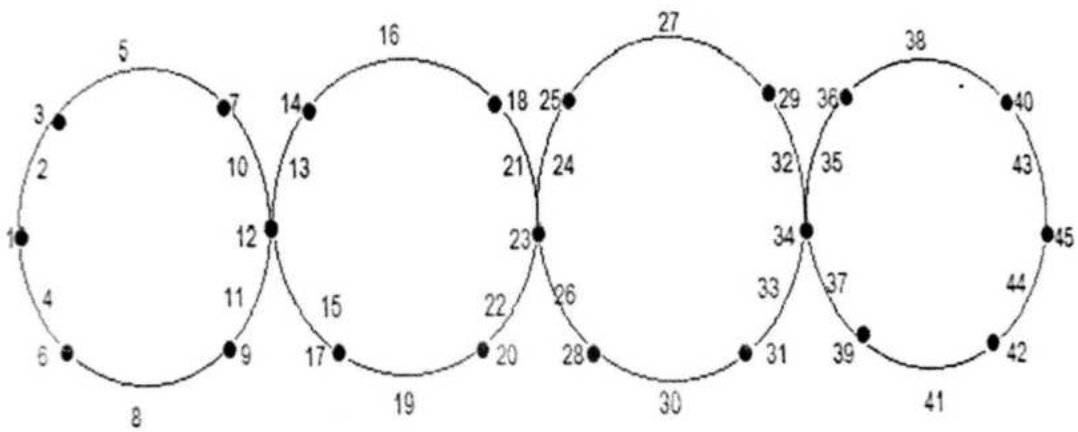
$$g^*\left(e_{i_j}\right) = f^*(e_{i_1}) + (j - 1)(2n - 1), \quad 1 \leq i \leq n, 2 \leq j \leq m$$

$$g^*(e_{i_{m+1}}) = f^*(e_{i_1}) + m(2n-1), \qquad 1 \le i \le n.$$

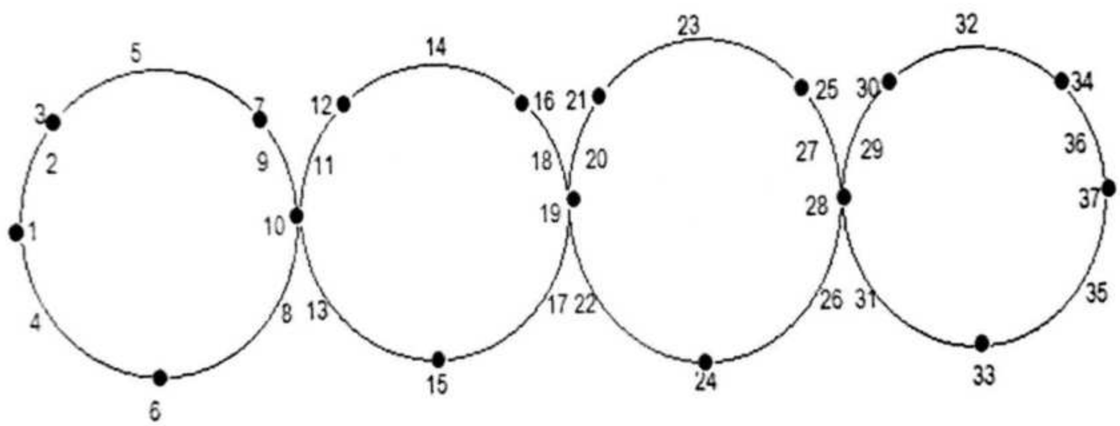
Then, g is a super mean labeling of $(m+1)C_n$-snake and hence $(m+1)C_n$- snake is a super mean graph.

Hence, mC_n – snake is a super mean graph for $m \geq 1, n \geq 3$ and $n \neq 4$.

For example, the super mean labeling of $4C_6$ - snake and, $4C_5$ - snake are shown in Figure3.7



$4C_6$- snake



$4C_5$- snake Figure 3.7

Theorem: 3.13

If G is a super mean graph, then $P_n(G)$ is also a super mean graph.

Proof:

Let f be a super mean labeling of G. Le $v_1, v_2, \dots, v_p$ be the vertices and $e_1, e_2, \dots, e_p$ be the edges of G and let $u_1, u_2, \dots, u_p$ and $E_1, E_2, \dots, E_{n-1}$ be the vertices and edges of P_n respectively.

Define g on $P_n(G)$ as follows:

$$g(v_i) = f(v_i) \qquad 1 \le i \le p$$

$$g(u_j) = p + q + 2j - 2, \qquad 1 \le j \le n.$$

For the vertex labeling g, the induced edge labeling g^* is defined as follows:

$$g^*(e_i) = f^*(e_i), \qquad 1 \le i \le q$$

$$g^*(E_j) = p + q + 2j - 1, \qquad 1 \le j \le n - 1.$$

Clearly, g is a super mean labeling of $P_n(G)$.Hence $P_n(G)$ is a super mean graph.

Definition: 3.14

The **dragon** is a graph formed by joining the end vertex of a path (P_n) to a vertex of the cycle (C_m) and is denoted by $P_n(C_m)$.

Corollary: 3.15

Dragon $P_n(C_m)$ is a super mean graph for $m \geq 3$ and $m \neq 4$.

Proof:

Since C_m is a super mean graph for $m \geq 3$ and $m \neq 4$.

By theorem 3.11, $P_n(C_m)$ for $m \geq 3$ and $m = 4$ is also a super mean graph.

For example, the super mean labeling of $P_5(C_6)$ is shown in Figure 3.9.

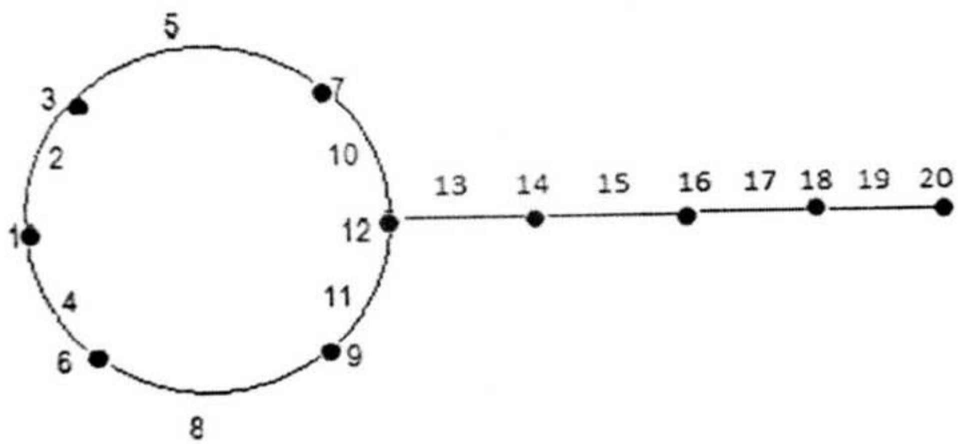


Figure 3.8

Remark : 3.16

The converse of the theorem 3.11 need not true.

For example, consider the graph $C_4.P_n(C_4)$ for $n \geq 3$ is a super mean graph butC_4 is not a super mean graph.

A super mean labeling of the graph $P_4(C_4)$ is shown in Figure 3.9.

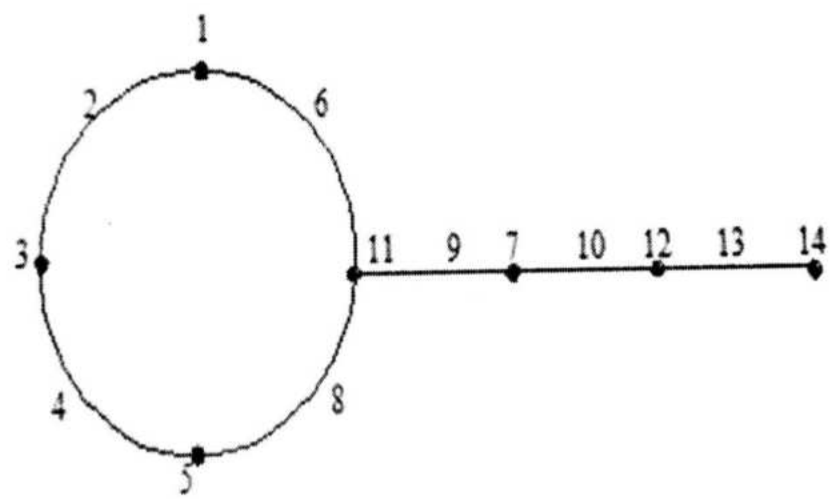


Figure 3.9

4. SUPER MEAN NUMBER OF GRAPHS

Definition: 4.1

Let $f: V(G) \to \{1,2,\dots,n\}$ be a function such that the label of the edge uv is $\frac{f(u)+f(v)}{2}$ or $\frac{f(u)+f(v)+1}{2}$ according as $f(u)+f(v)$ is even or odd and $f(V(G)) \cup \{f^*(e): \in E(G)\} \subseteq \{1,2,3,\dots,n\}$. If n is the smallest positive integer Satisfying all the above conditions that all the vertex and edge labels are distinct and there is no common vertex and edge labels, then n is called the Super mean number of a graph G and it is denoted by $S_m(G)$.

For example,$S_m(K_{1,4}) = 10$ is shown in Figure 4.1

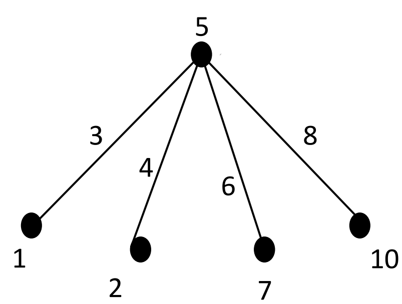


Figure 4.1

It is observed that $S_m(G) \geq p + q$, where p is the order and q is the size of the graph G.

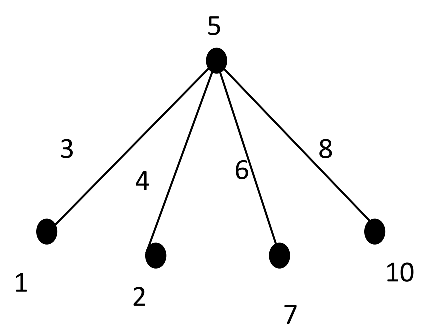
Clearly, the equality holds for all super mean graphs.Super Mean number of some standard graphs.

Theorem: 4.2

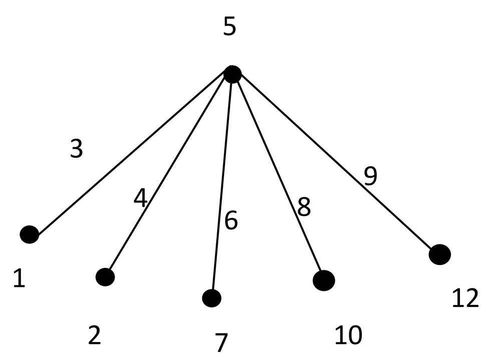
$S_m(K_{1,n}) = 2n + 2$ for $n = 4,5,6$.

Proof:

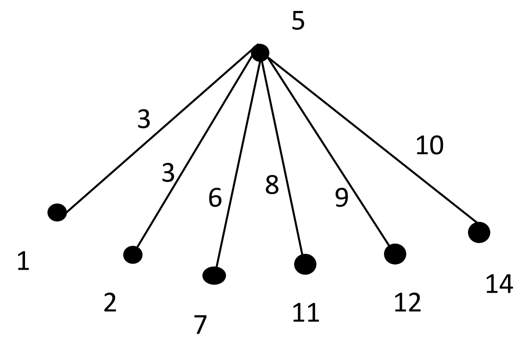
The vertex labeling and the corresponding induced edge labeling of $K_{1,4}, K_{1,5}, K_{1,6}$ are given in Figure 4.2



$K_{1,4}$



$K_{1,5}$



$K_{1,6}$

Figure 4.2

From the Figure 4.2, $S_m(K_{1,4}) = 10, S_m(K_{1,5}) = 12$ and $S_m(K_{1,6}) = 14$.

Thus we concluded $S_m(K_{1,n}) = 2n + 2$ for $n = 4,5,6$.

Hence the result follows:

Theorem: 4.3

$S_m(K_{1,n}) \leq 4n - 10, n \geq 7.$

Proof:

Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ and

$E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$

Define f on $V(K_{1,n})$ as follows:

$f(v) = 5,$

$f(v_i) = i, \qquad 1 \le i \le 2$

$f(v_3) = 7,$

$f(v_i) = 2i + 3, \qquad 4 \le i \le 5$

$f(v_i) = 15 + 4(i - 6), \qquad 6 \le i \le n - 1$

$f(v_n) = 4n - 10.$

Clearly, the vertex labels and edge labels are distinct and no vertex and edge labels are equal.

Hence, $S_m(K_{1.n}) \le 4n - 10$ for $n \ge 7$.

Remark: 4.4

The star $K_{1,n}$ is a super mean graph for $n \le 3$, by remark 3.1

Hence, $S_m(K_{1,n}) = 2n + 1$ for $n \le 3$.

Theorem: 4.5

$S_m(tK_{1,n}) \le (2n + 1)t + 1$ for $n = 5,6$ and $t > 1$.

Proof :

Let $v_0, v_{i_j}, 1 \le j \le t, 1 \le i \le n$ be the vertices and $v_{0_j}v_{i_j}, 1 \le j \le t, 1 \le i \le n$ be the edges of $tK_{1,n}$.

Define f on $V(tK_{1.n}), n = 5,6$ as follows:

When $t = 1$ an $n = 5$.

Define $f(v_{0_1}) = 5$,

$$f(v_{i_1}) = i, \quad 1 \leq i \leq 2.$$

$$f(v_{i_1}) = 7 + 3(i - 3), \; 3 \leq i \leq 4.$$

$$f(v_{5_1}) = 12.$$

When $t = 1$ and $n = 6$

$$f(v_{0_1}) = 5,$$

$$f(v_{i_1}) = i, \; 1 \leq i \leq 2,$$

$$f(v_{3_1}) = 7,$$

$$f(v_{4_1}) = 11,$$

$$f(v_{5_1}) = 12,$$

$$f(v_{6_1}) = 14.$$

For $t > 1$, label the vertices of $tK_{1,5}$ and $tK_{1,6}$ as follows:

$$f\left(v_{0_j}\right) = f(v_{0_1}) + (2n + 1)(j - 1), \;\; 2 \leq j \leq t,$$

$$f(v_{1_2}) = f(v_{1_1}) + 2n,$$

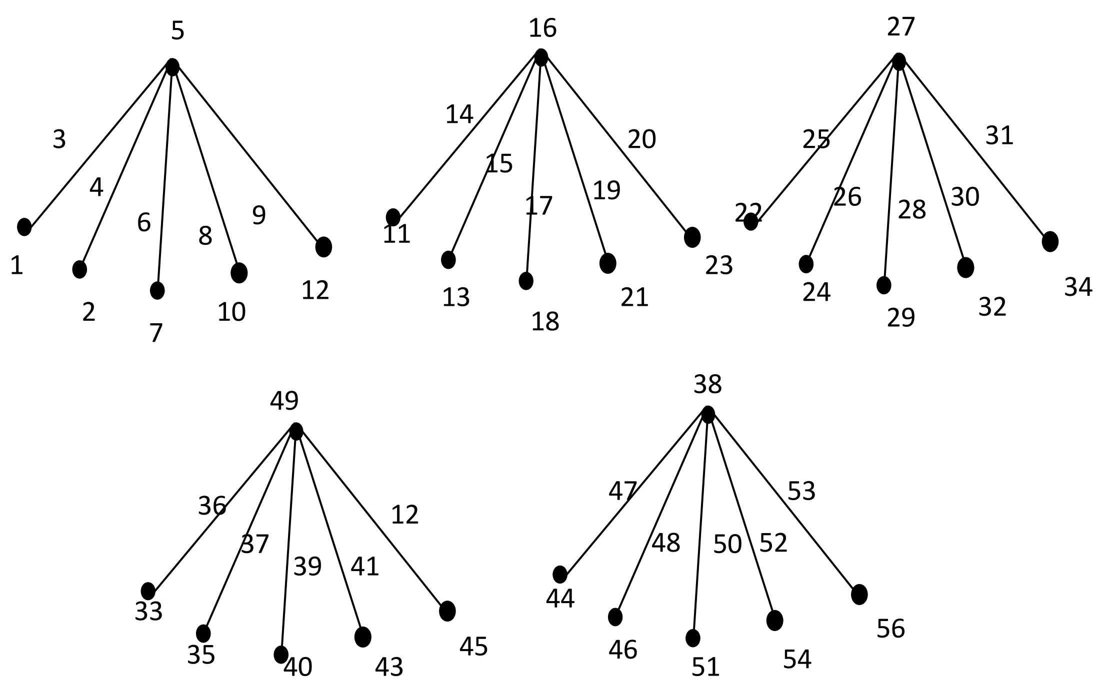
$f\left(v_{1_j}\right) = f(v_{1_2}) + (2n + 1)(j - 2), \;\; 3 \leq j \leq t$, and

$$f\left(v_{i_j}\right) = f\left(v_{i_1}\right) + (2n+1)(j-1), \quad 2 \le j \le t, 2 \le i \le n.$$

Clearly, the vertex labels are distinct. Also the vertex labeling f induces distinct edge labels and $f\left(E(G)\right) \subseteq \{1,2,3 \subseteq (2n+1)t+1\} - f\left(V(G)\right)$.

Hence $S_m\left(tK_{1,n}\right) \le (2n+1)t+1$.

According to theorem 4.3, in the following Figure 4.3, the labeling of $5K_{1,5}$ shows that $S_m\left(K_{1,5}\right) \le 56$.



$5K_{1,5}$

Figure 4.3

Theorem: 4.6

When t is an odd integer. $S_m(tK_{1,n}) \leq t(2n+2)+3$ for $n > 6$.

Proof:

Let $V(tK_{1,n}) = \left\{v_{0_j}, v_{i_j}: 1 \leq i \leq n, 1 \leq j \leq t\right\}$ and $E(tK_{1,n}) = \left\{\begin{matrix} v_{0_j}, v_{i_j}: 1 \leq i \leq n, \\ 1 \leq j \leq t \end{matrix}\right\}$

Let $t = 2k+1$ for some $k \in Z^+$.

Define f on $V(tK_{1,n})$ as follows:

For $1 \leq j \leq 2k$,

$$f\left(v_{0_{2j+1}}\right) = (4n+4)j+1,, \quad 0 \leq j \leq k-1.$$

$$f\left(v_{0_{2j}}\right) = (4n+3)j+j-1, \quad 1 \leq j \leq k.$$

$$f\left(v_{i_{2j+1}}\right) = (4n+4)j+4i-1, \quad 1 \leq j \leq k-1, 1 \leq i \leq n \text{ and}$$

$$f\left(v_{i_{2j}}\right) = (4n+4)(j-1)+4i+1, \quad 1 \leq j \leq k, 1 \leq i \leq n.$$

When $j = 2k+1$,

$$\left(v_{0_{2k+1}}\right) = 4+(2n+2)2k,$$

$$\left(v_{i_{2k+1}}\right) = i+(2n+2)2k-1, \qquad 1 \leq i \leq 2,$$

$$\left(v_{3_{2k+1}}\right) = 4+(2n+2)2k,$$

$$\left(v_{i_{2k+1}}\right) = 2i+(2n+2)2k+2, \qquad 4 \leq i \leq 5,$$

$\left(v_{i_{2k+1}}\right) = 4i + (2n+2)2k - 10, \quad 6 \le i \le n-1$ and

$\left(v_{n_{2k+1}}\right) = (2n+2)2k + 4n - 11.$

Clearly, the vertex labels and the induced edge labels are distinct and further $f\big(V(G)\big) \cap f\big(E(G)\big) = \emptyset$. $S_m\big(tK_{1,n}\big) \le t(2n+2) + 3$ for $n > 6$ and t is an odd integer.

Thoerem: 4.7

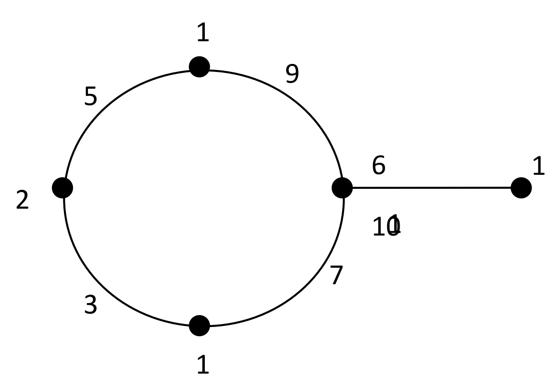
$\langle C_m, K_{1,n}\rangle$ is a super mean graph for $n \le 4$ and $m \ge 3$.

Proof:

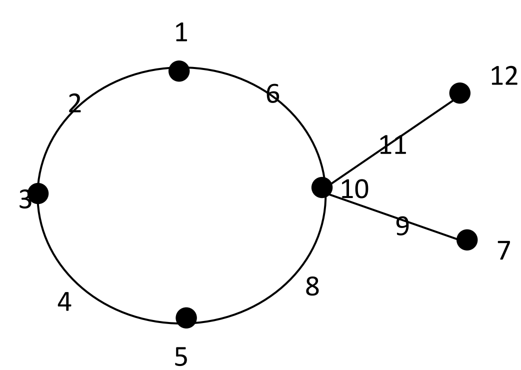
Let $V\big(\langle C_m, K_{1,n}\rangle\big) = \{u_1, u_2, \dots\dots, u_m, v_1, v_2, \dots\dots, v_n, v = u_1\}$ and

$E\big(\langle C_m, K_{1,n}\rangle\big) = \{u_1u_2, u_2u_3, \dots\dots, u_mu_1, u_1v_i : 1 \le i \le n\}.$

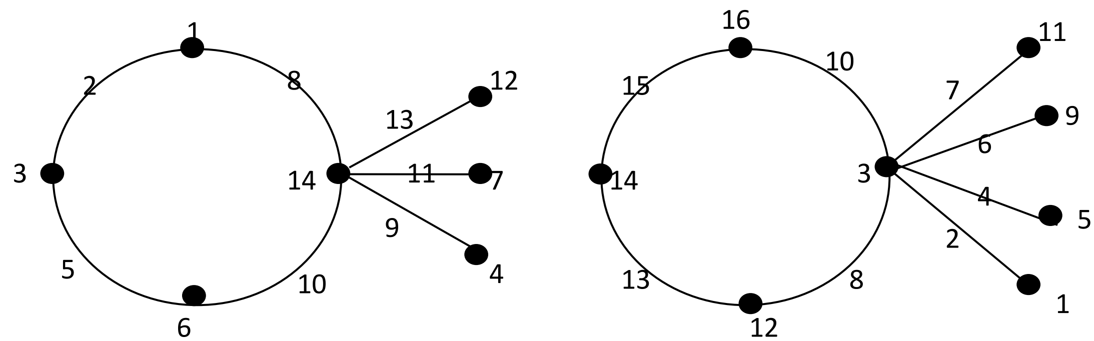
For $m = 4$, the super mean labeling of the graphs $\langle C_4, K_{1,1}\rangle$, $\langle C_4, K_{1,2}\rangle$, $\langle C_4, K_{1,3}\rangle$, and $\langle C_4, K_{1,4}\rangle$ are shown in Figure 4.4



$\langle C_4, K_{1,1}\rangle$



$\langle C_4, K_{1,1}\rangle$



$\langle C_m, K_{1,3}\rangle$ $\langle C_m, K_{1,4}\rangle$

Figure 4.4

Case (i) : $n = 1,2$

Define f on $V\langle\langle C_m, K_{1,n}\rangle\rangle, n = 1,22, m \geq 3$ and $m \neq 4$ as follows:

Subcase (i) When m is odd.

Let $m = 2k + 1, k \in Z^+$.

$f(v_i) = i, \quad 1 \leq i \leq n$

$f(u_1) = 2n + 1,$

$f(u_j) = 2n + 4j - 5, \quad 2 \leq j \leq k + 1$

$f(u_{k+1+j}) = 2n + 4k - 4j + 6, \quad 1 \leq j \leq k.$

Subcase (ii) When m is even.

Let $m = 2k, k \in Z^+$.

$f(v_i) = i, \quad 1 \leq i \leq n$

$f(u_1) = 2n + 1,$

$f(u_j) = 2n + 4j - 5, \qquad 2 \le j \le k.$

$f(u_{k+j}) = 2n + 4j - 3(j - \qquad 1 \le j \le 2.$

$f(u_{k+2+j}) = 2n + 4k - 4j - 2, \quad 1 \le j \le k - 2.$

Clearly, f induces distinct edge labels and it can be verified that f induces a super mean Labeling and hence $\langle C_m, K_{1,n} \rangle$. $n = 1,2, m \ge 3$ and $m \ne 4$ is a super mean graph.

Case (ii) $n = 3,4$

Define f on $V(\langle C_m, K_{1,n} \rangle), n = 3,4, m \ge 3$ and $m \ne 4$ as follows;

Label the vertices of $K_{1,n}, n = 3,4$ as

$f(v_i) = i, \qquad 1 \le i \le 2,$

$f(v_3) = 7,$ and

$f(v_4) = 11$ in the case of $n = 4$.

Label the vertices of C_m as follows:

Subcase (i) When m is odd

Let $m = 2k + 1, k \in Z^+$.

$f(u_1) = 5,$

$f(u_j) = 2n + 4j - 1, \qquad 2 \le j \le k.$

$$f(u_{k+j}) = 2n + 4k - 4j + 6, \qquad 1 \le j \le k$$

$$f(u_{2k+1}) = 2n + 4.$$

Subcase (ii) When m is even.

Let $m = 2k$ for some $k \in Z^+$

$$f(u_1) = 5,$$

$$f(u_j) = 2n + 4i - 1, \qquad 2 \le j \le k - 1$$

$$f(u_{k-1+j}) = 2n + 4k - 3(j-1), \qquad 1 \le j \le 2.$$

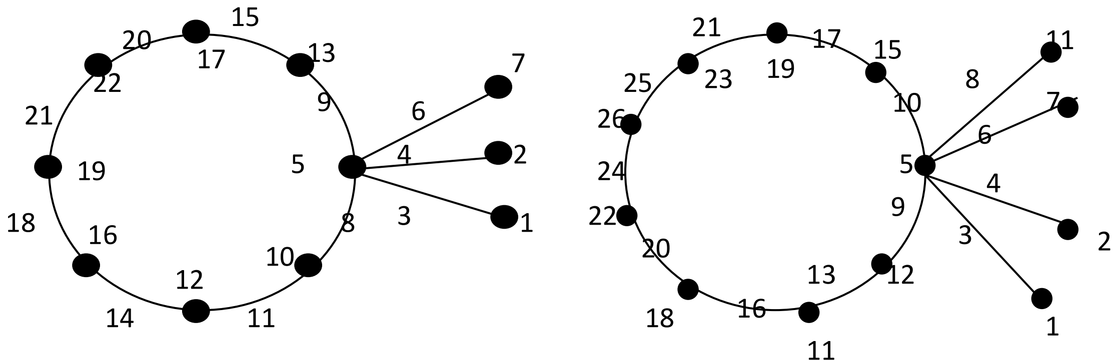
$$f(u_{k+1+j}) = 2n + 4k - 4j - 2, \qquad 1 \le j \le k - 2 \text{ and}$$

$$f(u_{2k}) = 2n + 4. \qquad 1 \le j \le k - 2.$$

Clearly, f induces distinct edge labels and it is easy to check that f generates a upper mean labeling and hence $\langle C_m, K_{1,n} \rangle, n = 3,4, m \ge 3$ and $m \ne 4$ is a super mean graph.

Thus, $\langle C_m, K_{1,n} \rangle$ is super mean graph for $n \le 4$ and $m \ge 3$.

Example: 4.8



$\langle C_8, K_{1,3} \rangle$ Figure 4.5 $\langle C_9, K_{1,4} \rangle$

A STUDY ON PRIME CORDIAL LABELING

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

N.JEBA PRIYADHARSHINI

Reg.No: 19SPMT09

Under the guidance of

Ms. I. ANBU RAJAMMAL M.Sc., M.phil., B.Ed., SET

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"PRIME CORDIAL LABELING"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **N. JEBA PRIYADHARSHINI**

(Reg. No: 19SPMT09)

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON PRIME CORDIAL LABELING"** Submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. I. Anbu Rajammal M.Sc., M.Phil., B.Ed., SET,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

N. Jeba Priyadharshini

Signature of the student

Station: Thoothukudi

Date: 10.4.2021

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Place: Thoothukudi

Date: 10.4.2021

CONTENT

CHAPTER 1

PRELIMINARIES

Definition: 1.1

A graph G contains of a pair $(V(G), X(G))$ where V (G) is non empty finite set whose elements are called points or vertices and X (G) is called a set of unordered pairs of distinct elements of V(G).The elements of X(G) are called lines or edges of the graph.

Definition: 1.2

If the vertices of the graph are assigned values subject to certain conditions is known as labeling.

Definition: 1.3

A bijection of a mapping that is both one-to-one and onto. A function which relates each member of a set S to a separate and distinct member of another set T, where each member in T also has a corresponding member in S.

Definition: 1.4

A subdivision of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with end points {u,v} yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, {u,w} and {w,v}.

Definition: 1.5

A vertex is a pendant if and only if it has a degree one. Deg (v).when (u, v) is an edge of the graph G with directed edges, u is said to be adjacent v and v is said to be adjacent from u.

Definition: 1.6

An edge of a graph is said to be pendant if one of its vertices is a pendant vertex.

Definition: 1.7

An edge induced subgraph is a subset of the edges of a graph, together with any vertices that are their end points.

Definition: 1.8

The fixed number of a graph G is the minimum k, such that every k-set of vertices of G is a fixing set of G.A graph G is called a k-fixed graph if their fixing numbers are both k.

Definition: 1.9

The joint of two vertex disjoint graphs G_1andG_2, denoted by $G_1 + G_2$ or $G_1 V G_2$, is a graph such that

1.$V(G) = V(G_1) \cup V(G_2)$

2. E (G) =$E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1), v \in V(G_2)\}$

Definition: 1.10

Bistar is the graph obtained by joining the apex vertices of two copies of star

$K_{1,n}$

Definition: 1.11

Let $G = (V(G), E(G))$ be a graph. A mapping $f: V(G) \to \{0,1\}$ is called Binary vertex labeling of G and f (v) is called the label of the vertex v of G underf. For an edge $e = uv,$ the induced edge labeling $f^*(e) = |f(u) - f(v)|$.

Definition: 1.12

A binary vertex labeling of a graph G is called a Cordial labeling if $\left|v_f(0) - v_f(1)\right| \leq 1$and $\left|e_f(0) - e_f(1)\right| \leq 1$. A graph G is cordial if it admits cordial labeling.

Definition: 1.13

A graph is said to be connected if there is a path between every pair of vertex to any other vertex, there should be some path to traverse. That is called the connectivity of graph.

Definition: 1.14

A graph is disconnected if it at least two vertices of the graph are not connected by a path. If a graph G is disconnected, then every maximal connected Subgraph of G is called a connected component of the graph G.

Definition: 1.15

A graph that contains no cycles is called an acyclic graph. A connected acyclic graph is called tree.

Definition: 1.16

An apex graph is graph possessing at least one vertex whose removal results in a planar graph.

Definition: 1.17

A graph is called planar if it can be drawn on a plane without intersecting edges.

Definition: 1.18

A subtree of tree which is a child of node. The name emphasizes that everything which is a descendant of a tree node is tree, and is a subset of the larger tree.

CHAPTER 2

PRIME CORDIAL LABELING FOR CYCLE RELATED GRAPHS

Definition: 2.1

A prime cordial labeling of a graph G with vertex set V (G) is a bijection

$f: V(G) \rightarrow \{1,2,3, \ldots, p\}$ defined by $f(e = uv) = 1$; if gcd $\big(f(u), f(v)\big)$

$f(e = uv) = 0$; otherwise

and $\left|e_f(0) - e_f(1)\right| \leq 1$. A graph which admits prime cordial labeling is called a prime cordial graph.

Example: 2.2

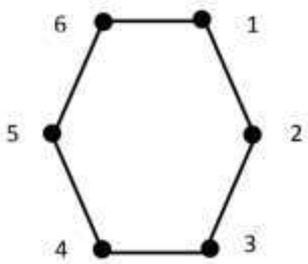


Figure 2.1: Cycle C_6

Definition: 2.3

Duplication of a vertex $v_k$by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G' Such that $N(v'_k) \cap N\big(v''_k\big) = v_k$.

Definition: 2.4

Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graphG' Such that $N(w) = \{u, v\}$.

Definition: 2.5

Let graphs $G_1, G_2, \ldots\ldots, G_n,\ n \geq 2$ be all copies of a fixed graph G. Adding an edge between G_i to G_{i+1} for i=1, 2,…, n−1 is called the path union of G.

Definition: 2.6

A Friendship graph F_n is a one point union of n copies of cycle C_3.

Theorem: 2.7

The graph obtained by duplicating each edge by a vertex in cycle $C_n$admits prime cordial labeling except for $n = 4$

Proof:

If $C_n{}'$be the graph obtained by duplicating an each edge by a vertex in a cycle $C_n$then let $v_1, v_2, \ldots.., v_n$ be the vertices of cycle C_n and $v'_1, v'_2, \ldots.., v'_n$ be the added verticesto obtain C'_n corresponding to the vertices $v_1, v_2, \ldots, v_n$ in C_n.

To define $f : V(C'_n) \to \{1,2, \ldots.,2p\}$, we consider following two cases.

Case 1: n is odd

Sub case1: n $=$ 3,5

The prime cordial labeling of C'_n for n=3, 5 is as shown in figure 2.1

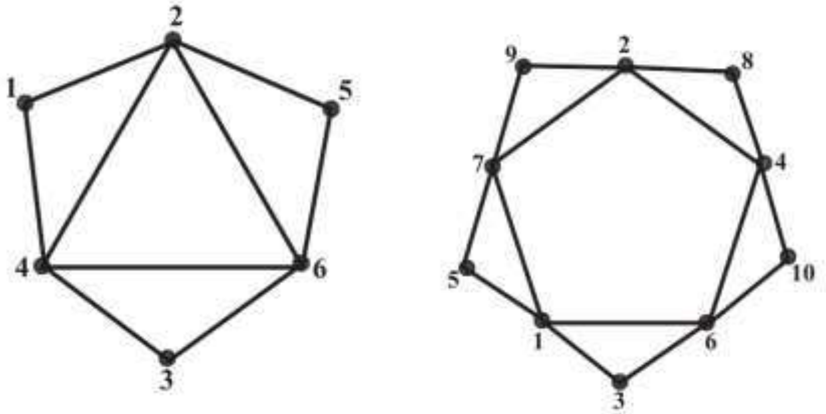


Fig 2.2 Prime cordial labeling of C'_3 and C'_5

$$f(v_i') = f\left(v_{\lfloor \frac{n}{2} \rfloor}\right) + 2i; \qquad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$f\left(v'_{\lfloor \frac{n}{2} \rfloor + 1}\right) = 3,$$

$$f\left(v'_{\lfloor \frac{n}{2} \rfloor + 1 + i}\right) = 4i + 1; \qquad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

In the view of the labeling pattern defined above we have

$$e_f(0) + 1 = e_f(1) = 3\left\lfloor \frac{n}{2} \right\rfloor + 2$$

Case 2: n is even

Sub Case 1: n=4

For the graph $C'_4$the possible pairs of labels of adjacent vertices are (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8).

Then obviously $e_f(0) = 5, e_f(1) = 7$. That is, $e_f(1) - e_f(0) = 2$ and in all other possible Arrangement of vertex labels $|e_f(0) - e_f(1)| \geq 2$. Thus C'_4 is not a prime cordial graph.

Sub Case 2: n= 6, 8, 10

The prime cordial labeling of C'_6, C'_8 and C'_{10} is as shown in Figure 2.2

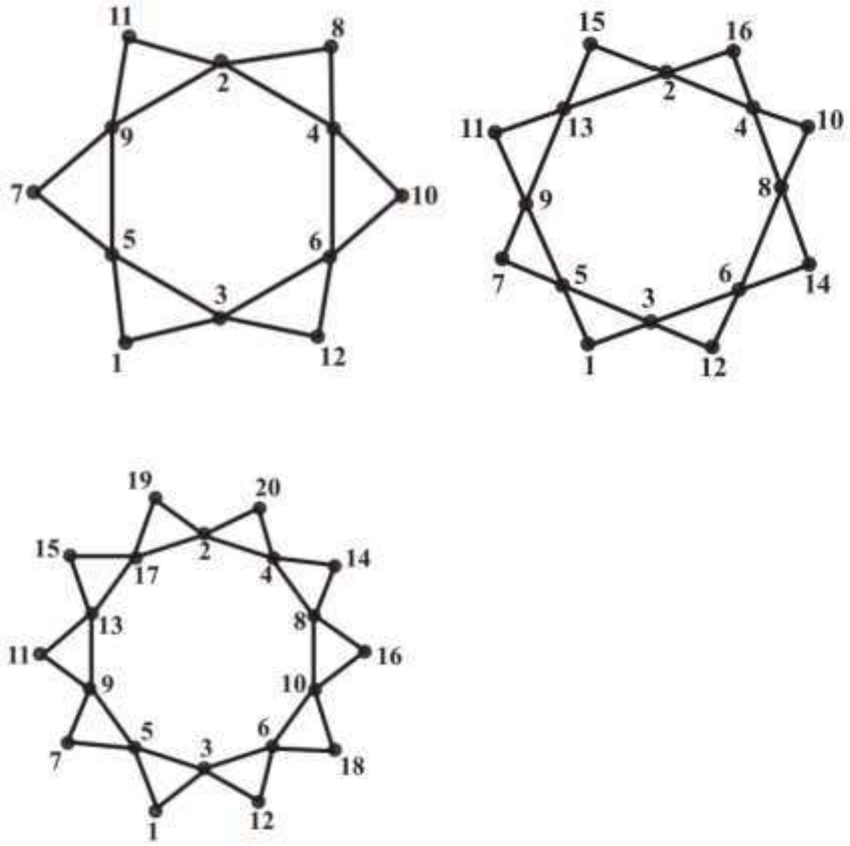


Fig 2.3

Prime cordial labeling of C'_6 , C'_8 and C'_{10}

Sub Case 3: n≥ 12

` $f(v_1) = 2,\ f(v_2) = 4,\ f(v_3) = 8,\ f(v_4) = 10,\ f(v_5) = 14,$

$$f(v_{5+i}) = 14 + 2i; 1 \leq i \leq \frac{n}{2} - 6$$

$$f\left(v_{\frac{n}{2}}\right) = 6,$$

$$f\left(v_{\frac{n+1}{2}+1}\right) = 3,$$

$$f\left(v_{\frac{n}{2}+1+i}\right) = 4i + 1; 1 \leq i \leq \frac{n}{2} - 1$$

$$f(v'_1) = 2n,$$

$$f(v'_{1+i}) = f\left(v_{\frac{n}{2}-1}\right) + 2i; 1 \leq i \leq \frac{n}{2} - 2$$

$$f\left(v'_{\frac{n}{2}}\right) = 12,$$

$$f\left(v'_{\frac{n}{2}+1}\right) = 1,$$

$$f\left(v'_{\frac{n}{2}+1+i}\right) = 4i + 3;\ \ 1 \leq i \leq \frac{n}{2} - 1$$

In the above two cases we have $\left|e_f(0) - e_f(1)\right| \leq 1$

Hence the graph obtained by duplicating each edge by a vertex in a cycle C_n admits prime cordial labeling except for $n = 4$.

Theorem: 2.8

The graph obtained by duplicating a vertex by an edge in cycle C_n is prime cordial graph.

Proof:

If C'_n be the graph obtained by duplicating a vertex by an edge in cycle C_n then Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n and $v'_1, v'_2, \dots., v'_{2n}$ be the added vertices to obtain C'_ncorresponding to the vertices $v_1, v_2, \dots, v_n$ in C_n.

To define $f : V(C'_n) \to \{1,2,3, \dots., 3p\}$, we consider following two cases.

Case 1: n is odd

Sub Case 1: n=3, 5

The prime cordial labeling of $C'_n$for n=3, 5 is shown in figure 2.3

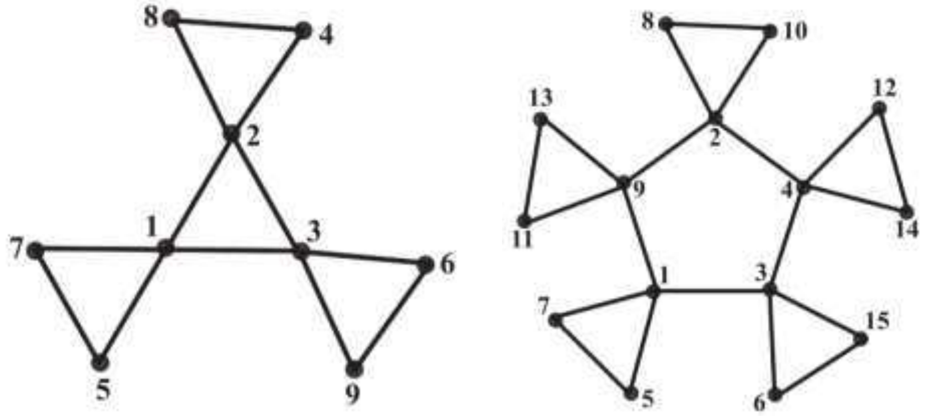


Fig 2.4

Prime cordial labeling of C'_3 and C'_5

Sub Case 2: n≥ 7

$f(v_1) = 2,\ f(v_2) = 4,$

$$f(v_{2+i}) = 6 + 2i;\ \ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 2$$

$$f\left(v_{\frac{n+1}{2}}\right) = 3,$$

$$f\left(v_{\frac{n+1}{2}+1}\right) = 1,$$

$$f\left(v_{\left\lfloor \frac{n}{2} \right\rfloor+2+i}\right) = 6i + 5;\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$$

$$f(v'_i) = f\left(v_{\left\lfloor \frac{n}{2} \right\rfloor}\right) + 2i;\ 1 \le i \le 2\left\lfloor \frac{n}{2} \right\rfloor$$

$$f\left(v'_{2\left\lfloor \frac{n}{2} \right\rfloor+1}\right) = 6, \qquad f\left(v'_{2\left\lfloor \frac{n}{2} \right\rfloor+2}\right) = 9,$$

$$f\left(v'_{2\left\lfloor \frac{n}{2} \right\rfloor+3}\right) = 5, \qquad f\left(v'_{2\left\lfloor \frac{n}{2} \right\rfloor+4}\right) = 7,$$

$$f\left(v'_{2\left\lfloor \frac{n}{2} \right\rfloor+4+2i-1}\right) = 6i + 7; 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$$

$$f\left(v'_{2\left\lfloor \frac{n}{2} \right\rfloor+4+2i}\right) = 6i + 9;\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$$

Case 2: n is even

Sub Case 1: n= $4, 6$

The prime cordial labeling of C'_n for $n = 4, 6$ is shown in figure 2. 4

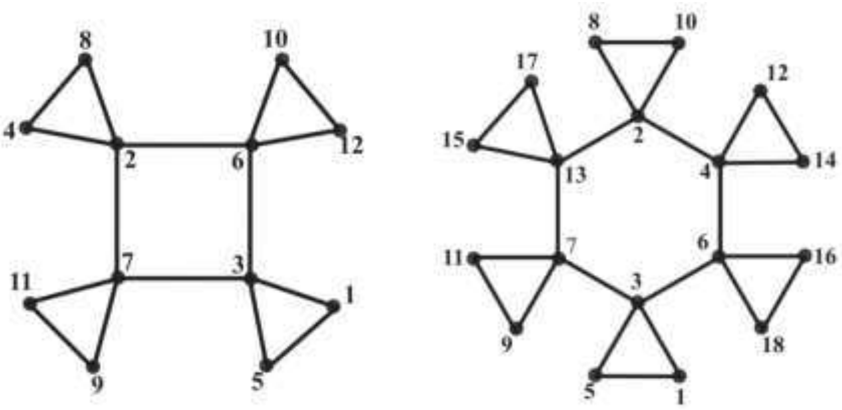


Fig 2.5

Prime cordial labeling of C'_4 and C'_6

Sub Case 2: n ≥ 8

$f(v_1) = 2,\ f(v_2) = 4,$

$f(v_{2+i}) = 6 + 2i; 1 \le i \le \frac{n}{2} - 3$

$f\left(v_{\frac{n}{2}}\right) = 6,$

$f\left(v_{\frac{n}{2}+1}\right) = 3,$

$f\left(v_{\frac{n}{2}+1+i}\right) = 6i + 1;\ 1 \le i \le \frac{n}{2} - 1$

$f(v'_i) = f\left(v_{\frac{n}{2}-1}\right) + 2i;\ 1 \le i \le n$

$f(v'_{n+1}) = 1, \qquad f(v'_{n+2}) = 5,$

$f(v'_{n+1+2i}) = 6i + 3; 1 \le i \le \dfrac{n}{2} - 1$

$f(v'_{n+2+2i}) = 6i + 5; 1 \le i \le \dfrac{n}{2} - 1$

Thus in both the cases defined above we have $e_f(0) = e_f(1) = 2n$

Hence C'_n admits prime cordial labeling.

Theorem: 2.9

The friendship graph F_n is a prime cordial graph for $n \geq 3$.

Proof:

Let v_1 be the vertex common to all the cycles. Without loss of generality we start the label assignment fromv_1.

To define $f : V(F_n) \to \{1,2,3, \ldots .,2n+1\}$, we consider following two cases.

Case 1: n is even

Let p be the highest prime such that $3p \leq 2n+1$,

$$f(v_1) = 2p,$$

Now label the remaining vertices from 1to $2n+1$ except 2p.

In the view of the labeling pattern define above we have

$$e_f(0) = e_f(1) = \frac{3n}{2}$$

Case 2: n is odd

Let p be the highest prime such that $2p \leq 2n+1$,

$$f(v_1) = 2p,$$

Now label the remaining vertices from 1 to $2n+1$ except $2p$.

In the view of the labeling above defined we have

$$e_f(0) + 1 = e_f(1) = 3\left\lfloor\frac{n}{2}\right\rfloor + 2$$

Thus in above two cases $|e_f(0) - e_f(1)| \leq 1$

Hence friendship graph admits prime cordial labeling.

Consider the friendship graph F_8. The labeling is as shown in figure 2.5.

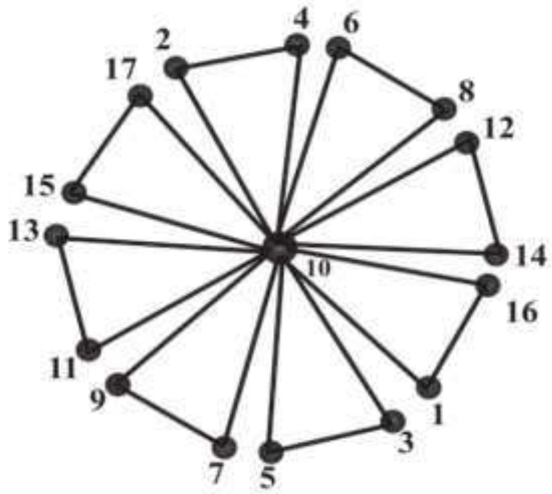


Fig 2.6

Prime cordial labeling of F_8

Theorem: 2.10

$C_m \cup P_n$ is prime cordial if $m \leq 5$ is odd and $n \geq 6$.

Proof:

Let $G = (V, E, f)$ be disconnected graph $C_m \cup P_n$ with order $p = m + n$ and size $q = m + n - 1$. Here $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ are the vertices where$u_1, u_2, \ldots, u_m$ be the vertices of the cycle C_3 and $v_1, v_2, \ldots, v_n$ be the vertices of path P_n.

Define the function $f: (V(G)) \to \{1,2, \ldots., m+n\}$ as follows,

Case (i) $\boldsymbol{C_3 \cup P_n}$

If n is even,

$f(u_1) = 1, f(u_2) = 5, f(u_3) = 7, f(v_n) = 3, f(v_{n-1}) = 9.$

$f(v_i) = 2i, i = 1,2, \ldots\ldots, \frac{n+2}{2}$

$f(v_i) = f(v_{i-1}) + 2, \qquad i = n-2, n-3, \ldots., \frac{n+4}{2}$

If n is odd,

$f(u_1) = 1, f(u_2) = 5, f(u_3) = 7, f(v_n) = 3, f(v_{n-1}) = 9.$

$f(v_i) = 2i, i = 1,2, \ldots\ldots, \frac{n+3}{2}$.

$f(v_i) = f(v_{i-1}) + 2, \ i = n-2, n-3, \ldots \ldots, \frac{n+5}{2}$

Case (ii) $\boldsymbol{C_5 \cup P_n}$

If n is even,

$f(u_1) = 1, f(u_2) = 3, f(u_3) = 9.$

$f(u_{i+2}) = 2i + 1, \ i = 2,3.$

$f(v_i) = 2i, i = 1,2, \ldots, \frac{n+4}{2}.$

$f(v_n) = 11.$

$f(v_i) = f(v_{i+1}) + 2, \quad i = n-1, n-2, \ldots, \frac{n+6}{2}.$

If n is odd.

$f(u_1) = 1, f(u_2) = 3, f(u_3) = 9.$

$f(u_{i+2}) = 2i + 1, \; i = 2,3.$

$f(v_i) = 2i, i = 1,2,..,\frac{n+5}{2}.$

$f(v_n) = 11.$

$$f(v_i) = f(v_{i+1}) + 2, \quad i = n-1, n-2, \ldots, \frac{n+7}{2}.$$

Then the above function f admits the prime cordial labeling. Hence $C_m \cup P_n$ are prime cordial labeling. The generalized graph of $C_m \cup P_n$ is shown in figure 2.7

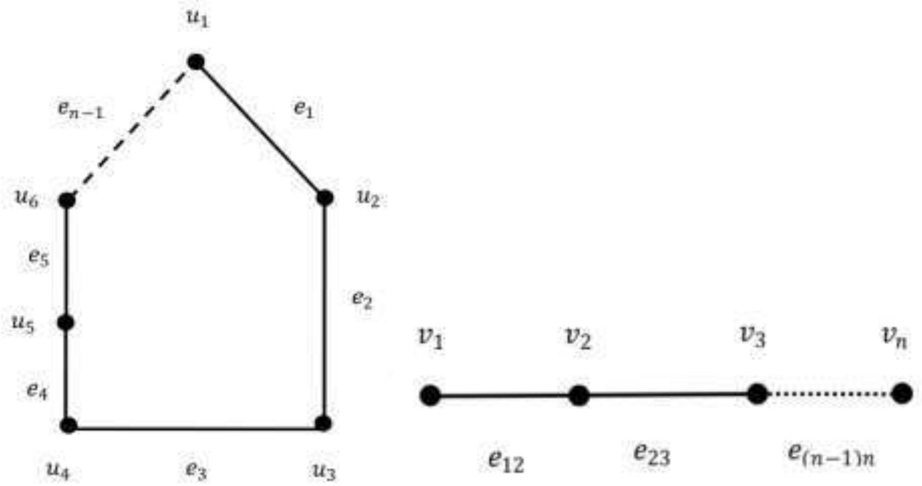


Figure 2.7 Disconnected graphs $C_m \cup P_n$

CHAPTER 3

PRIME CORDIAL LABELING FOR

WHEEL RELATED GRAPHS

Definition: 3.1

A wheel graph W_n is join of C_n and K_1. i.e. W_n=C_n+K_1. Here the edges of C_n are the rim edges of W_n.

Definition: 3.2

The gear graph G_n is obtained from the wheel W_n by subdividing each of its rim edge.

Definition: 3.3

The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex of w_n.

Definition: 3.4

The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex of H_n to from a cycle.

Theorem: 3.5

The graph obtained by joining two copies of wheel graph W_n by a path of arbitrary length is prime cordial.

Proof:

Let G be the graph obtained by joining two copies of wheel graph W_n by a path P_k of length $k-1$. Let u_0 be the apex vertex and $u_1, u_2, \dots, u_n$ be the successive rim vertices of first copy of wheel W_n. Let w_0 be the apex vertex and $w_1, w_2, \dots, w_n$ be the successive rim vertices of second copy of wheel W_n. Let $v_1, v_2, \dots, v_k$ be the vertices of path P_k with $v_1 = u_0$ and $v_k = w_0$. Here we define labeling function

$f: V(G) \rightarrow \{1,2, \dots, 2n+k\}$ as follows.

$$f(u_0) = 1, f(u_1) = 3, f(u_n) = 5$$

$$f(u_i) = 4i+1;\ 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$= 4(n-1)+3;\ \left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \le i \le n-1$$

$$f(v_j) = 2(n+j)-1;\ 2 \le j \le \left\lceil \frac{k}{2} \right\rceil$$

$$= 2\left(j - \left\lceil \frac{k}{2} \right\rceil\right);\ \left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \le j \le k$$

$$f(w_i) = 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 4(i-1);\ 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 2\left(\left\lceil \frac{k}{2} \right\rceil + 1\right) + 4(n-i);\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n$$

One can observe that the labeling defined above satisfies the conditions of prime cordial labeling and graph under consideration is prime cordial graph.

Theorem: 3.6

The graph obtained by joining two copies of gear graph G_n by a path of arbitrary length is prime cordial.

Proof:

Let G be the graph obtained by joining two copies of gear graph G_n by a path P_k of length $k-1$. Let u_0 be the apex vertex and $u_1, u_2, \ldots, u_n$ be the rim vertices of W_n corresponding to the first copy of G_n. Let $u_1', u_2', \ldots, u_n'$ be the vertices of first copy of G_n which make subdivision of edges of corresponding W_n, where u_i' is adjacent to u_i and u_{i+1}, $i = 1,2,\ldots,n_1$; $u_n'$is adjacent to u_n and u_1.Similarly let w_0 be the apex vertex, $w_1, w_2, \ldots, w_n$ be the rim vertices of W_n corresponding to the second copy of G_n. Let $w_1', w_2', \ldots, w_n'$ be the vertices of second of G_n which makes subdivision of the edges of correspondingW_n, where w'_i is adjacent to w_i and w_{i+1}, $i = 1,2,\ldots,n-1$; w_n' is adjacent to w_n and w_1.Let $v_1, v_2, \ldots, v_k$ be the vertices of path P_k with $v_1 = u_0$ and $v_k = w_0$. We define the labeling function

$f : V(G) \to \{1,2,\ldots,4n+k\}$

Case 1: $k = 2$

$$f(u_0) = 6,\ f(w_0) = 1, f(u_1) = 2, f(u_1') = 4$$

$$f(u_i) = 2(i+2);\ 2 \le i \le n$$

$$f(u_i') = 2(n+i+1); 2 \le i \le n$$

$$f(w_i) = 8i - 5;\ 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 8(n-i) + 9; \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n$$

$$f(w_i') = 8i - 1; 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$= 8(n-i) + 5;\ \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \le i \le n$$

Case 2: $k = 3$

$$f(u_0) = 6,\ f(w_0) = 1,\ f(u_1) = 2,\ f(u_1') = 4, f(v_2) = 3$$

$$f(u_i) = 2\left(\left\lfloor \frac{k}{2} \right\rfloor + i + 1\right);\ 2 \le i \le n$$

$$f(u_i') = 2\left(\left\lfloor \frac{k}{2} \right\rfloor + n + i\right); 2 \le i \le n$$

$$f(w_i) = 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8i - 7; 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8(n-i) + 7; \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n$$

$$f(w_i') = 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8i - 3; 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$= 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8(n-i) + 3; \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \le i \le n$$

Case: 3 $k \ge 4$

$$f(w_0) = 1, f(u_1) = 2$$

$$f(u_1') = 4, f\left(v_{\left\lfloor \frac{k}{2} \right\rfloor}\right) = 6$$

$$f(u_i) = 2\left(\left\lfloor \frac{k}{2} \right\rfloor + i + 1\right); 2 \le i \le n$$

$$f(u_i') = 2\left(\left\lfloor \frac{k}{2} \right\rfloor + n + i\right); 2 \le i \le n$$

$$f(v_j) = 2j + 6; 1 \le j \le \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)$$

$$= 2\left(j - \left\lfloor \frac{k}{2} \right\rfloor\right) + 1; \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \le j \le k - 1$$

$$f(w_i) = 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8i - 7; 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$= 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8(n - i) + 7; \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \le i \le n$$

$$f(w_i') = 2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8i - 3; 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$=2\left(\left\lceil \frac{k}{2} \right\rceil\right) + 8(n - i) + 3; \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \le i \le n$$

One can observe that in each case the labeling defined above satisfies the conditions of prime cordial labeling and the graph under consideration is prime cordial graph.

Example: 3.7

For the graphG_{20}, $|V(G_{20})|$=41 and $|E(G_{20})| = 60$. It is easy visualize that

$e_f(0) = 30 = e_f(1)$

The prime cordial labeling is shown in figure 3.1

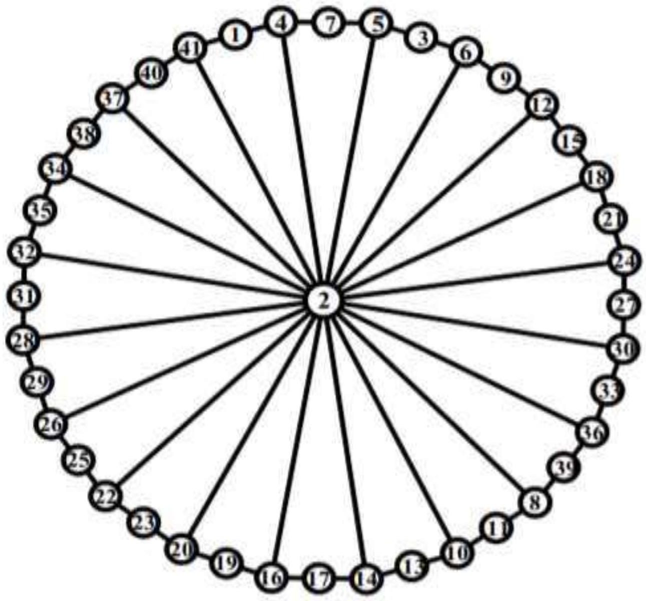


Fig 3.1

Theorem: 3.8

The graph obtained by joining two copies of Helm H_n by a path of arbitrary length is prime cordial.

Proof:

Let G be graph obtained by joining two copies of helm graph H_n by a path P_k of length $k-1$. Let u_0 be the apex vertex, $u_1, u_2, \ldots, u_n$ be the rim vertices and $u_1', u_2', \ldots, u_n'$ be the pendant Vertices of first copy of helm H_n. Let w_0 be the apex vertex, $w_1, w_2, \ldots, w_n$ be the rim vertices and $w_1', w_2', \ldots, w_n'$ be the pendent vertices of second copy of helm H_n. Let $v_1, v_2, \ldots, v_k$ be the vertices of path P_k with $v_1 = u_0$ and $v_k = w_0$.

To define labeling function $f : V(G) \to \{1,2, \ldots, 4n + k\}$ we consider following cases.

Case 1: $k = 2$

$$f(u_0) = 2, f(w_0) = 1$$

$$f(u_i) = 2(i+1); 1 \le i \le n$$

$$f(u_i') = 2i + 1; 1 \le i \le n$$

$$f(w_i) = 2(n + i + 1); 1 \le i \le n$$

$$f(w_i') = 2(n + i) = 1; 1 \le i \le n$$

Case 2: $k = 3$

$$f(u_0) = 2, f(w_0) = 1, f(v_2) = 4n + 3, f(u_1') = 5, f(u_2') = 3$$

$$f(u_i) = 2(i+1); 1 \le i \le n$$

$$f(u_i') = 2i + 1; 3 \le i \le n$$

$$f(w_i) = 2(n + i + 1); 1 \le i \le n$$

$$f(w_i') = 2(n + i) + 1; 1 \le i \le n$$

Case 3: $k \ge 4$

$$f(w_0') = f(v_k) = 1, f(u_1') = 5, f(u_2') = 3$$

$$f\left(v_{\left\lfloor\frac{k}{2}\right\rfloor}\right) = 2$$

$$f(u_i) = 2(i + 1); 1 \le i \le n$$

$$f(u_i') = 2i + 1; 3 \le i \le n$$

$$f(v_j) = 4n + 2(j + 1); 1 \le j \le \left(\left\lfloor\frac{k}{2}\right\rfloor - 1\right)$$

$$= 4n + 2\left(j - \left\lfloor\frac{k}{2}\right\rfloor\right) + 1; \left(\left\lfloor\frac{k}{2}\right\rfloor + 1\right) \le j \le k - 1$$

$$f(w_i) = 2(n + i + 1); 1 \le i \le n$$

$$f(w_i') = 2(n + i) + 1; 1 \le i \le n$$

One can observe that in each case the labeling defined above satisfies the conditions of prime cordial Labeling and the graph under consideration is prime cordial graph.

Example: 3.9

The graph H_{13} and its prime cordial labeling is shown in Fig 3.2

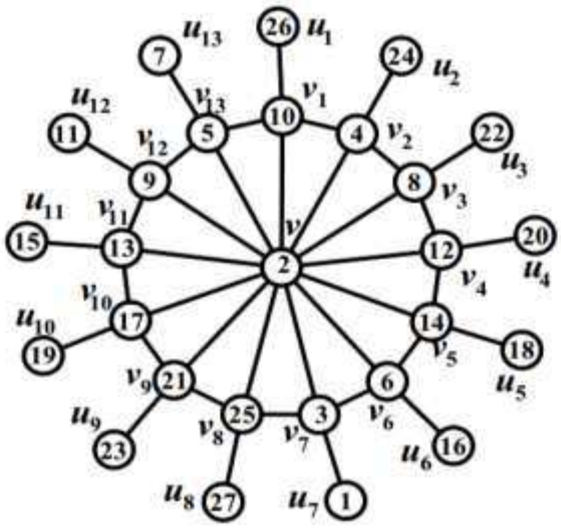


Fig 3.2

Theorem: 3.10

The graph obtained by joining two copies of closed helm $\boldsymbol{C\,H_n}$ by a path of arbitrary length is prime cordial.

Proof:

Let G be the graph obtained by joining two copies of helm graph $\boldsymbol{C\,H}_n$ by a path P_k of length $k-1$. Let u_0 be the apex vertex, $u_1, u_2, \dots, u_n$ be the vertices of inner cycle and $u_1', u_2', \dots, u_n'$ be the vertices of outer cycle of the first copy of closed helm$\boldsymbol{C\,H}_n$. Similarly let w_0 be the apex vertex, $w_1, w_2, \dots, w_n$ be the vertices of inner cycle and $w_1', w_2', \dots, w_n'$ be the vertices of outer cycle of second copy of closed helm$\boldsymbol{C\,H}_n$. Let $v_1, v_2, \dots, v_k$ be the vertices of path P_k with $v_1 = u_0$ and$v_k = w_0$.

To define labeling function $f : V(G) \to \{1,2,\dots,4n+k\}$ we consider the following cases.

Case 1: $k = 2$

$$f(u_0) = 6, f(w_0) = 1, f(u_1) = 4, f(u_1') = 2$$

$$f(u_i) = 2\left(\left\lfloor\frac{k}{2}\right\rfloor\right) + 8(i-1); 2 \leq i \leq \left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)$$

$$= 2\left(\left\lfloor\frac{k}{2}\right\rfloor + 2\right) + 8(n-i+1); \left(\left\lfloor\frac{n}{2}\right\rfloor + 2\right) \leq i \leq n$$

$$f(u_i') = 2\left(\left\lfloor\frac{k}{2}\right\rfloor - 1\right) + 8(i-1); 2 \leq i \leq \left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)$$

$$=2\left(\left\lfloor\frac{k}{2}\right\rfloor + 1\right) + 8(n-i+1);\ \left(\left\lfloor\frac{n}{2}\right\rfloor + 2\right) \leq i \leq n$$

$$f(w_i) = 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8i - 5; 1 \leq i \leq \left\lceil\frac{n}{2}\right\rceil$$

$$= 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8(n-i+1) - 1;\ \left(\left\lceil\frac{n}{2}\right\rceil + 1\right) \leq i \leq n$$

$$f(w_i') = 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8i - 7; 1 \leq i \leq \left\lceil\frac{n}{2}\right\rceil$$

$$=2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8(n-i+1) - 3; \left(\left\lceil\frac{n}{2}\right\rceil + 1\right) \leq i \leq n$$

Case 2: $k = 3$

$$f(u_0) = 6, f(w_0) = 1, f(u_1) = 4, f(u_1') = 2, f(v_2) = 3$$

$$f(u_i) = 2\left(\left\lfloor\frac{k}{2}\right\rfloor\right) + 8(i-1); 2 \leq i \leq \left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)$$

$$= 2\left(\left\lfloor\frac{k}{2}\right\rfloor + 2\right) + 8(n-i+1); \left(\left\lfloor\frac{n}{2}\right\rfloor + 2\right) \leq i \leq n$$

$$f(u_i') = 2\left(\left\lfloor\frac{k}{2}\right\rfloor - 1\right) + 8(i-1); 2 \leq i \leq \left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)$$

$$= 2\left(\left\lfloor\frac{k}{2}\right\rfloor + 1\right) + 8(n-i+1); \left(\left\lfloor\frac{n}{2}\right\rfloor + 2\right) \leq i \leq n$$

$$f(w_i) = 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8i - 5;\, 1 \le i \le \left\lceil\frac{n}{2}\right\rceil$$

$$= 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8(n - i + 1) - 1;\, \left(\left\lceil\frac{n}{2}\right\rceil + 1\right) \le i \le n$$

$$f(w_i') = 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8i - 7;\, 1 \le i \le \left\lceil\frac{n}{2}\right\rceil$$

$$= 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8(n - i + 1) - 3;\, \left(\left\lceil\frac{n}{2}\right\rceil + 1\right) \le i \le n$$

Case 3: $k \ge 4$

$$f(u_1) = 4, f(w_0) = 1, f(u_1') = 2, f\left(v_{\left\lfloor\frac{k}{2}\right\rfloor}\right) = 6$$

$$f(u_i) = 2\left(\left\lfloor\frac{k}{2}\right\rfloor\right) + 8(i - 1);\, 2 \le i \le \left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)$$

$$= 2\left(\left\lfloor\frac{k}{2}\right\rfloor + 2\right) + 8(n - i + 1);\, \left(\left\lfloor\frac{n}{2}\right\rfloor + 2\right) \le i \le n$$

$$f(u_i') = 2\left(\left\lfloor\frac{k}{2}\right\rfloor - 1\right) + 8(i - 1);\, 2 \le i \le \left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)$$

$$= 2\left(\left\lfloor\frac{k}{2}\right\rfloor + 1\right) + 8(n - i + 1);\, \left(\left\lfloor\frac{n}{2}\right\rfloor + 2\right) \le i \le n$$

$$f(v_j) = 2(j + 3);\, 1 \le j \le \left(\left\lfloor\frac{k}{2}\right\rfloor - 1\right)$$

$$= 2\left(j - \left\lfloor\frac{k}{2}\right\rfloor\right) + 1;\, \left(\left\lfloor\frac{k}{2}\right\rfloor + 1\right) \le j \le k - 1$$

$$f(w_i) = 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8i - 5;\, 1 \le i \le \left\lceil\frac{n}{2}\right\rceil$$

$$= 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8(n - i + 1) - 1;\, \left(\left\lceil\frac{n}{2}\right\rceil + 1\right) \le i \le n$$

$$f(w_i') = 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8i - 7;\, 1 \le i \le \left\lceil\frac{n}{2}\right\rceil$$

$$= 2\left(\left\lceil\frac{k}{2}\right\rceil\right) + 8(n - i + 1) - 3;\, \left(\left\lceil\frac{n}{2}\right\rceil + 1\right) \le i \le n$$

One can observe that in each case the labeling defined above satisfies the conditions of prime cordial labeling and the graph under consideration is prime cordial graph.

Example: 3.11

The graph $\boldsymbol{C\,H_{10}}$ and its prime cordial labeling is shown in Fig 3.3

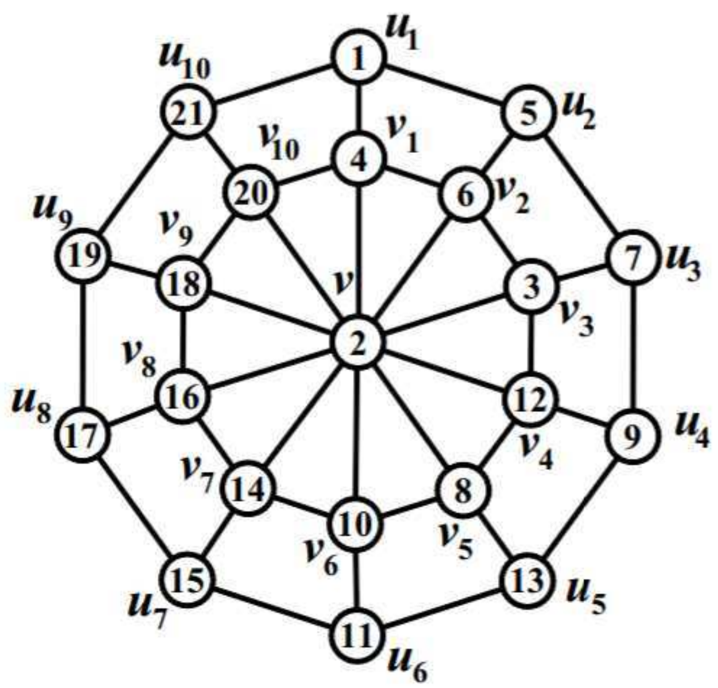


Fig 3.3

Theorem: 3.12

W_n is a prime cordial graph for $n \geq 8$.

Proof:

Let $v_0$be the apex vertex of wheel W_n and $v_1, v_2, \dots, v_n$ be the rim vertices.

To define $f: V(W_n) \to \{1,2,\dots,2n\}$, we consider the following these cases.

Case 1: $n = 8,9,10$

The graphs W_8,$W_9 and\ W_{10}$ are to be dealt separately and their prime cordial labeling is shown figure 3.4

Case 2: $n\ is\ even, n \geq 12$

$f(v_0) = 2,$

$f(v_1) = 5,$

$f(v_2) = 10,$

$f(v_3) = 4.$

$f(v_4) = 8,$

$f(v_{4+i}) = 12 + 2(i - 1);\ \ 1 \leq i \leq \frac{n}{2} - 5$

$f\left(v_{\frac{n}{2}}\right) = 6,$

$f\left(v_{\frac{n}{2}+1}\right) = 3,$

$f\left(v_{\frac{n}{2}+2}\right) = 9,$

$f(v_{n-1}) = 1,$

$f(v_n) = 7,$

$f\left(v_{\frac{n}{2}+2+i}\right) = 11 + 2(i - 1);\ \ 1 \leq i \leq \frac{n}{2} - 4$

In the view of the above defined labeling pattern we have $e_f(0) = n = e_f(1)$.

Case 3: n *is odd*, $n \geq 11$

$$f(v_0) = 2,$$

$$f(v_1) = 10,$$

$$f(v_2) = 4,$$

$$f(v_3) = 8,$$

$$f(v_{3+i}) = 12 + 2(i-1); \qquad 1 \leq i \leq \frac{n-1}{2} - 4$$

$$f\left(v_{\frac{n-1}{2}}\right) = 6,$$

$$f\left(v_{\frac{n+1}{2}}\right) = 3,$$

$$f\left(v_{\frac{n+3}{2}}\right) = 1,$$

$$f(v_{n+1-i}) = 5 + 2(i-1); \qquad 1 \leq i \leq \frac{n-3}{2}$$

In the view of the above defined labeling pattern we have $e_f(0) = n = e_f(1)$.

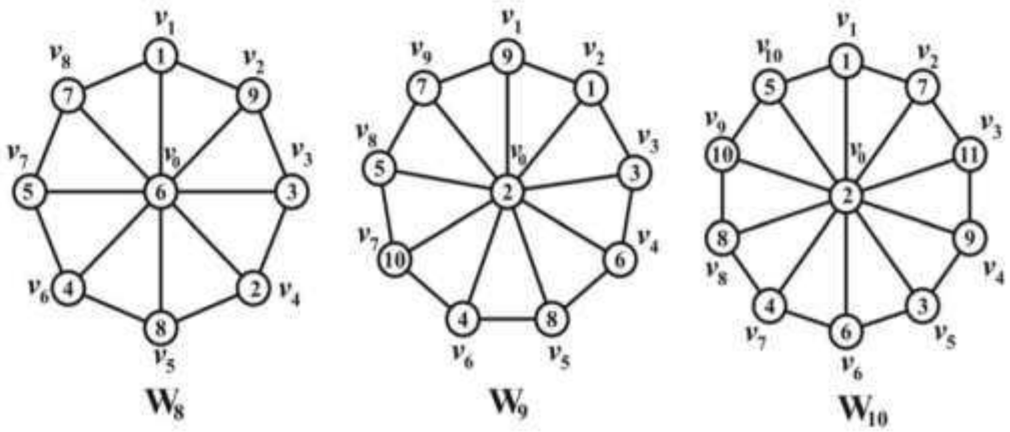


Figure 3.4

Thus in all the cases we have $|e_f(0) - e_f(1)| \leq 1$.

Hence W_n is a prime cordial graph for$n \geq 8$.

CHAPTER 4

PRIME CORDIAL LABELING FOR SOME SPECIAL GRAPHS

Definition: 4.1

Let $G = (V, E)$ be a graph. Let $e = uv$ be an edge of G and w is not vertex of G. The edge e is Sub divided when it is replaced by edges $e' = uw$ and $e'' = wv$. Let $G = (V, E)$ be a graph. If every edge of graph G is sub divided, then the resulting graph is called barycentric subdivision of graph G.

Definition: 4.2

Let G be the graph obtained by joining two copies of $c_n(c_n)$ by a path of length one and it denoted by $K_2 \Theta C_n(C_n)$.

Definition: 4.3

Let graphs $G_1, G_2, \dots, G_n, n \geq 2$ be all copies of a fixed graph G. Adding an edge between G_i to G_{i+1} for $i = 1,2, \dots., n-1$ is called the path union of G.

Definition: 4.4

The double star $K_{1,n,n}$ is a tree obtained from the $K_{1,n}$ by adding a new pendent edge to each of the existing n pendent vertices.

Definition: 4.5

$K_{1,m} \Theta K_{1,n}$ is a tree obtained by adding n pendent edges to each pendent vertices of $K_{1,m}$. It has totally $m(n+1)+1$ vertices.

Definition: 4.6

A shell S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n on n vertices. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan F_{n-1}. i.e., $S_n = F_{n-1} = P_{n-1} + K_1$. Consider two shells $S_n^{(1)}$and $S_n^{(2)}$ then $G =< S_n^{(1)}: S_n^{(2)} >$ Obtained by joining apex vertices of shells to a new vertex$x$.

Definition: 4.7

For a graph G the split graph is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted as $spl(G)$.

Definition: 4.8

For a simple connected graph G the square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition: 4.9

The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Algorithm: 4.10

Step 1: $V = \{u_1, u_2, u_3, \dots, u_n\} \cup \{u_1', u_2', u_3', .., u_n'\}$

Step 2: $E = \{u_i'u_i : 1 \leq i \leq n\} \cup \{u_j u_{j+1}' : 1 \leq j \leq n-1\} \cup \{u_n u_1'\}$

Step 3: $f(u_1') = 1, f(u_1) = 2$

Step 4: $for\ i = 1\ to\ n, f(u_i) = 2i$

Step 5: $for\ j = 1\ to\ n, f(u_j') = 2j - 1$

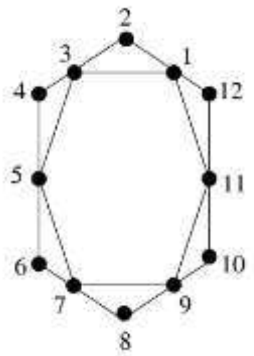


Figure 4.1: prime labeling for $c_6(c_6)$

Theorem 4.11

The barycentric subdivision of cycle C_n is prime.

Proof:

Let $\{u_1, u_2, \ldots, u_n\}$ be the vertices of cycle C_n and $\{u_1', u_2', u_3', \ldots., u_n'\}$ be the newly inserted vertices to obtain barycentric subdivision of cycleC_n. To define prime labeling $f: V(G) \to \{1,2, \ldots ,2n\}$ is defined in algorithm 4.7 which gives us $f(u_1') = 1$ and $f(u_1) = 2.$ For any edge $e = u_i u_i': 1 \leq i \leq n.$ $\gcd\big(f(u_i'), f(u_i)\big) = 1$ Since$f(u_1')$ and $f(u_i)$ and are labeled with consecutive positive integer which implies that they are relatively prime.

For any edge $e = u_j u'_{j+1}: 1 \leq j \leq n-1, \gcd\left(f(u_j), f(u'_{j+1})\right)$=1.

For any edge $e = u'_1 u_n, \gcd(f(u'_1), f(u_n)) = 1$, since $(1, x) = 1$ for any positive integer x. Hence the barycentric subdivision of the cycle C_n is prime.

Theorem: 4.12

The graph $K_{1,m}\Theta\, K_{1,n}$ for all $m, n \geq 1$ is prime labeling.

Proof:

Case (i):

Let a_0 be the root of tree. Let $a_1, a_2, .., a_m$be the children of the root .Each sub tree a_i, $1 \leq i \leq m$ will have n number of vertices which are considered as leaves of the graph namely $a_{i1}, a_{i2}, \ldots, a_{in}; 1 \leq i \leq m$ leaves. Let $a_0 = 2$. The intern al vertices $a_1, a_2, a_3, .., a_m$ are labeled with Consecutive largest prime numbers less than or equal to $mn + m + 1$. The vertices that acts as a leaves of the graph $K_{1,m}\Theta\, K_{1,n}$ are labeled with the remaining numbers other 2, $a_1, a_2, \ldots, a_m$ in the consecutive order. The greatest common divisor of a_0 with $a_1, a_2, a_3, \ldots, a_m$ is one because $a_1, a_2, a_3, \ldots, a_m$ all are prime numbers. The greatest common divisor of a_i with $a_{i1}, a_{i2}, \ldots, a_{in}; 1 \leq i \leq m$ is one, sincea_i's are all consecutive largest prime numbers. Hence that the graph $K_{1,m}\Theta K_{1,n}$ for all $m, n \geq 1$ is prime.

Case (ii):

In $K_{1,m}\Theta\, K_{1,n}$ when m is too large and n is two small then the construction in the Theorem 4.8 need not hold. For example the above theorem holds for certain restriction for m i.e., When $n = 2,\ m \leq 12;\ n = 3,\ m \leq 56;\ n = 4,\ m \leq 72;$ $n = 5,\ m \leq 189.$

For further values of n, m Can be calculated using program to list prime numbers and their count in C programming

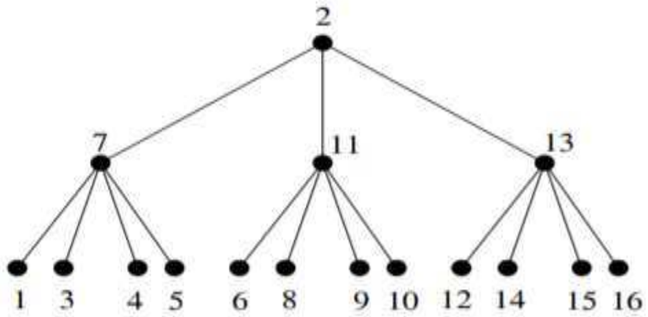


Figure 4.2

Theorem : 4.13

The graph $K_2 \Theta C_n(C_n)$ admits prime cordial labeling if $n \equiv 0,2(mod\ 3)$.

Proof:

Let G be the graph $K_2 \Theta C_n(C_n)$ obtained by joining two copies of $c_n(c_n)$ by a path of Length one. Let $u_1, u_2, .., u_n$ be the vertices of cycle c_n and $u'_1, u'_2, .., u'_n$ be the corresponding vertices of the cycle which is obtained by joining newly inserted vertices of adjacent edges in cyclec_n. Next denote the corresponding vertices in second copy of $c_n(c_n)$ by $v_1,\ v_2, \dots, v_n$ and $v'_1, v'_2, .., v'_n$ respectively. Let u_1 and

v_1 be the vertices of path p_2. Let the vertex and the edges sets are defined as follows:

$V = \{\{u_1, u_2, \dots, u_n\}, \{u'_1, u'_2, \dots, u'_n\}, \{v_1, v_2, \dots, v_n\}, \{v'_1, v'_2, \dots, v'_n\}\}$

$E = E_1 \cup E_2 \cup \dots \cup E_{10}$ Where

$E_1 = \{u_i u_i'; 1 \leq i \leq n-1\}$

$E_2 = \{u_j' u_{j+1}; 1 \leq j \leq n-1\}$

$E_3 = \{u_1 u_n'\}$

$E_4 = \{v_i v_i'; 1 \leq i \leq n-1\}$

$E_5 = \{v_j' v_{j+1}; 1 \leq j \leq n-1\}$

$E_6 = \{v_1 v_n'\}$

$E_7 = \{u_i' u_{i+1}'; 1 \leq i \leq n\}$

$E_8 = \{u_1' u_n'\}$

$E_9 = \{v_i' v_{i+1}'; 1 \leq i \leq n-1\}$

$E_{10} = \{v_1' v_n'\}$

Let $f : V \to \{1,2,3, \dots, |v|\}$ be the bijective function defined as

$f(u_i) = 4i - 2; 1 \leq i \leq n$

$f(u_j') = 4j; 1 \leq j \leq n$

$f(v_i) = 4i - 3; 1 \leq i \leq n$

$f(v_j') = 4j - 1; 1 \leq i \leq n$

$g(u_i u_i') = 0;\ 1 \leq i \leq n-1$

$g(u_j' u_{j+1}) = 0; 1 \leq j \leq n-1$

$g(v_i v_i') = 1; 1 \leq i \leq n-1$

$g(v_j' v_{j+1}) = 1; 1 \leq j \leq n-1$

$g(u_i' u_{i+1}') = 0; 1 \leq i \leq n$

$g(u_1' u_n') = 0$

$g(v_i' \, v_{i+1}') = 1; 1 \leq i \leq n$

$g(v_1' v_n') = 1$

$g(u_1 v_1) = 1$

$g(u_1 u_n') = 0$

$g(v_1 v_n') = 1.$

The total number of edges labeled with 1's is given by $K_1 = 3n + 1$ and the total number of edges labeled with 0's is given by $K_2 = 3n$. Therefore the total difference between 1's and 0's is given by $|K_1 - K_2| = |(3n+1) - 3n|$ and they differ by one. This proves that the graph $K_2 \Theta C_n(C_n)$ Prime cordial labeling.

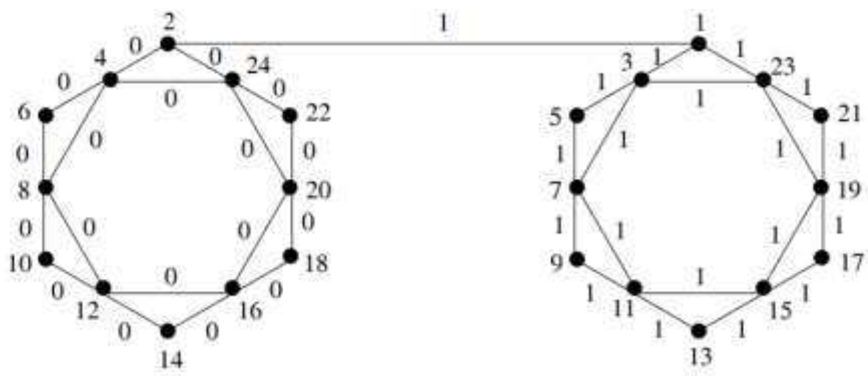


Figure 4.3: The graph $K_2 \Theta C_6(C_6)$

Theorem: 4.14

The graph $< K_{1,n,n} >$ for $n \geq 3$ admits prime cordial labeling.

Proof:

Let the vertex set and edge set of $< K_{1,n,n} >$ be defined as $V = [v_1, v_2, \dots, v_{2n+1}\}$ and $E = \{E_1 \cup E_2 \cup E_3\}$ Where

$E_1 = \{v_1v_2, v_1v_{2n+1}, v_4v_5, v_6v_3\}$, $E_2 = \{v_2v_{2i+2}; 1 \leq i \leq n-1\}$ and

$E_3 = \{v_{2i+6}v_{2i+5}; 1 \leq i \leq n-3\}$.

Let $f : V \to \{1,2,3, \dots, |V|\}$ be the bijective function defined as $f(v_i) = i; 1 \leq i \leq 2n+1$. We compute the edge labeling defined by

$$g(u,v) = 0 \quad if\ \big(f(u), f(v)\big) > 1$$

$$1 \quad if\ \big(f(u), f(v)\big) = 1$$

$$g(v_1v_2) = g(v_1v_{2n+1}) = g(v_4v_5) = 1$$

$$g(v_6v_3) = 0$$

$$g(v_2v_{2i+2}) = 0; 1 \leq i \leq n-1$$

$$g(v_{2i+6}v_{2i+5}) = 1; 1 \leq i \leq n-3$$

The total number of edges labeled with 1's is given by $K_1 = n$ and the total numbers of edges Labeled with 0's are given by$K_2 = n$. Therefore the total difference between 1's and 0's is given by $K_1 - K_2 = n - n$ and they differ by zero. This proves that $< K_{1,n,n} >$ for $n \geq 3$ is a prime cordial labeling.

Theorem: 4.15

The graph $S_n^{(1)}:S_n^{(2)}$ for all $n > 2$ admits prime cordial labeling.

Proof:

Let the vertex set and edge set of $S_n^{(1)}:S_n^{(2)}$ be defined as $V = \{v_1, v_2, v_3, \dots, v_{2n+1}\}$ and $E = \{E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5\}$ Where

$E_1 = \{v_1v_3, v_3v_6, v_2v_4, v_4v_8, v_6v_2, v_6v_4\}$

$E_2 = \{v_6v_{2i}; 1 \leq i \leq n\}$

$E_3 = \{v_{2i+6}v_{2i+8}; 1 \leq i \leq n-4$

$E_4 = \{v_1v_{2i+3}; 1 \leq i \leq n-1\}$

$E_5 = \{v_{2i+3}v_{2i+5}; 1 \leq i \leq n-2\}$.

Let $f: V \to \{1,2, \dots, |v|\}$ be the bijective function defined as

$f(v_i) = i; 1 \leq i \leq 2+1.$

We compute the edge labeling defined by

$g(u,v) = 0 \;\; if \;\; \big(f(u), f(v)\big) > 1 \qquad and$

$\qquad\qquad 1 \;\; if \;\; \big(f(u), f(v)\big) = 1$

$g(v_1v_3) = 1$

$g(v_6v_3) = g(v_2v_4) = g(v_4v_8) = g(v_6v_2) = g(v_6v_4) = 0$

$g(v_6v_{2i}) = 0; 4 \leq i \leq n$

$g(v_{2i+6}v_{2i+8}) = 0; 1 \leq i \leq n-4$

$g(v_1 v_{2i+3}) = 1; 1 \leq i \leq n-1$

$g(v_{2i+3} v_{2i+5}) = 1; 1 \leq i \leq n-2$

The total numbers of edges labeled with 1's are given by $K_1 = 2n-2$ and the total numbers of edges labeled with 0's are given by $K_2 = 2n-2$. Therefore the total difference between 1's and 0's is given by $|K_1 - K_2| = |(2n-2)-(2n-2)|$ and they differ by zero. This proves that the graph $S_n^{(1)}: S_n^{(2)}$ for $n > 2$ as a prime cordial labeling.

Theorem: 4.16

The full binary tree admits prime cordial.

Proof:

The root a_0 is called the special vertex in tree. Let N denotes the number of levels in full binary tree. The root has edges to n other vertices called children. The children of root are said to be on level one. There are $2^{N+1}-1$ vertices and $2^{N+1}-2$ edges in full binary tree.

Case (i):

Let a_1 and a_2 be the children ofa_0. The vertices on the last level N have no children and are leaves. The vertices are not leaves are said to be internal vertices. Let $a_0 = 1, a_1 = 2 \; and \; a_2 = 3$ are fixed. The vertices a_1 and a_2 are divided into two sub trees. The left most subtree a_1 are labeled with consecutive even numbers from top to bottom. The greatest common divisor of any two numbers on the leftmost subtree is greater than one and the edges are labeled with zero. In the second level the vertices on the right most subtree a_3 are labeled with 7 and 5 and the remaining levels of the right most subtree are labeled with consecutive odd numbers starting with the number 9. The right most vertices of level 2 and level 3 are connected by an edge with label zero since $\gcd(5,15) > 1$ and all other edges on the rightmost subtree a_3 are labeled with one. The root a_0 with their children $a_1 and\ a_2$ are connected by the edges labeled with one.

The total number of edges labeled with 0 is given by $k_1 = 2^n - 1$ and the total number of edges labeled with 1 is given by $k_2 = 2^n - 1$. Therefore the total difference between 1's and 0's is given by $|K_1 - K_2| = |(2^n - 1) - (2^n - 1)|$ and they differ by zero. This proves that the binary tree is prime cordial labeling.

Case (ii):

The full binary tree with the second level is prime cordial when $a_0 = 2, a_1 = 4\ and\ a_2 = 6$. In the second level the vertices are numbered as 1, 5, 3, and 7. The total number of edges labeled with 0 is given by $k_1 = 3$ and the total number of edges labeled with 1 is given by$k_2 = 3$. Therefore the total difference between 1's and 0's is given by $|K_1 - K_2| = |3 - 3|$ and they differ by zero.

Theorem: 4.17

$spl(K_{1,n})$ is a prime cordial graph.

Proof:

Let $v_1,\ v_2, \dots, v_n$ be the pendant vertices, v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, \dots, u_n$ are the vertices corresponding to $v, v_1,\ v_2, \dots, v_n$ in $spl\left(K_{1,n}\right)$.

Denoting $spl\left(K_{1,n}\right) = G$ then $|V(G)| = 2n + 2$ and $|E(G)| = 3n$.

To define $f: V(G) \to \{1,2,\dots,2n + 2\}$, we consider following two cases.

Case: 1 $n = 2, 3$

The graphs $spl(K_{1,2})$ and $spl(K_{1,3})$ are to be dealt separately and their prime cordial labeling is shown in Fig 4.4

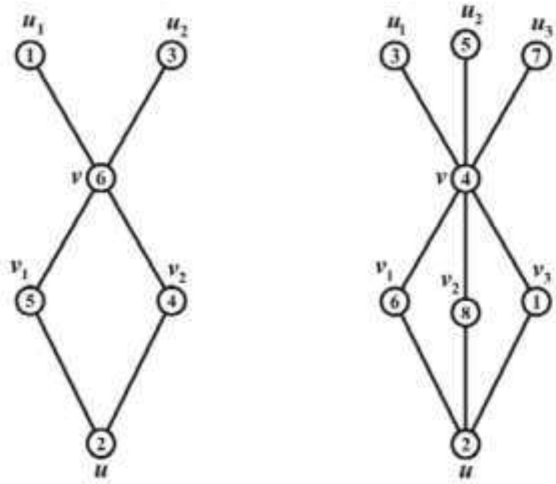


Figure 4.4 $spl\left(K_{1,2}\right)$ and $spl(K_{1,3})$

Case: 2 $n \geq 4$

$f(v) = 4,$

$f(u) = 2,$

$f(v_1) = 2i + 4; \quad 1 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor + 1$

$f\left(v_{\left\lfloor\frac{n}{2}\right\rfloor+1+i}\right) = 2i - 1; \quad 1 \leq i \leq \left\lceil\frac{n-2}{2}\right\rceil$

$f(u_i) = f(v_n) + 2i; \quad 1 \leq i \leq n + 1 - \left\lceil\frac{n-2}{2}\right\rceil$

$f(u_{n+1-i}) = 2(n + 2 - i); \quad 1 \leq i \leq \left\lceil\frac{n-4}{2}\right\rceil$

In the view of the labeling pattern defined above we have

$e_f(0) = \frac{3n}{2} = e_f(1)$ for n even and

$e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor = e_f(1) - 1$ for n odd. Thus we have $\left| e_f(0) - e_f(1) \right| \leq 1$.

Hence G is prime cordial graph.

Theorem: 4.18

$spl\ (B_{n,n})$ is prime cordial graph.

Proof:

Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i, v_i are pendant vertices. In order to obtain $spl(B_{n,n})$ and u', v', u_i', v_i' vertices corresponding to u, v, u_i, v_i where $1 \leq i \leq n$. If $G = spl(B_{n,n})$ then $|V(G)| = 4(n+1)$ and

$|E(G)| = 6n + 3$. we define vertex labeling

$f: V(G) \to \{1, 2, .., 4(n+1)\}$ as follows.

$f(u) = 6,$

$f(v) = 2,$

$f(u') = 4,$

$f(v') = 1,$

$f(u_i) = 8 + 2(i-1);\ \ 1 \leq i \leq n$

$f(u_i') = f(u_n) + 2i;\ \ \ 1 \leq i \leq n-1$

$f(u_n') = 3,$

$f(v_i) = 5 + 2(i-1);\ \ 1 \leq i \leq n$

$f(v_i') = f(v_n) + 2i;\ \ 1 \le i \le n$

In view of pattern defined above we have

$e_f(0) = 3n + 2 = e_f(1) + 1.$

That is $|e_f(0) - e_f(1)| \le 1.$

Hence G is a prime cordial graph.

Prime cordial labeling of the graph $spl(B_{5,5})$ is shown in Figure 4.5

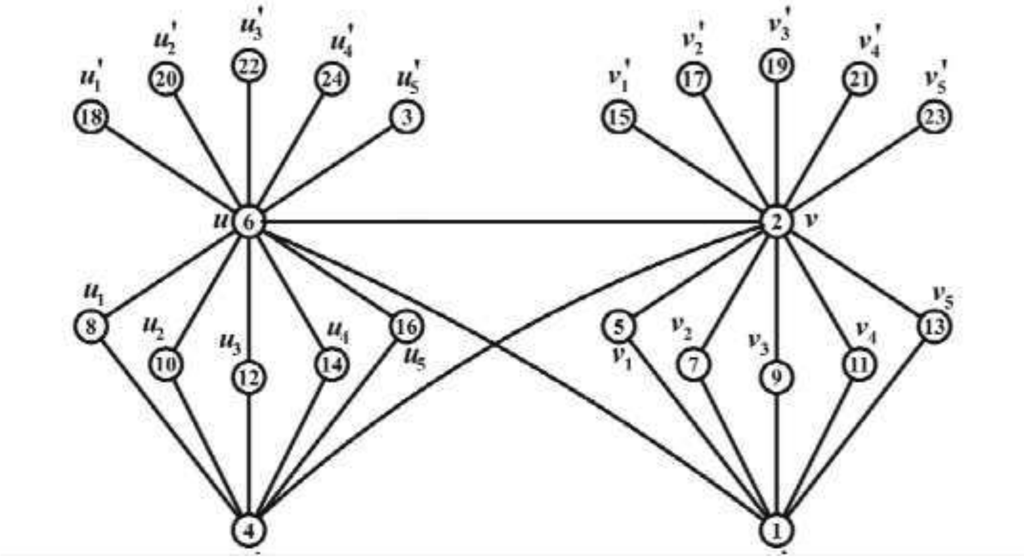


Figure 4.6

Theorem :4.19

$B^2_{n,n}$ is a prime cordial graph .

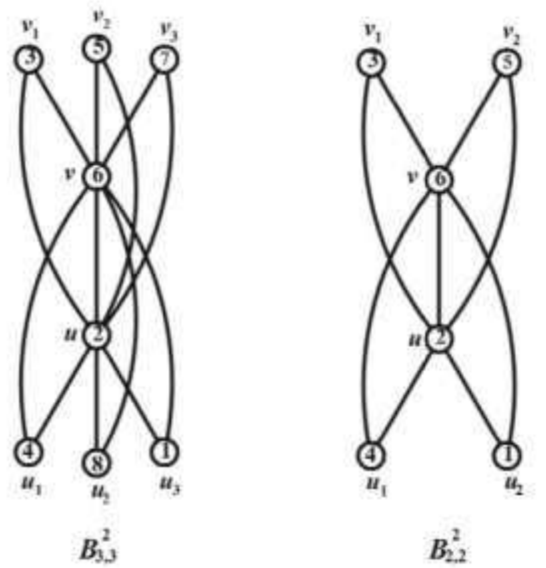
Proof :

Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \le i \le n\}$ where u_i, v_i pendant vertices. Let G be a graph $B^2_{n,n}$ then$|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1.$

To define $f: V(G) \rightarrow \{1,2,3, \ldots, 2n+2\}$, we consider following two cases.

Case 1: $n = 2, 3$

The graphs $B_{2,2}^2$ and $B_{3,3}^2$ are to be dealt separately and their prime cordial labeling is shown in figure 4.7



Case :2 $n \geq 4$

Choose a prime number p such that $3p \leq 2n+2 < 5p$,

$$f(u) = 2,$$

$$f(v) = 1,$$

$f(u_i) = m$, where m is distinct even numbers between 4 and $2n+2$ except $2p$ with $2 \leq i \leq n$.

$$f(v) = 2p,$$

$$f(v_i) = 3 + 2(i-1) \text{ for } 1 \leq i \leq n.$$

In view of the above defined labeling pattern we have

$$e_f(o) - 1 = 2n = e_f(1).$$

Thus in both the cases we have $|e_f(0) - e_f(1)| \leq 1$.

Hence G is a prime cordial graph.

A STUDY ON GRACEFUL COLOURING GRAPHS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

X.JENO SELES

Reg. No: 19SPMT10

Under the guidance of

Dr. C. REENA M.Sc., B.Ed., M.Phil., SET., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON GRACEFUL COLOURING GRAPHS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **X . JENO SELES (Reg. No: 19SPMT10)**

Signature of the Guide **Signature of the Coordinator**

Joyce
Signature of the Director
Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose
Signature of the Principal
Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON GRACEFUL COLOURING GRAPHS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Dr. C. Reena M.Sc., B.Ed., M.Phil., SET., Ph.D.,** Assistant Professor, Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi .**

X·Jeno Seles

Station: Thoothukudi **Signature of the Student**

Date: 10·4·2021

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I express my sincere gratitude and heartfelt thanks to our principal **Rev. Dr. Sr. A.S.J. Lucia Rose M.Sc., PGDCA., M.Phil., Ph.D.,** and to our Director **Rev. Dr. Sr. F. Mary Joyce Baby M.A., M.Phil., Ph.D.,** for kindly permitting me to do this project.

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Place: Thoothukudi

Date: 10·4·2021

CONTENT

CHAPTER 1

PRELIMINARIES

In this chapter I have given the basic definitions on graph theory which are needed for the subsequence chapters.

Definition 1.1 :

A **graph** G consists of a finite nonempty set of points, denoted by V (G), called **vertices**, and a collection of pairs of elements, E(G), called **edges**.

The elements of E(G) are pairs of vertices.

Definition 1.2 :

The **degree** (or **vertex degree**) of a vertex v, deg(v), is equivalent to the number of edges incident with that vertex. For a graph G the largest degree in the graph, that is the largest value of deg(v) for all $v \in G$, is denoted as Δ

(it can also be referred to as $\Delta(G)$, especially if it is unclear which graph it relates to).

Definition 1.3 :

A vertex with a degree equal to one is called an **end point** (or **end vertex**).

Definition 1.4 :

A **walk** in a graph consists of an alternating sequence of vertices and edges, beginning at a vertex called the **initial vertex** and ending at a final vertex known as the **terminal vertex**.

Definition 1.5 :

If the vertices in the walk are all distinct then it is called a **path**.

Definition 1.6 :

If a walk has $v_1 = v_n$ and all edges in the walk are distinct, then we get what is known as a **cycle**.

Definition 1.7 :

A **subgraph** H of a graph G is a graph whose vertices and edges lie within V(G) and E(G) respectively.

Definition 1.8 :

A graph G is **connected** if and only if there exists a path between every pair of vertices.

Definition 1.9 :

A **path graph** by P_n, where n is the total number of vertices in the graph and n − 1 is the number of edges, it is also connected. The vertices of P_n can be written as v_1, v_2 , v_3 , ..., v_n where an edge connects v_i to v_{i+1} , for $1 \leq i \leq n-1$, in the case where i = n, v_n is adjacent to v_{n-1} . Hence, all the vertices have a degree less than or equal to 2 (in fact only two vertices, the end points v_1 and v_n, will have a degree of value 1).

Definition 1.10 :

A **star graph**, S_n, is a graph with n vertices and n−1 edges, such that one vertex, call this the **central vertex**, has the degree Δ = n − 1 while all other vertices are end points with degree of value 1.

Definition 1.11 :

The graph C_n is a **cycle graph** with n vertices and n edges, this is such that every vertex has degree 2 and the graph is connected.

Definition 1.12 :

The **complete graph** K_n is the graph with n vertices such that every pair of vertices in the graph is adjacent. This makes $|E(K_n)| = \frac{n(n-1)}{2}$.

Definition 1.13 :

A **wheel graph**, W_n, has n vertices with n ≥ 4. It consists of the cycle graph C_{n-1} where every vertex is adjacent to another additional single vertex, called the **central vertex**, displayed in the middle of the cycle. Hence every vertex has a degree of value 3 except the central vertex which has a degree equal to n−1.The total number of edges in a wheel graph W_n is $2(n-1)$.

Definition 1.14 :

A **bipartite graph** is also a graph whose vertices can be split into two sets, A and B, such that vertices in set A are only adjacent to a vertex in set B, and vice-versa, however unlike a complete bipartite graph, not all vertices in set A are adjacent to vertices in set B.

Definition 1.15 :

A **complete bipartite graph**, $k_{a,b}$, is a graph whose vertices can be split into two sets, call these set A and set B, such that every vertex in set A is adjacent to every vertex in set B only (and vice-versa). The total number of vertices in set A is a and the number of vertices in set B is b, hence the graph $k_{a,b}$ has a+b vertices in total and ab edges.

Definition 1.16 :

For a graph G if the vertices can be allocated one of k colours such that no adjacent vertices have the same colour, then G is called **k-colourable** and is described to have a **proper vertex colouring**.

Definition 1.17 :

The **chromatic number** of a graph G, denoted χ(G), is k if G is k-colourable but not (k−1)-colourable. Therefore, if a graph has n vertices the chromatic number cannot be greater than n.

Definition 1.18 :

A graph G is called **k-edge-colourable** if the edges of the graph can be coloured in k colours such that no adjacent edges are the same colour. G is said to have a **k-edge-colouring** and is described to have a **proper edge colouring**. If G is k-edge-colourable but is not (k −1) edge-colourable it is said that the **chromatic index** of G is k, denoted as $\chi'(G) = k$.

Definition 1.19 :

The **chromatic polynomial**, $P_G(\lambda)$, of a graph G is the number of ways to colour the vertices of the graph with λ or fewer colours.

Definition 1.20 :

A graph G is called **Eulerian** if there exists a closed walk, that is a walk that begins and ends at the same vertex, which contains every edge in the graph exactly once. This walk is then referred to as an **Eulerian circuit** for G.

Definition 1.21 :

A **tree** is a connected graph with no cycles. If a graph is not connected but still contains no cycles then it is called a **forest**, which has components that are all trees.

Definition 1.22 :

Any vertices in a tree with a degree of one is called a leaf.

CHAPTER 2

GRACEFUL GRAPHS

Definition 2.1 :

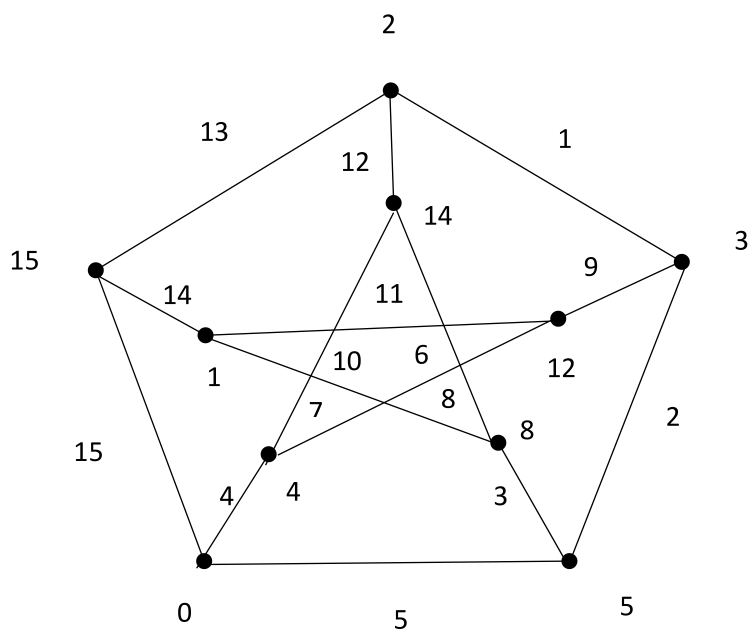
A **graceful labelling** of a graph G consists of:

A) A labelling of the vertices in G from the open set of integers {0, . . . , m}, where m is the number of edges;

B) A labelling of the edges of G, where the edge label corresponds to the absolute difference of the vertex labels of the vertices that the edge is adjacent to, such that the set of edge labels is the closed set {1,...,m}.

G is then known as a **graceful graph**.

Example 2.2 :



The gracefully labelled Petersen graph

Theorem 2.3 :

Let G be a graph with n vertices and m edges. If G is graceful then it is possible to partition the vertices into two sets, A and B, such that the number of edges connecting set A to set B is $\frac{m}{2}$, if m is an even integer, or $\frac{(m+1)}{2}$, if m is an odd integer.

Proof :

Let G be a graph with a graceful labelling.

We can partition G's vertices into two sets, one being the set of vertices with even vertex labels, A, and the other being the set with odd vertex labels, B.

The m edges will be labelled from 1 to m, hence if m is even, $\frac{m}{2}$ of the edge labels will be odd integers or if m is odd, $\frac{(m+1)}{2}$ of the edge labels will be odd integers.

Conclusively, any edge with an odd edge label must be incident with one even valued vertex label and one odd valued vertex label, since the edge labels are determined by the absolute difference of the vertices an edge is adjacent with.

Therefore, in both cases, the edges labelled with odd integers represent the set of edges connecting vertices in A to B, so the theorem holds.

Theorem 2.4 :

All paths are graceful.

Proof :

Let P_n be a graph with n vertices and m edges.

Label the first vertex in the path (so a vertex of degree 1) as 0.

From this point skip the next adjoining vertex in the path and label the vertex following this as 1.

Continue to do this along the path, increasing the vertex label by one each time, so that alternating vertices along the path are labelled from 0 to $\frac{m}{2}$, if m is even, or 0 to $\frac{(m-1)}{2}$, if m is odd.

Then starting from the second vertex in the path (adjacent to vertex 0) label this as m.

From here skip the next adjoining vertex, which would now be labelled as 1, and label the following vertex as m − 1.

Continue to do this along the path, decreasing the vertex label by one each time.

This newly labelled set of vertices should go from m down to $\frac{(m+2)}{2}$, if m is even, or $\frac{(m+2)}{2}$, if m is odd.

Therefore P_n has m + 1 vertices all with a distinct label from the set {0,...,m}. Consequently, all edges will have a distinct label from the set.

As a result, by having the vertices of the path labelled this way the edge labels are calculated to be: m, m − 1, m − 2, ..., 2, 1, as you traverse along the path starting at vertex 0.

This is a requirement for a graph to be graceful, hence P_n has a graceful labelling, thus proving the theorem.

Theorem 2.5 :

All star graphs, S_n, are graceful for all n.

Proof :

A star graph has n vertices and n − 1 edges.

It consists of one central vertex of degree n − 1 and has the remaining vertices, all with degree 1, connected to it.

Considering this let the central vertex be labelled with the vertex label 0;

therefore every other vertex in the graph may be labelled with distinct vertex labels from 1 to n − 1.

As a result the set of absolute differences along each edge,

i.e. the edge labels, will be the complete set of integers from 1 to n − 1,

hence the graph is graceful for all n in S_n.

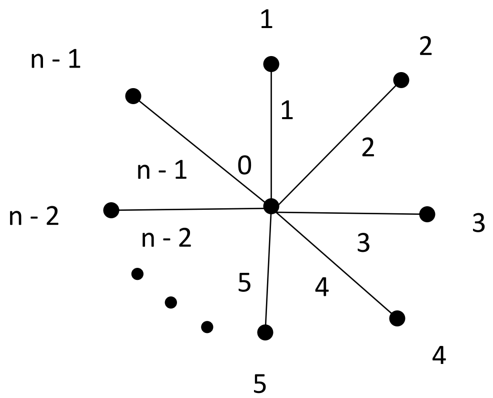


Figure 2.6

Theorem 2.7 :

The complete bipartite graph $k_{a,b}$, where a and b are positive integers, is graceful for all values of a and b.

Proof :

This theorem will be proven by showing a graceful labelling for $k_{a,b}$.

Note that this graph has a + b vertices and ab edges.

Consider the two sets of vertices, A and B, consisting of a and b elements respectively.

Let the vertices in set A be labelled with the integers: 0, 1, ..., a − 1,

and the vertices in set B be labelled with the integers: a, 2a, . . . , ba.

This type of labelling means that every number from 1 to ab can be calculated to be a distinct edge label for the edges of the graph;

this is because every edge uniquely connects a vertex in B with a vertex in A such that every individual vertex in A is adjacent to every vertex in B, and vice-versa.

Hence, $k_{a,b}$ has a graceful labelling.

Remark 2.8 :

The star graph S_n is in fact a complete bipartite graph, $k_{1,n}$

Theorem 2.9 :

The graph W_n is graceful for all n ≥ 4.

Proof :

For a wheel graph W_n , n denotes the total number of vertices in the graph.

In this proof we shall let k = n − 1, furthermore, the central vertex will always be called v_0 and the outer vertices will be represented by the closed cycle

$\{v_1, v_2, \ldots, v_k\}$.

We will separate the proof into two cases, one where k is even and one where k is odd.

First we will examine the case where k is even.

When k = 4 we can give the graph W_5 the graceful labelling in figure 2.10

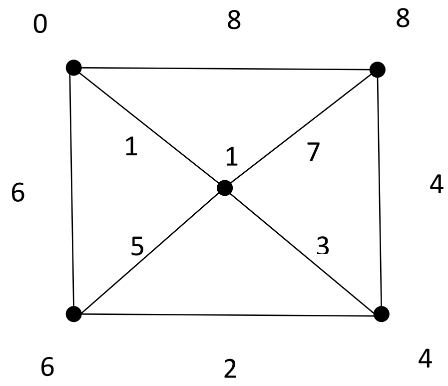


Figure 2.10

The gracefully labelled graph W_5.

For cases where k ≥ 6 we get a graceful labelling of W_n using the following formula, f_e, which will allocate numbers to the vertices v_i, where $0 \leq i \leq k$:

$$f_e(v_i) = \begin{cases} 2k - i - 1 & \text{if } i = 2, 4, 6, \ldots, k-2 \\ 2 & \text{if } i = k-1 \\ i & \text{if } i = 1,3,5, \ldots, k-3, \text{ also if } i = 0 \\ 2k & \text{if } i = k \end{cases}$$

From here the edge labels are calculated by taking the absolute difference of the vertices that edge is incident with.

When k = 8 we can derive the graceful labelling of the graph W_9 in figure 2.11

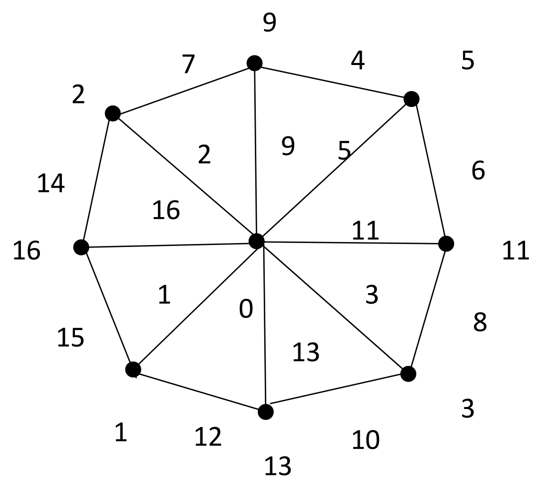


Figure 2.11

The graceful labelling of the graph W_9

Now we will look at the case where k is odd.

For k = 3 the wheel W_4 is also the complete graph k_4, a graceful labelling in figure 2.12

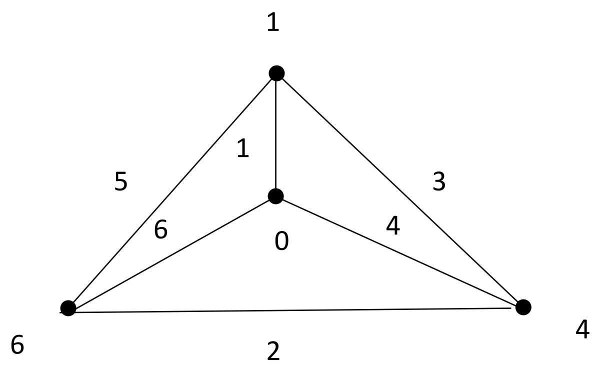


Figure 2.12

The graceful labelling of the graph $W_4 = k_4$

When k ≥ 5, a graceful labelling can be given to W_n using the following formula, f_0, which will allocate numbers to the vertices v_i , where 0 ≤ i ≤ k:

$$f_e(v_i) = \begin{cases} 2i & \text{if } i = 0,\ 1 \text{ or } k \\ k + i & \text{if } i = 2, 4, 6, \ldots, k-1 \\ k + 1 - i & \text{if } i = 3, 5, 7, \ldots, k-2 \end{cases}$$

Again, from here the edge labels are calculated accordingly.

We have shown a way to gracefully label all wheel graphs W_n.

Theorem 2.13 :

Any complete graph K_n with n > 4 cannot be graceful.

Proof :

When n > 4 the graph K_n has its total number of edges m ≥ 10.

Assume K_n can have a graceful labelling.

Then the vertices of the graph can be labelled using a set of values from the open set {0,1,2,...,m} such that all the edges of the graph can be assigned distinct labels from the closed set {1,2,...,m}.

For K_n to have an edge labelled with m, both 0 and m must be vertex labels of vertices in that graph.

(This would be the case for the graph K_2 with m = 1).

Following this, for there to be an edge labelled with m − 1, either 1 or m − 1 must be a vertex label of a vertex in that graph.

The vertex label 1 may be selected since for a graceful graph G with m edges, every vertex label v_i can be replaced by m − v_i and yet all the edge labels would still remain the same, so there is no loss of generality.

Hence we will proceed with the case where the vertex label 1 was chosen knowing that if m − 1 had been chosen a similar overall outcome would have been reached.

To get an edge labelled with m − 2 the vertex label m − 2 must be added to a vertex in that graph, given that we already have vertex labels 0, 1 and m included.

If the vertex label m − 1 had be chosen instead, to get the edge label m−2 allocated to the edge connecting the vertices m − 1 and 1, there would then be two edges with the label 1,

i.e. the edge incident with the vertices labelled 0 and 1 and the edge incident with the vertices labelled m − 1 and m, which is not allowed.

Similarly, if the vertex label 2 was added to get the edge label m − 2 for the edge incident with m and 2, this would again result in two edges having the edge label 1, namely the edges incident with vertices 0 and 1 and vertices 1 and 2.

Hence the vertex label m − 2 must be chosen.

Now with the vertices labelled as 0, 1, m − 2 and m we get the set of edge labels {1,2,m − 3,m − 2,m − 1,m}.

To get an edge labelled with m − 4 a vertex with label 4 must be added.

Other choices of values for this vertex are dismissed using the same principles previously demonstrated.

We now have vertices with the labels 0, 1, 4, $m-2$ and m giving us the set of edge labels $\{1,2,3,4,m-6,m-4,m-3,m-2,m-1,m\}$.

Following this, there is no way to add another labelled vertex to obtain an edge with the label $m-5$.

This is because all the ways to obtain the $m-5$ edge label, as a difference of the two vertices it is incident with, creates duplicate edge labels in the rest of the graph,

i.e. there are no vertex label options from the set

$\{2,3,5,m-5,m-4,m-3,m-1\}$ that can be selected.

This contradicts the statement that K_n is graceful for all cases where $m-5>4$,

i.e. when $n \geq 5$.

Hence the theorem holds.

Theorem 2.14 :

C_5 is not graceful.

Proof :

For C_5 to be graceful the graph must be able to be labelled with a selection of five distinct vertex labels from the set $\{0,1,2,3,4,5\}$.

Every edge must then have a distinct edge label from the closed set

$\{1,2,3,4,5\}$, where the allocated edge label is the absolute difference of the vertex labels the edge incidents with.

Assume C_5 is graceful.

The only way an edge label of value 5 can feature on the graph is if the two vertices the edge incidents with have the vertex labels 0 and 5.

Assign two adjacent vertices on the graph with these labels in figure 2.15

Note that due to the symmetry of the graph it doesn't matter which two vertices are selected.

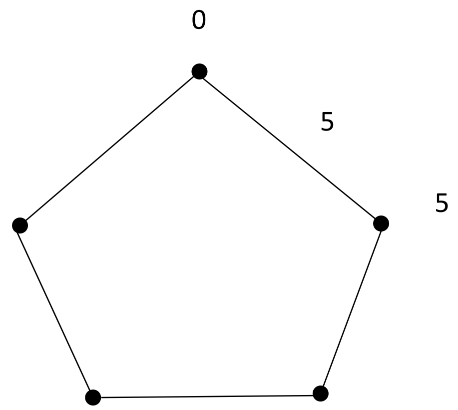


Figure 2.15

The first step attempting to gracefully label C_5.

Next we can determine that there are only two ways for the edge label 4 to be added to the graph.

This is by either allowing the vertex adjacent to vertex 0 to be allocated the vertex label 4 (call this Case A) or by the vertex adjacent to vertex 5 being given the vertex label 1 (call this Case B).

Now let us first investigate Case A.

Case A :

Say we add a vertex label of value 4 to the vertex adjacent to vertex 0 to get an edge label of value 4.

We now need to add an edge label of value 3 to the graph. Here there are again only two possible options:

either allocate the vertex adjacent to vertex 4 with the vertex label 1

(call this Option 1) or allocate the vertex adjacent to vertex 5 with the vertex label 2 (call this Option 2).

Let us select Option 1 and add the vertex label 1 to the graph, we are now left with one remaining vertex without a label, call this vertex w in figure 2.16

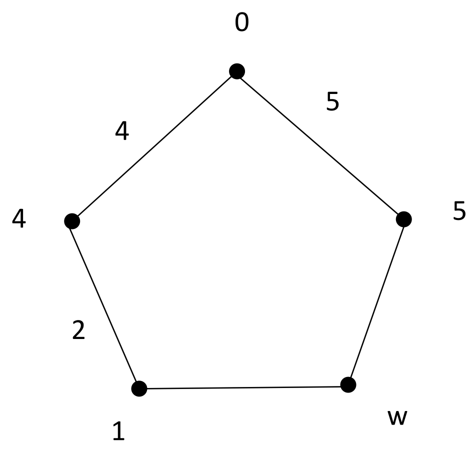


Figure 2.16

The second step attempting to gracefully label C_5.

There are now only two vertex labels left that can be given to w, label 2 or 3. However, if w was given a label of value 2 the edge label 1 can be added to the graph on the edge incident with vertices 1 and 2.

But then the graph will not feature an edge label of value 2 seeing as the remaining unlabelled edge would be incident with vertices 2 and 5, for which the absolute difference is 3.

On the other hand if w was given a vertex label of value 3, the vertices 1 and 5, that w is incident with, will both have the absolute difference of their vertex labels calculated to be 2.

This not only means that no edge label of value 1 can be added to the graph but also that a duplication of an edge label value, in this case 2, would appear.

Hence no graceful labelling can be found this way.

We will now examine Option 2.

So we still add a vertex label of value 4 to the graph in figure 2.15 , but this time we allocate the vertex adjacent to vertex 5 with the vertex label 2 to get an edge label of 3 added to the graph.

There again remains one unlabelled vertex, call this vertex x, in figure 2.17.

Vertex x can now only be given a vertex label of value 1 or 3.

However, similar to what occurred in Option 1, if the value 1 is selected to be this vertex label the edge label 1 can be added to the graph (between vertices 1 and 2) but not the edge label of value 2.

Consequently if the vertex label 3 is given to x the edge label 1 can again be added to the graph (but this time on the edge incident with vertices 3 and 4).

Nonetheless, the edge label of value 2 again cannot be added to the graph. Therefore the graph cannot be gracefully labelled this way.

Case B :

We now look back at the graph in figure 2.15 and add the vertex label 1 to the unlabelled vertex adjacent to vertex 5 to get an edge label of value 4.

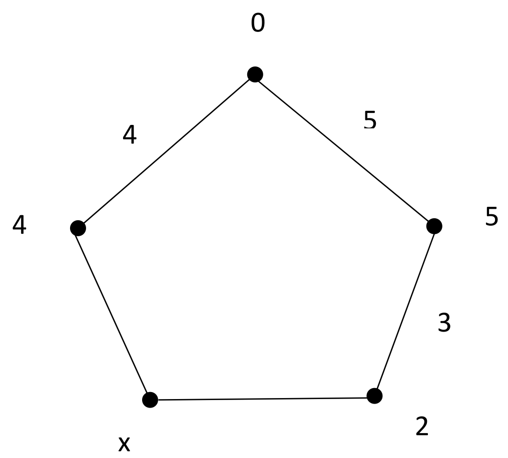


Figure 2.17

The third step attempting to gracefully label C_5.

From here we want to add the edge label of value 3 to the graph.

The only two options this time is to either add the vertex label 4 to the vertex adjacent to vertex 1 (call this Option 3) or allocate the vertex adjacent to vertex 0 with vertex label 3 (call this Option 4).

Say we select Option 3 and add the vertex label 4 to the graph, we are left with one unlabelled vertex again call this y in figure 2.18.

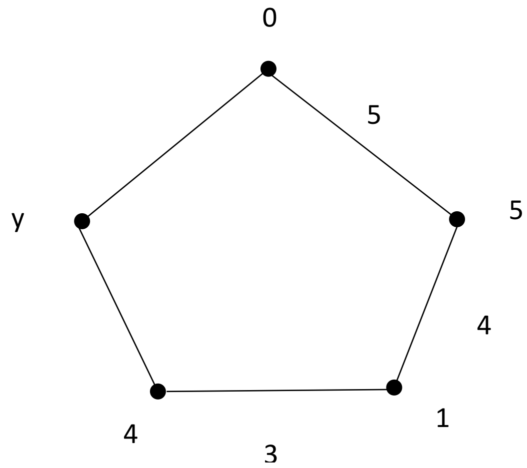


Figure 2.18

The fourth step attempting to gracefully label C_5.

From here we can see that vertex y can either be given a vertex label of value 2 or 3.

Nevertheless, if the vertex label 2 was given to y the edge label 2 can be assigned to the graph on the edge between vertices 0 and 2 (and vertices 2 and 4)

but the graph would not feature an edge label of 1.

If the vertex label of value 3 was given to y instead, the edge incident with vertices 3 and 4 could be allocated the edge label of value 1,

however no edge label of value 2 could be added to the graph

(there also would be a duplication of the edge label 3 between vertices 0 and 3).

Therefore no graceful labelling can be given.

Finally we will look at Option 4, this involves still adding a vertex label with value 1 to Figure 2.15, as done in Option 3,

but now instead we add the vertex label 3 to the vertex adjacent to vertex 0 to get an edge label of value 3.

Here we are left with an unlabelled vertex, call this z in figure 2.19

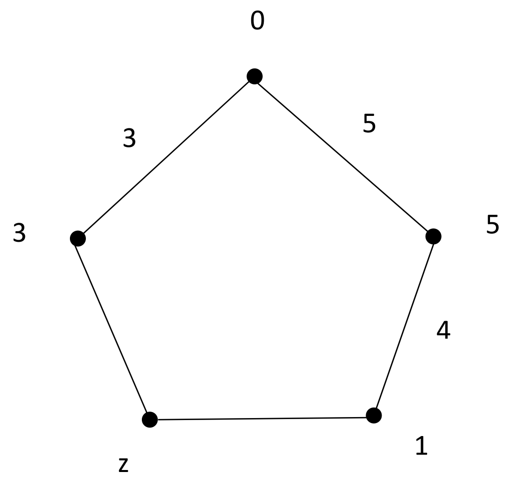


Figure 2.19

The final step attempting to gracefully label C_5.

Vertex z can now either be labelled with value 2 or 4.

If the vertex label 2 was given to z then the edge label 1 would need to be added to the graph for both the edge incident with vertices 1 and 2 and the edge incident with vertices 2 and 3,

this is duplication which is not allowed in a graceful graph.

On the other hand, if z was given the vertex label 4 the edge label of value 1 could be assigned to the edge connecting vertices 3 and 4 but no edge label of value 2 could be added to the graph.

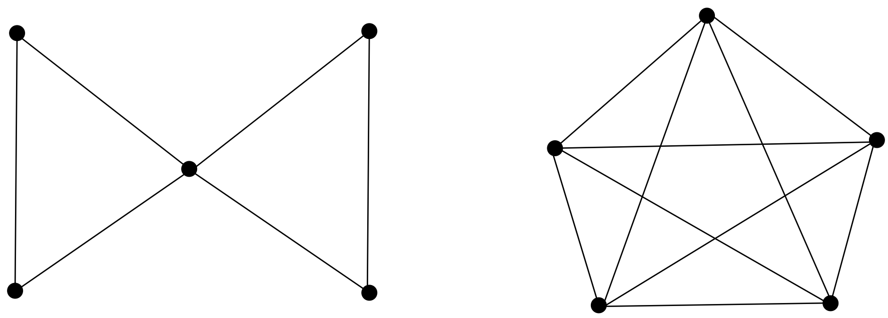
Therefore it cannot be gracefully labelled.

As a result, every possible way of labelling an edge in the graph C_5 with the edge label 4 has been exhausted and no graceful labelling of the graph has been found.

Hence, C_5 is not a graceful graph.

Remark 2.20 :

C_5 is actually one of three graphs containing 5 or fewer vertices that is not graceful, the other two graphs are shown. All other graphs consisting of 5 or fewer vertices can have a graceful labelling.



Two of three graphs containing 5 or fewer vertices that are not graceful.

Theorem 2.21 :

If G is a graceful Eulerian graph with n vertices, then n ≡ 0 (mod 4) or n ≡ 3 (mod 4).

Proof :

Let C be an Eulerian circuit in the graph G which follows the walk: $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$.

Let a graceful labelling of G be such that the integer a_i, where $0 \leq a_i \leq$ n, is assigned to v_i , where $0 \leq$ i $\leq$ n, with $a_i = a_j$ if $v_i = v_j$.

Therefore the label given to the edge incident with vertices v_{i-1} and v_i is the absolute difference of a_i and a_{i-1}, i.e. $|a_i - a_{i-1}|$.

Notice that, $|a_i - a_{i-1}\ | \equiv (a_i - a_{i-1})(\text{mod } 2)$ for $1 \leq i \leq n$.

This implies that the sum of the edge labels in G is,

$\sum_{i=1}^{n} |a_i - a_{i-1}\ | \equiv \sum_{i=1}^{n} (a_i - a_{i-1}\ ) \equiv 0 \ (\text{mod } 2)$,

which means the sum of the edge labels in G is even.

However the sum of the edge labels is

$$\sum_{i=1}^{n} \frac{n(n+1)}{2},$$

so $\frac{n(n+1)}{2}$ is even.

As a result, $4|n(n+1)$, which suggests $4|n$ or $4|(n+1)$,

leading to $n \equiv 0 \ (\text{mod } 4)$ or $n \equiv 3 \ (\text{mod } 4)$.

Theorem 2.22 :

The graph C_n is graceful if and only if $n \equiv 0 \ (\text{mod } 4)$ or $n \equiv 3 \ (\text{mod } 4)$.

Proof :

It is known that C_n is an Eulerian graph for all n,

Therefore by Theorem 2.13 if $n \equiv 1 \ (\text{mod } 4)$ or $n \equiv 2 \ (\text{mod } 4)$ then C_n is not graceful.

It will now be shown that if $n \equiv 0 \ (\text{mod } 4)$ or $n \equiv 3 \ (\text{mod } 4)$ then C_n is graceful.

Let C_n be the cycle: $\{v_1, v_2, \ldots, v_n\}$.

First let $n \equiv 0 \ (\text{mod } 4)$, we therefore can assign the vertex v_i in C_n the label a_i where,

$$a_i = \begin{cases} \frac{i-1}{2} & \text{if i is odd} \\ n+1-\frac{i}{2} & \text{if i is even and i} \leq \frac{n}{2} \\ n-\frac{i}{2} & \text{if i is even and i} > \frac{n}{2} \end{cases}$$

Implementing this formula will result in a graceful labelling.

Next we look at the case where n ≡ 3 (mod 4), we assign a vertex v_i in C_n with the label b_i where,

$$b_i = \begin{cases} n+1-\frac{i}{2} & \text{if i is even} \\ \frac{i-1}{2} & \text{if i is odd and i} < \frac{(n-1)}{2} \\ \frac{i+1}{2} & \text{if i is odd and i} > \frac{(n-1)}{2} \end{cases}$$

This also gives a graceful labelling.

CHAPTER 3

COLOURING OF GRACEFUL GRAPHS

Definition 3.1 :

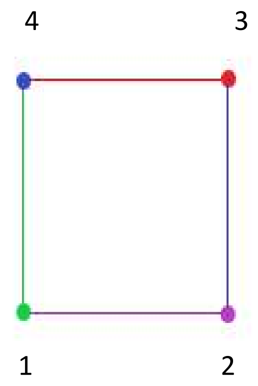
A **rainbow vertex colouring** of a graph G with n vertices and m edges, involves assigning distinct colours from the set {0,...,m} to all the vertices of G - denote this as f(v), where v ∈ V (G). Colours are then allocated from the set {1,...,m} to the edges of G by applying the function f '(uv) = |f(u)−f(v)|, where uv is the edge incident with vertices u, v ∈ G.

If all the edges have distinct colours then we can say that a rainbow vertex colouring has led to a **rainbow edge colouring** of G. We can then say G has a **rainbow colouring**.

If f ' : E(G) → {1,2,...,m}, where every element is distinct and the set is closed, then this implies G has a graceful labelling.

Example 3.2 :

Below we have the cycle graph C_4 which has been given a rainbow vertex colouring. Using the function defined previously this has induced a rainbow edge colouring hence depicting a graceful labelling of C_4. Note that the colour 4 is red, the colour 3 is blue, the colour 2 is green and the colour 1 is violet.



A rainbow colouring on C_4

Theorem 3.3 :

For any natural number n, there exists a graceful graph G such that

$\chi(G) = n$.

Proof :

When n = 1 the result is trivial, so assume $n \geq 2$.

Let v_1, ..., v_n denote the vertices in the complete graph k_n, for every i, such that

$1 \leq i \leq n$.

We will let 2^i be the label of v_i, note that we are purposefully exceeding the usual vertex label bounds at this stage but we will clarify this step later in the proof.

First we prove that when these vertex labels are given and the absolute difference is calculated to provide edge labels for the graph, all the edges of k_n have different labels.

Suppose that $v_i v_j$ and $v_k v_l$, for some i , j, k , l, are two edges with the same labels.

Assume that i > j and k > l, then we deduce that the edge labels of $v_i v_j$ and $v_k v_l$ are $2^i - 2^j$ and $2^k - 2^l$ respectively.

Hence these must be distinct values, unless i = k and j = l which would mean the edges $v_i v_j$ and $v_k v_l$ are the same edge.

Now we note that the largest vertex label in k_n is 2^n for vertex v_n.

It is obvious that for each natural number n we have $2^n > \frac{n(n-1)}{2}$.

Therefore we add $2^n - \frac{n(n-1)}{2}$ new vertices to k_n by connecting them all to v_n.

Call this newly formed graph G.

We will now show that G has a graceful labelling.

For every x, x ∈ {1,..., 2^n} that does not exist as a label of an edge in k_n label one of the new vertices with $2^n - x$.

It is clear that all the new edge labels that can then be calculated will have different values (call this Claim A) .

Moreover, all the vertex labels will be from the set {1,..., 2^n} and will be distinct.

This is because if two vertices had the same label then two edges

(both of which would be incident with v_n) would have the same edge label,

this contradicts Claim A.

Hence a graceful graph exists for any arbitrary number n such that χ(G) = n.

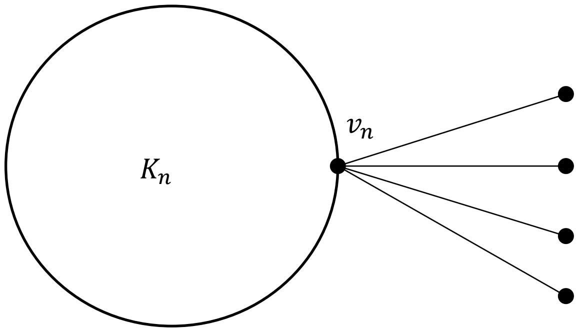


Figure 3.4

Definition 3.5 :

A **graceful k-colouring** of a graph G is a vertex colouring $c : V(G) \to \{1,2,...,k\}$, where $k \geq 2$ and each integer represents a different colour, that induces an edge colouring $c' : E(G) \to \{1,...,k-1\}$ defined by $c(uv) = |c(u) - c(v)|$, where u and v are vertices in G and uv is the edge connecting them.

Any vertex colouring c for a graph G that is a graceful k-colouring for some integer k is called a **graceful colouring** of G.

The **graceful chromatic number** of G, denoted by $\chi_g(G)$, is the minimum value of k that gives G a graceful k-colouring.

Proposition 3.6 :

If G is a graph with n vertices such that $n \geq 3$ and has a diameter of atmost 2, then $\chi_g(G) \geq n$.

Proposition 3.7 :

If the graph H is a subgraph of a graph G, then $\chi_g(H) \leq \chi_g(G)$.

Proposition 3.8 :

If G is a graph with n vertices where n ≥ 3, then we have that

χ_g(G) ≥ max{χ(G),χ'(G)}+ 1.

Theorem 3.9 :

If G is a complete bipartite graph with n vertices such that n ≥ 3, then

χ_g(G) = n.

Proof :

Let G = $k_{a,b}$ be the complete graph where n = a + b such that the set A consists of a vertices with A = {u_1,u_2,...,u_a} and the set B consisting of b vertices with B = {$v_1 v_2$,..., v_b}.

Since the diameter of G is 2 we know by Proposition 3.5 that χ_g(G) ≥ n.

Now consider the colouring c : V (G) →{1,2,...,n} where c(u_i) = i,

for 1 ≤ i ≤ a, and c(v_j) = a + j, for 1 ≤ j ≤ b.

Therefore, c'($u_i w_j$) = |a + $(j-1)$|.

Note that if i is fixed and 1 ≤ $j_1 \neq j_2$ ≤ b, then |a + (j_1 − i)| ≠ |a + (j_2− i)|,

Similarly, if j is fixed 1 ≤ $i_1 \neq i_2$ ≤ a, then |a + (j − i_1)| ≠ |a + (j − i_2)|.

Hence the edges of G have a distinct colouring such that no adjacent edges share the same colour and χ_g(G) = n.

Remark 3.10 :

To examine the graceful chromatic number for cycle graphs we will need to introduce some notation for this we let C_n = (v_1,v_2, . . . , v_n, v_{n+1} = v_1) , where n ≥ 3. For an edge in C_n we let e_i = $v_i v_{i+1}$ for $i = 1, 2, \ldots, n$. Then for the colouring of the vertices, c, in C_n we let $s_c = (c(v_1), c(v_2), \ldots, c(v_n))$.

And similarly for the colouring of the edges, c', of C_n we let

$s_{c'} = (c'(e_1), c'(e_2), \ldots, c'(e_n))$.

Theorem 3.11 :

For n ≥ 4 we have, if n = 5, $\chi_g(C_5)$ = 5 and if n ≠ 5, $\chi_g(C_5)$ = 4.

Proof :

Let C_n be defined as notated above and first suppose n = 5.

Since the diameter of C_5 is 2, we know by Proposition 3.5 that $\chi_g(C_5)$ ≥ 5.

Now define a colouring of the vertices, c, to be the following: s_c = (1, 5, 3, 4, 2).

This then makes the resulting edge colouring c' to be defined as $s_{c'}$= (4, 2, 1, 2, 1).

Therefore, this shows a graceful 5-colouring of C_n so $\chi_g(C_5)$ = 5.

Next we examine the case where n ≠ 5.

To begin with we want to show that in this case $\chi_g(C_n)$ ≥ 4.

Let's assume that in fact a graceful 3-colouirng of C_n exists and say c(v_1) = 1.

Since this is a graceful colouring we must have (c(v_2),c(v_n)) = (2,3),

So let's suppose c(v_2) = 2 and c(v_n) = 3.

This would then mean c(v_3) = 3, as if it was equal to 1 then v_2 would have two edges adjacent to it with the same number hence resulting in two adjacent edges being given the same value, which isn't allowed.

However, this would then lead to c'(v_1v_2) = c'(v_2v_3) = 1 which would go against it resulting in a graceful 3-colouring since again two adjacent edges share the same colour.

Hence $\chi_g(C_n) \geq 4$.

Now will define graceful 4-colourings of C_n for different values of n.

n ≡ 0 (mod 4).

We use the following pattern to denote a graceful 4-colouring of C_n:

for n = 4, let s_c = (1, 2, 4, 3) such that $s_{c'}$= (1, 2, 1, 2).

Then for n ≥ 8, let s_c = (1, 2, 4, 3, ..., 1, 2, 4, 3) which implies that $s_{c'}$ = (1, 2, ..., 1, 2).

n ≡ 1 (mod 4).

We have, when n = 9 let s_c = (1, 2, 4, 1, 2, 4, 1, 2, 4)

so that $s_{c'}$ = (1, 2, 3, 1, 2, 3, 1, 2, 3).

For n $\geq$ 13, let s_c = (1, 2, 4, 3, ..., 1, 2, 4, 3, 1, 2, 4, 1, 2, 4, 1, 2, 4)

which means $s_{c'}$ = (1, 2, 1, 2, ..., 1, 2, 1, 2, 3, 1, 2, 3, 1, 2, 3).

n ≡ 2 (mod 4).

If n = 6 then s_c = (1, 2, 4, 1, 2, 4) leading to $s_{c'}$ = (1, 2, 3, 1, 2, 3).

Then for n ≥ 10, let s_c = (1, 2, 4, 3, ..., 1, 2, 4, 3, 1, 2, 4, 1, 2, 4) so that $s_{c'}$ = (1, 2, 1, 2, ..., 1, 2, 1, 2, 3, 1, 2, 3).

n ≡ 3 (mod 4).

For all cases where n ≥ 7 let s_c = (1, 2, 4, 3, ..., 1, 2, 4, 3, 1, 2, 4).

Then $s_{c'}$ = (1, 2, 1, 2, ..., 1, 2, 1, 2, 3).

Hence in all cases when n ≠ 5 there is a graceful 4-colouring and so $\chi_g(C_n)$ = 4.

Theorem 3.12 :

For a path P_n, where n ≥ 5, $\chi_g(P_n)$ = 4.

Proof :

Denote $P_n = (v_1, v_2, ..., v_n)$.

For n = 5 the graceful 4-colouring, call this c∗,

is such that (c∗(v_1),c∗(v_2),c∗(v_3),c∗(v_4),c∗(v_4)) = (1, 2, 4, 1, 2),

it is easy to see that a graceful 3-colouring is not possible for P_5 ,

hence $\chi_g(P_5)$ = 4.

For n ≥ 6 we observe that P_n is a subgraph of C_n,

therefore by using Proposition 3.7 and theorem 3.11 we can determine that since the graceful chromatic number of C_n when n ≥ 6 is 4 we must have that $\chi_g(P_n) \leq 4$.

We will now show that P_n does not have a graceful 3-colouring.

Suppose $\chi_g(P_n)$ = 3 for some vertex colouring c.

We can see that c(v_3) ≠ 2 since the possible integers for c(v_2) and c(v_4) would result in the edges v_2v_3 and v_3v_4 having the same value, which is not allowed.

Hence let c(v_3) = 1.

Therefore (c(v_2),v(c_4)) = (2,3), so let c(v_2) = 2 which therefore means we must have c(v_1) = 3.

However this results in c'(v_1v_2) = c'(v_2v_3) = 1, which again is not allowed.

Hence no graceful 3-colouring is possible and we can conclude that

$\chi_g(P_n) = 4$.

Theorem 3.13 :

If W_n is a wheel graph with n ≥ 6, then $\chi_g(W_n)$ = n.

Proof :

The graph W_n is the join of the graphs C_{n-1} and K_1 ,

So let $C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_n = v_1)$ and denote the central vertex as v_0 .

By Proposition 3.6 we know that $\chi_g(W_n) \geq$ n,

hence we will show that W_n has a graceful n-colouring.

The graceful n-colouring of the graph W_n when n = 6, 7 and 8 means that the central vertex is coloured as 1 and the graceful n-colouring for W_7 and W_8 was obtained using the graceful (n−1)-colouring of W_{n-1} for n = 6 and 7 respectively.

This was achieved by inserting a new vertex into the cycle C_{n-2} in W_{n-1} and connecting this to the central vertex, then giving this new vertex the colour n.

Now we will prove that for n ≥ 7 there is a graceful (n − 1)-colouring of W_{n-1}, with the central vertex, v_0 , coloured as 1, where an edge xy (the edge connecting some vertex x to some vertex y) in the cycle C_{n-2} in W_{n-1} can have a new vertex, call this v, inserted into it and joined to v_0 to produce W_n.

Furthermore, v can be coloured as n to produce a graceful n-colouring for the resulting graph W_n.

So suppose there exists a graceful (n − 1)-colouring, c, of W_{n-1} for some n ≥ 7, with the central vertex coloured with 1.

We want to show that there is an edge xy in C_{n-2} where c(x) and c(y) satisfying the following two conditions:

i) c(x) ≠ (n + 1)/2 and c(y) ≠ (n + 1)/2,

ii) if (x',x,y,y') is a path in C_{n-2}, then c(x) ≠ c(x') + $\frac{n}{2}$ and c(y) ≠ c(y') + $\frac{n}{2}$.

To show this we let $C_{n-2} = (v_1, v_2, \ldots, v_{n-2}, v_{n-1} = v_1)$; the diameter of W_{n-1} is 2 so all the vertices in W_n must be assigned different colours by the vertex colouring c.

Therefore, if for some i, $c(v_{i+1}) = \frac{c(v_{i+2})+n}{2}$ then $c(v_j) \neq \frac{c(v_{i+2})+n}{2}$ for all j ≠ i + 1.

(Note, here the subscripts are denoted as integers modulo n − 2).

We now consider the two cases where n is an odd integer and when n is an even integer.

Case 1

Let n be an odd integer.

Suppose that for some i, $c(v_{i+2}) = \frac{c(v_{i+2}+n)}{2}$.

This would mean the edge $v_i v_{i+1}$ makes the condition ii) invalid.

We know that n = 2c(v_{i+1})c(v_{i+2}) is odd, hence we can deduce that c(v_{i+2}) is odd.

There are $\frac{n-3}{2}$ vertices in C_{n-2} which will be assigned odd colours by c, since the central vertex is coloured with 1, therefore at most $\frac{n-3}{2}$ edges in C_{n-2} will fail condition ii).

Conclusively this means there will be at least (n − 2) − $\frac{n-3}{2} = \frac{n-1}{2}$ edges (which will have a value ≥ 3) in C_{n-2} that satisfy condition ii).

Amongst these edges - those which satisfy condition ii) - at most two of them will fail condition i).

Hence, there will be a least one edge xy in C_{n-2} for which c(x) and c(y) satisfy both i) and ii).

Case 2

We will now assume n is even.

Suppose that for some i, $c(v_{i+2}) = \frac{c(v_{i+2}+n)}{2}$.

We can see that n = 2c(v_{i+1})c(v_{i+2}) is even, therefore it follows that c(v_{i+2}) is even.

There are then $\frac{n-2}{2}$ vertices in C_{n-2} which will be assigned even colours by c, since again the central vertex is coloured with 1, which means that at most $\frac{n-2}{2}$ edges in C_{n-2} will fail condition ii).

Consequently there will be at least (n−2) − $\frac{n-2}{2} = \frac{n-2}{2}$ edges

(which will have a value ≥ 4) in C_{n-2} that satisfy condition ii).

Since the value of $\frac{n+1}{2}$ is not an integer, all of these edges will satisfy condition i).

Hence, there will be a least one edge xy in C_{n-2} for which c(x) and c(y) satisfy both i) and ii).

CHAPTER 4

GRACEFUL TREES

Theorem 4.1 :

All caterpillars are graceful

Proof :

Let T be a caterpillar with n vertices.

Call the path that is formed from T when all the leaves are removed H.

Select an endpoint of H (a vertex with degree 1) and call this x_0,

then name the next adjacent vertex to x_0 in H as x_1, continue to do this along H;

such that the vertices of H are given names from x_0 to x_k for some integer k.

Denote X to be the set of vertices in T whose distance from x_0 is even, this includes x_0 itself.

Then let Y denote the set of vertices in T which are of odd distance from x_0.

Here we can note that every edge connects together two vertices, one of which is in X and one that is in Y.

Now assign the label n − 1 to x_0.

Label the neighbours of x_0 with 0, 1, 2, ..., where x_1 is the neighbour that receives the greatest label.

Next assign labels n − 2, n − 3, ..., to the neighbours of x_1 - giving the largest label to x_2.

From here we continue as follows:

after x_{2i} (for some value i) receives its label assign its neighbours with increasing integer labels starting with the smallest available integer that has not yet been used as a label,

do not include its neighbour x_{2i-1} as this will already have a label allocated, and give the largest label to x_{2i+1}.

Then assign decreasing integer labels to the neighbours of x_{2i+1}, not including x_{2i}, starting with the largest unused integer that is smaller than n and giving the smallest value of these labels to x_{2i+2}.

This will result in all members of the set X receiving the labels:

$n-1, n-2, ..., n-|X|$, whereas all member of the set Y will receive the labels

$0, 1, 2, ..., |Y|$.

It can be easily observed that the graph has a graceful labelling.

Definition 4.2 :

A **spider tree** is a tree graph which has exactly one vertex with degree larger than or equal to 3, such a vertex is called the **branch point** of the tree. The paths that lead from the branch point to a leaf in the tree are referred to as **legs**.

Remark 4.3 :

The length of a leg is of value m, this means that m edges were crossed from the branch point at the start of the path to reach the vertex at the end of the path.

Theorem 4.4 :

Let T be a spider tree with l legs. If each leg has a length of either m or m + 1 for some m ≥ 1, then T is graceful.

Proof :

Let us assume $l \geq 3$, otherwise T would simply be a path which as we have shown previously is graceful.

First we will look at the case where l is odd.

Let $l = l_0 + l_1$, where l_i is the number of legs with length m + i for i ∈ {0,1}.

We can then calculate the number of vertices, n, in T such that $n = lm + l_1 + 1$

(this is calculated by accounting that all l legs are of length of at least m, plus l_1 of them have an extra vertex at the end, whilst noting that all the legs originate from the same single vertex, the branch point).

These vertices will be given labels from the set {0,1,...,k}, where k in this case is equivalent to the number of edges in T.

Next we give names to the legs using $L_1, L_2, \ldots, L_l$, here $L_1, L_2, \ldots, L_{l_1}$ are the legs of length m + 1 and $L_{(l_1)+1}, L_{(l_1)+2}, \ldots, L_l$ are the legs of length m.

Let v^* denote the branch point of T and $v_{i,j}$ denote a vertex in L_i of distance from v^*.

We now will define the following labelling function, Φ:

i) let $\Phi(v^*) = 0$,

ii) if i and j are both odd, then $\Phi(v_{i,j}) = k - \frac{i-1}{2} - \frac{(j-1)l}{2}$;

iii) if i and j are both even, then $\Phi(v_{i,j}) = k - \frac{l-1}{2} - \frac{i}{2} - \frac{(j-2)l}{2}$;

iv) if i is even and j is odd, then $\Phi(v_{i,j}) = \frac{i}{2} + \frac{(j-1)l}{2}$;

v) if i is odd and j is even, then $\Phi(v_{i,j}) = \frac{l-1}{2} + \frac{i+1}{2} - \frac{(j-1)l}{2}$.

The Φ labelling puts a 0 label at the spider's branch point and then, by traversing along the spider's longer legs first, it give labels to the rest of the vertices.

It does this by alternating between the highest and lowest remaining unused labels and spiralling away from the centre.

An example of this is shown in Figure 4.5 where $l_0 = 2$, $l_1 = 3$ and m = 4.

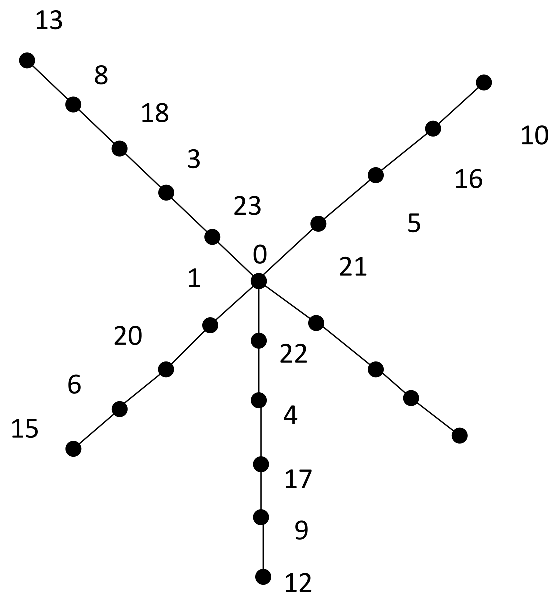


Figure 4.5

To compute the edge labels induced by the newly allocated vertex labels we recognise that the local maxima of Φ occurs at $v_{i,j}$ when i and j have the same parity,

So when both i and j are odd or when both i and j are even.

For when this is the case for i and j we have,

$$\Phi(v_{i,j}) - \Phi(v_{i,j+1}) = k - \frac{l-1}{2} - i + (1-j)l > 0, \qquad (1)$$

$$\Phi(v_{i,j}) - \Phi(v_{i,j-1}) = k - \frac{l-1}{2} - i + (2-j)l > 0. \qquad (2)$$

When looking for a contradiction suppose that two distinct edges share the same edge label.

Consider the indices of the vertices these two edges incident with.

It can be deemed that distinct pairs of indexes (i, j) and (i', j') can be chosen such that i and j have the same parity, i' and j' likewise have the same parity and an edge incident with $v_{i,j}$ shares the same label as a different edge incident with $v_{i',j'}$.

Hence, this would imply one of the following three case could occur:

$$\Phi(v_{i,j}) - \Phi(v_{i,j+1}) = \Phi(v_{i',j'}) - \Phi(v_{i',j'+1}), \qquad (3)$$

$$\Phi(v_{i,j}) - \Phi(v_{i,j+1}) = \Phi(v_{i',j'}) - \Phi(v_{i',j'-1}), \qquad (4)$$

$$\Phi(v_{i,j}) - \Phi(v_{i,j-1}) = \Phi(v_{i',j'}) - \Phi(v_{i',j'-1}). \qquad (5)$$

We will examine the case where (3) holds.

By equation (1) we obtain that $i - i' + (j - j')l = 0$, however $j \neq j'$ otherwise $i = i'$ which contradicts the assumption that $(i, j) \neq (i', j')$.

Therefore we can write $= \frac{i-i'}{j'-j}$.

Hence, $|i - i'| < l$ and $|j - j'| \geq 1$, and $l = |\frac{i-i'}{j-j'}| < \frac{l}{1} = l$, another contradiction.

Similarly, when equations (4) and (5) hold they result in contradictions.

Therefore two distinct edges cannot have the same edge labels and Φ gives a graceful labelling.

We will now look at the case where l is even.

Without loss of generality, assume L_l is a leg with length m.

Removing this results in a tree, call this T_0, with an odd number of legs, $l-1$.

The previous construction yields the graceful labelling Φ_0 of T_0 such that $\Phi_0(v^*) = 0$.

Now we let $|V(T_0)| = k' + 1$ and define a new graceful labelling , Φ'_0 , on T_0

where $\Phi_0(v) = k' - \Phi_0(v)$ for all v.

Next we construct a new tree, T_1 , by adding a new vertex, call this w_1, to T_0's branch point.

Define Φ_1 on V (T_1) by $\Phi_1(w_1) = 0$ and $\Phi_1(v) = \Phi'_0(v) + 1$ for all $v \in V(T_0)$

Then also define Φ'_1 on T_1 by $\Phi'_1(v) = k' + 1 - \Phi_1(v)$ for all v, noting that

$\Phi'_1(w_1) = n' + 1$.

Following this we add a new vertex w_2 to w_1 and construct the graceful labellings

Φ_2 from Φ'_1 and Φ'_2 from Φ_2,and so on, until we have the full reconstruction of

$L_l = w_1, w_2, \ldots, w_m$ which recovers our original graph T.

This will then mean T will have a graceful labelling.

Lemma 4.5 :

All trees with a diameter of at most 4 are graceful.

Proof :

A tree with no diameter is just a single vertex so the result is trivial.

A tree with a diameter of 1 will simply be a tree with one edge (and two vertices) hence again is trivial.

If a tree has a diameter of 2 it is either the path P_2 or any star S_n, where n is the total number of vertices, which we have previously proven to be graceful.

A tree with a diameter of 3 will be a caterpillar, if all the leaves were removed from this graph we would be left with a path consisting of a single edge and two vertices, as we have shown in Theorem 4.1 this is graceful.

Finally, all trees with a diameter of 4 have been proven to be graceful

Remark 4.6 :

If T is a caterpillar with a maximum degree Δ, where $\Delta \geq 2$, then

$\Delta + 1 \leq \chi_g (T) \leq \Delta + 2.$

CHAPTER 5

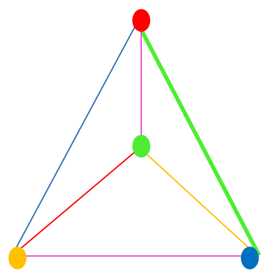
TOTAL COLOURING OF GRACEFUL GRAPHS

Definition 5.1 :

A **total colouring** of a graph G assigns colours to the vertices and edges of G, such that: no pair of adjacent edges or vertices share the same colour and no edge and vertex that are incident with each other are the same colour. If such a colouring for G can be achieved using k colours then G is said to be **k-total-colourable** and have received a **k-total-colouring**.

The **total chromatic number**, $\chi''(G)$, for the graph G is the minimum number of colours needed to produce a total colouring of G.

Example 5.2 :



Total colouring of the wheel graph W_4 with the total chromatic number is 5.

Conjecture 5.3 :

For every graph G, $\chi''(G) \leq \Delta + 2$, where Δ is the largest vertex degree in the graph.

Lemma 5.4 :

For any graceful graph, G, where the degree of the vertex labelled 0 is 1,

$\chi''(G) \leq m + 1$ where m is the total number of edges in G.

Proof :

We know that for a graceful graph, G, other than at vertex 0, no edge has the same label value as any vertices it is incident with.

Additionally, all edges and vertices have distinct labels.

Using these two facts we can begin to create a total colouring for G.

Firstly, let any edge or vertex labelled i, for $1 \leq i \leq m-1$, be coloured with the colour i.

Next colour the vertex m with the colour m and the edge m with the colour $m + 1$.

All edges of the graph should now be coloured and the only vertex not assigned a colour is vertex 0.

From here it is easy to see that since vertex 0 has a degree of 1, the only edge adjacent to it is coloured with $m + 1$ whilst the only vertex adjacent to it is coloured with m.

This means vertex 0 can be coloured with any colour i, where $1 \leq i \leq m-1$.

Hence, the graceful graph receives a $(m + 1)$-total-colouring.

Lemma 5.5 :

For any graceful graph, G, where the degree of the vertex labelled 0 is greater than 1, $\chi''(G) \leq m + 1$ where m is the total number of edges in G.

Proof :

By analysing the vertex 0 and both the edges and vertices adjacent to it.

Firstly we colour the vertex 0 with the colour m + 1.

Let us then say vertex 0 has a degree of value d.

This means that there are d vertices incident with vertex 0, call these v_j for each $1 \le j \le d$.

Now we can colour the vertices v_j with the colour of their vertex labels i, where $1 \le i \le m$.

We next want to colour the edges that adjoin the vertices v_j to vertex 0.

To do this we apply the following rule:

For an edge e_j , that is an edge connecting vertex v_j to vertex 0, colour this with the colour of vertex v_j +1 for $1 \le j \le d - 1$.

In the case where j = d, let e_d be coloured with the colour of v_1 . $(*)$

We move onto colouring the reminder of the graph.

Again, we note the fact that no edge has the same label value as any vertices it is incident with (except at vertex 0) and all vertex and edge labels are distinct.

This means all the remaining vertices and edges labelled i can be coloured with colour i, where $1 \le i \le m - 1$.

(Note that no values of i previously used in the step before will be repeated and we know that the vertex labelled m must be adjacent to vertex 0 hence will already be coloured).

Following these steps will provide a valid (m + 1)-total-colouring for the graceful graph.

Lemma 5.6 :

For a path P_n, χ"(P_n) = 3.

Proof :

For a path P_n where n and m are the total number of vertices and edges in the graph respectively, denote P_n by the sequence consisting of its vertices v_i

(where 1 ≤ i ≤ n) and edges v_j (where 1 ≤ j ≤ m = n − 1) in the order they are traversed along the path from the initial vertex v_1 to the terminal vertex v_n.

This is as follow: v_1, e_1 , v_2 , e_2, v_3 , e_3 , v_4, e_4 , ..., e_{n-1} , v_n.

Now if we allocate colours to the elements of the sequence we can see by inspection that in order for no adjacent elements to share the same colour,

as well as no v_i to share the same colour as v_{i+1} and v_{i-1} (for 2 ≤ i ≤ n−1)

and no e_j to share the same colour as e_{j+1} and e_{j-1} (for 2 ≤ i ≤ n−2),

three colours are needed.

Hence, v_1 can be coloured with colour 1, e_1 can be coloured with colour 2, v_2 coloured with colour 3, then e_2 can be coloured with colour 1 again and so on.

Theorem 5.7 :

A star graph S_n has χ"(S_n) = Δ + 1 = n, where Δ is the largest degree in the graph.

Proof :

First we note that the central vertex is adjacent to all the outer vertices in the graph, hence every edge must be coloured a different colour.

(Here at least n colours are needed, that is, n − 1 colours for the edges and an additional colour for the central vertex).

To achieve the total colouring of S_n we colour the central vertex (labelled 0) with colour n.

We next let all the outer vertices be coloured with colour i, such that i is the value of their vertex label and 1 ≤ i ≤ n−1.

Then using the same process described by the rule (∗) in Lemma 4.5,

here vertex 0 is now defined as the central vertex of S_n, we assign colours to the

edges of S_n, these will all be colours from the set i.

Hence, χ"(S_n) = n.

Theorem 5.8 :

All wheel graphs, W_n , with n vertices (n ≥ 4) have χ"(W_n) = Δ + 1 = n, where Δ is the largest vertex degree in W_n.

Proof :

We can recognise that a subgraph of W_n is S_n.

Therefore to begin this proof, apply the total colouring derived in Theorem 5.7 to the subgraph S_n in W_n , recalling also the procedure used in Lemma 5.5 ($*$).

This results in a colour being allocated to all the vertices and inner edges of W_n using $\Delta + 1$ colours.

Next we must assign colours to the outer edges of W_n.

To achieve this we colour the edge connecting vertex v_j to v_{j+1} with the colour of vertex v_{j+2}, for $1 \leq j \leq j - 2$.

In the case where $j = j - 1$ we colour the edge connecting v_{j-1} to v_j with the colour of vertex v_1.

Then for the case where $j = d$ (in this case $d = \Delta = n - 1$) we colour the edge connecting v_j to v_1 with the colour of vertex v_2.

The result will be a $(\Delta+1)$-total-colouring for W_n.

Theorem 5.9 :

A complete bipartite graph, $K_{a,b}$, has $\chi''(K_{a,b}) = \Delta + 2$, where Δ is the largest degree in the graph.

Proof :

Given that the chromatic number of a bipartite graph is 2

(as the vertices of the graph can be split into two sets, A and B, such that vertices in set A are only adjacent to vertices in set B), we begin by colouring all the vertices in set A with what we will for now refer to as colour A and all vertices in set B with the colour B.

Next we recognise that since every vertex in set A is adjacent to every vertex in set B no edges in the graph can be coloured with colour A or B.

Furthermore we observe that $\Delta(K_{a,b})$ = max{a,b} = k, for some integer k,

hence at least k colours are needed to colour the edges of the graph.

Note that from this point on the colours 1 to k do not consist of the colours A or B.

Considering this, for the set where the vertices have degree Δ = k, label them as

$v_{1,1}$, $v_{1,2}$, ... , $v_{1,s}$, where s is the integer a or b (note that s $\neq$ k unless a = b).

For the case where a = b, hence every vertex in $K_{a,b}$ has degree k, either set A or B can be chosen to be labelled in this way.

Following this, label the second set of vertices as $v_{2,1}$, $v_{2,2}$ ..., $v_{2,k}$.

We will now begin to assign the edges colours starting with those incident with$v_{1,1}$.

Here, let $v_{1,1}v_{2,x}$ denote the edge connecting vertex $v_{1,1}$ with vertex $v_{2,x}$, where

$1 \leq x \leq k$.

Then for each value of x we assign the colour x to the edge $v_{1,1}v_{2,x}$

Next we look at the edges incident with $v_{1,2}$, here we assign the colour x + 1 to the edge $v_{1,2}v_{2,x}$, letting colour k + 1 = 1.

From this we can formulate the following rule:

For the edge $v_{1,y}v_{2,x}$, where $1 \leq y \leq s$ and $1 \leq x \leq k$, assign the colour x + (y − 1) to the edges, noting that whenever the colour is k + (y − 1) let this be equivalent to the colour y − 1.

Following this process will result in a proper edge colouring for $k_{a,b}$ using k colours. Therefore, by noting that two further colours (colours A and B) were used for the vertices of the graph to give a proper vertex colouring, when coloured in this way, both the proper edge and vertex colourings result in a proper total colouring.

Hence $\chi''(K_{a,b}) = \Delta + 2$.

A STUDY ON DIFFERENCE CORDIAL LABELING IN GRAPHS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

R. JEYASHRI

Reg. No: 19SPMT11

Under the guidance of

Ms. K. AMBIKA M.Sc., B.Ed., S.E.T.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON DIFFERENCE CORDIAL LABELING IN GRAPHS"** is submitted to **St.Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **R. JEYASHRI (Reg. No: 19SPMT11)**

K. Jyothi.

Signature of the Guide

P. Anbarasi Rodrigo

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous,
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON DIFFERENCE CORDIAL LABELING IN GRAPHS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. K. Ambika M.Sc., B.Ed., S.E.T.,** Assistant Professor , Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi.**

R. Jeyashri

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER - 1

PRELIMINARIES

Definition: 1.1

A graph G consists of a pair $(V(G), X(G))$ where $V(G)$ is a non-empty finite set whose elements are called **points or vertices** and $X(G)$ is set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called **lines or edges** of the graph G.

Definition: 1.2

If two vertices of a graph are joined by an edge then these vertices are called **adjacent vertices**. If two or more edges of a graph have a common vertex then these edges are called **incident edges**.

Definition: 1.3

A edge of a graph that joins a vertex to itself is called a **loop**. A loop is an edge $e = v_i v_i$. If two vertices of a graph are joined by more than one edge then these edges are called **multiple edges**. A graph which has neither loops nor parallel edges is called a **simple graph**.

Definition: 1.4

A graph in which any two distinct points are adjacent is called a **complete graph**.

Definition: 1.5

A graph G is called a **bigraph or bipartite graph** if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point V_2. (V_1,V_2) is called a **bipartition of G**. If further G contains every line joining the points of $V_1$to the points of V_2 then G is called a **complete bigraph**.

Definition: 1.6

Degree of a vertex v of any graph G is defined as the number of edges incident with v. It is denoted by deg (v) or d(v).

Definition: 1.7

For any graph G, we define $\delta(G) = min\{\deg v / v \in V(G)\}$ and $\Delta(G) = max\{\deg v / v \in V(G)\}$.If all the points of G have the same degree r then $\delta(G) = \Delta(G) = r$ and this G is called a **regular graph of degree r**.

Definition: 1.8

A graph $H = (V_1, X_1)$ is called a **subgraph** of $G = (V, X)$ if $V_1 \subseteq V$ and $X_1 \subseteq X$. If H is a subgraph of G we say that G is a supergraph of H. H is called a **spanning subgraph** of G if $V_1 = V$. H is called a **induced subgraph** of G if H is the maximal subgraph of G with point set V_1. Thus, if H is an induced subgraph of G, two points are adjacent in H iff they are adjacent in G.

Definition: 1.9

A **walk** of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \dots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line $x_i$is incident with $v_{i-1}$and v_i. A walk is called a **trail** if all its lines are distinct and is called a path if all points are distinct. A $v_0 - v_n$ walk is called **closed** if $v_0 = v_n$. A closed walk $v_0, v_1, v_2, \dots, v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, v_2, \dots, v_{n-1}$ are distinct of length n is called a **cycle**. A cycle with n vertices is denoted as C_n.

Definition: 1.10

Two points u and v of a graph G are said to be connected if there exist a $u - v$ path in G.

A graph G is said to be **connected** if every pair of its points are connected.

Definition: 1.11

Denote, $\lceil n \rceil$ – Largest integer greater than or equal to n.

$\lfloor n \rfloor$ – Smallest integer less than or equal to n.

Definition: 1.12

A graph that contains no cycles is called an **acyclic graph**. A connected acyclic graph is called a **tree**.

Definition: 1.13

A complete bipartite graph $k_{1,n}$ is called a star and it has n+1 vertices and n edges. $k_{1,n}$ is the graph obtained by the subdivision of the edge of the star $k_{1,n}$.

Definition: 1.14

Bistar is a graph obtained from a path P_2 by joining the root of stars S_m and S_n at the terminal vertices of P_2. It is denoted by $B_{m,n.}$

Definition: 1.15

The **wheel** $\boldsymbol{W_n}(n \in \mathbb{N}, n \geq 3)$ is a join of the graphs C_n and K_1.i.e. $W_n = C_n + K_1$.The vertex corresponding to $K_1$is called as apex vertex. The vertices corresponding to C_n are called as rim vertices and C_n is called rim of W_n.

Definition: 1.16

The **fan** $\boldsymbol{F_n}$ is the graph obtained by taking n concurrent chords in cycle C_{n+1}. The vertex at which all the chords are concurrent is called the apex vertex. It is also given by $F_n = P_n + K_1$.

Definition: 1.17

The **double fan** $\mathbf{DF_n}$ is defined as $P_n + 2K_1$.

Definition: 1.18

A **gear graph** $\boldsymbol{G_n}(n \geq 3)$ is obtained from the wheel W_n between adding a vertex every pair of adjacent vertices of rim of W_n.

Definition: 1.19

A **web graph** is the graph obtained by joining the pendant of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

Definition: 1.20

A **helm** $\boldsymbol{H_n}(n \geq 3)$ is the graph obtained from the wheel W_n by adding a pendant edge at each vertex on the rim of W_n.

Definition: 1.21

A **crown** $\boldsymbol{C_n \odot K_1}(n \in \mathbb{N}, n \geq 3)$ is obtained by joining a pendant edge to each vertex of C_n.

Definition: 1.22

A **chord of a cycle** $\boldsymbol{C_n}$ is an edge joining two non-adjacent vertices of cycle C_n.

Definition: 1.23

The **corona** of G with H, G ⊙ H is the graph obtained by taking one of G and p copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H.

Definition: 1.24

The **triangular snake** $\boldsymbol{T_n}$ is obtained from the path $P_n$by replacing each edge of the path by a triangle C_3.

Definition: 1.25

The **quadrilateral snake** $\boldsymbol{Q_n}$ is obtained from the path P_n by replacing each edge of the path by a cycle C_4.

Definition: 1.26

An **alternate triangular snake** $\boldsymbol{A(T_n)}$ is obtained from a path $u_1u_2 \dots u_n$ by joining u_i and u_{i+1}(alternatively) to new vertex v_i. That is every alternative edge of a path is replaced by C_3.

Definition: 1.27

A **double triangular snake** $\boldsymbol{DQ_n}$ consists of two triangular snakes that have a common path. That is a double triangular snake is obtained from a path $u_1u_2 \dots u_n$ by joining u_i and u_{i+1} to a new vertex v_i $(1 \le i \le n - 1)$ and to a new vertex $w_i(1 \le i \le n -)$.

Definition: 1.28

A **double quadrilateral snake** $\boldsymbol{DQ_n}$ consists of two triangular snakes that have a common path.

Definition: 1.29

A **double alternate triangular snake** $\boldsymbol{DA(T_n)}$ consists of two alternate triangular snakes that have a common path. That is, a double alternate triangular snake is obtained from a path $u_1u_2 \dots u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i.

Definition: 1.30

A **double alternate quadrilateral snake** $\boldsymbol{DA(Q_n)}$ consists of two alternate quadrilateral snakes that have a common path. That is, it is obtained from a path

$u_1u_2 \dots u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i and w_i, y_i respectively and adding the edges v_iw_i and x_iy_i.

Definition: 1.31

Labeling or valuation or numbering of a simple graphs G is an assignment of integer to the vertices or edges or both subjects to certain condition.

Definition: 1.32

A binary vertex labeling of a graph G is called a **cordial labeling** if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

Definition: 1.33

Let G be a (p, q) graph. Let f be a map from $V(G)$ to $\{1,2, \dots, p\}$. For each edge uv, assign the label $|f(u) - f(v)|$. $\boldsymbol{f}$ is called **difference cordial labeling** if f is $1-1$ and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled and labeled with 0 respectively. A graph with a difference cordial labeling is called a **difference cordial graph.**

Definition: 1.34

Let G be a (p, q) graph. Let k be an integer with $2 \leq k \leq p$ and $f: V(G) \rightarrow \{1,2, \dots, k\}$ be a map. For each edge uv, assign the label $|f(u) - f(v)|$. The function f is called a ***k*-difference cordial labeling of G** if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with $x (x \in \{1,2, \dots, k\})$, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with 1 and labeled with 0. A graph with a k-difference cordial labeling is called a k**-difference cordial graph.**

CHAPTER -2

DIFFERENCE CORDIAL LABELING OF GRAPHS

Definition: 2.1

Let G be a (p,q) graph. Let f be a map from V(G) to {1,2,3,…,p}. For each edge uv, assign the label $|f(u) - f(v)|$. f is called difference cordial labeling if f is $1 - 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and labeled with 0 respectively. A graph with a difference cordial labeling is called a difference cordial graph.

The following is simple example of a difference cordial graph.

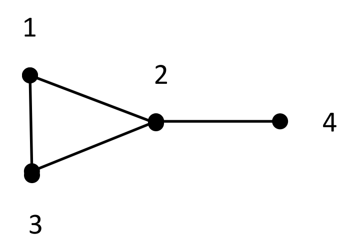

Figure 2.1

Theorem: 2.2

Every graph is a subgraph of a connected difference cordial graph.

Proof:

Let G be a given (p,q) graph.

Let $N = \frac{p(p-1)}{2} - 2p + 2$.

Consider the complete graph K_P and the cycle C_N.

Let $V(K_p) = \{u_1, u_2, \ldots, u_p\}$ and C_N be the cycle $v_1 v_2 \ldots v_N v_1$.

We construct the super graph G^* of G as follows:

Let $V(G^*) = V(K_p) \cup V(C_N)$ and $E(G^*) = E(K_p) \cup E(C_N) \cup \{u_p v_1\}$.

Clearly G is a subgraph of G^*.

Assign the label i to u_i $(1 \leq i \leq p)$ and $p + j$ to v_j $(1 \leq j \leq N)$.

Therefore, $e_f(0) = N + p$ and $e_f(1) = N + p - 1$.

Hence G^* is a difference cordial graph.

Theorem: 2.3

If G is a (p,q) difference cordial graph, then $q \leq 2p - 1$.

Proof:

Let f be a difference cordial labeling of G.

Obviously, $e_f(1) \leq p - 1$.

This implies $e_f(0) \geq q - p + 1$ …(1)

Case (i): $e_f(0) = e_f(1) + 1$.

From (1), $q \leq e_f(0) + p - 1$

$$= e_f(1) + 1 + p - 1$$

$$\leq 2p - 1$$

Therefore, $q \leq 2p - 1$ …(2)

Case 2(i): $e_f(1) = e_f(0) + 1$

From (1), $q \leq e_f(0) + p - 1$

$$= e_f(0) + p - 1$$

$$\leq 2p - 2$$

Therefore, $q \leq 2p - 2$ …(3)

Case 2(ii): $e_f(1) = e_f(0) + 1$

From (1), $q \leq e_f(0) + p - 1$

$$= e_f(1) - 1 + p - 1$$

$$\leq 2p - 3$$

Therefore, $q \leq 2p - 2$ …(4)

From (2), (3) and (4), $q \leq 2p - 1$.

Theorem: 2.4

If G is a r-regular graph with r ≥ 4 then G is not difference cordial.

Proof:

Let G be a (p,q) graph.

Suppose G is difference cordial.

Then by theorem 2.3 $q \leq 2p - 1$.

This implies $q \leq \frac{4q}{r} - 1$.

Hence $q \leq q - 1$.

This is impossible.

Theorem: 2.5

Any Path is difference cordial graph.

Proof:

Let P_n be path $u_1 u_2 \dots u_n$.

The following table 2.1 gives the difference cordial labeling of $P_n, n \leq 8$.

e	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
1	**1**							
2	**1**	**2**						
3	**1**	**3**	**2**					
4	**1**	**2**	**4**	**3**				
5	**1**	**2**	**4**	**3**	**5**			
6	**1**	**2**	**3**	**5**	**4**	**6**		
7	**1**	**2**	**3**	**5**	**7**	**6**	**4**	
8	**1**	**2**	**3**	**4**	**6**	**8**	**7**	**5**

Table 2.1

Assume $n < 8$.

Define a map $f: V(P_n) \to \{1, 2, \dots, n\}$ as follows:

Case (i): $n \equiv 0(mod\ 4)$.

Define,

$$f(u_i) = i \qquad\qquad 1 \le i \le \frac{n+2}{2}$$

$$f\left(u_{\frac{n+2}{2}+i}\right) = \frac{n+2}{2} + 2i \qquad 1 \le i \le \frac{n}{4} - 1$$

$$f\left(u_{\frac{3n}{4}+i}\right) = \frac{n+2}{2} + 2i - 1 \qquad 1 \le i \le \frac{n}{4}$$

Case (ii): $n \equiv 1(mod\ 4)$.

Define,

$$f(u_i) = i \qquad 1 \le i \le \frac{n+1}{2}$$

$$f\left(u_{\frac{n+1}{2}+i}\right) = \frac{n+1}{2} + 2i \qquad 1 \le i \le \frac{n-1}{4}$$

$$f\left(u_{\frac{3n+1}{4}+i}\right) = \frac{n+1}{2} + 2i - 1 \qquad 1 \le i \le \frac{n-1}{4}$$

Case (iii): $n \equiv 2(mod\ 4)$.

Define,

$$f(u_i) = i \qquad 1 \le i \le \frac{n+2}{2}$$

$$f\left(u_{\frac{n+2}{2}+i}\right) = \frac{n+2}{2} + 2i \qquad 1 \le i \le \frac{n-2}{4}$$

$$f\left(u_{\frac{3n+2}{4}+i}\right) = \frac{n+2}{2} + 2i - 1 \qquad 1 \le i \le \frac{n-2}{4}$$

Case (iv): $n \equiv 3(mod\ 4)$.

Define,

$$f(u_i) = i \qquad\qquad 1 \le i \le \frac{n+1}{2}$$

$$f\left(u_{\frac{n+1}{2}+i}\right) = \frac{n+1}{2} + 2i \qquad\qquad 1 \le i \le \frac{n+1}{4} - 1$$

$$f\left(u_{\frac{3n-1}{4}+i}\right) = \frac{n+1}{2} + 2i - 1 \qquad\qquad 1 \le i \le \frac{n+1}{4}$$

The following table 2.2 proves that f is a difference cordial labeling.

Nature of n	$\boldsymbol{e_f(0)}$	$\boldsymbol{e_f(1)}$
$\boldsymbol{n \equiv 0 (mod\ 2)}$	$\boldsymbol{\frac{n-2}{2}}$	$\boldsymbol{\frac{n}{2}}$
$\boldsymbol{n \equiv 1 (mod\ 2)}$	$\boldsymbol{\frac{n-1}{2}}$	$\boldsymbol{\frac{n-1}{2}}$

Table 2.2

Corollary: 2.6

Any Cycle is a difference cordial graph.

Proof:

The function f in theorem 2.5 is also a difference cordial labeling of the cycle $C: u_1 u_2 \dots u_n u_1$.

Theorem: 2.7

The Star $K_{1,n}$ is difference cordial iff $n \leq 5$.

Proof:

Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$, $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$.

Table 2.3 shows that the star $K_{1,n}, n \leq 5$ is difference cordial.

$\boldsymbol{n}$	$\boldsymbol{u}$	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	$\boldsymbol{u_3}$	$\boldsymbol{u_4}$	$\boldsymbol{u_5}$
1	**1**	**2**				
2	**1**	**2**	**3**			
3	**1**	**2**	**3**	**4**		
4	**2**	**1**	**3**	**4**	**5**	
5	**2**	**1**	**3**	**4**	**5**	**6**

Table 2.3

Assume $n > 5$.

Suppose f is a difference cordial labeling of $K_{1,n}$, $n > 5$.

Without loss of generality assume that $f(u) = x$.

To get the edge label 1, the only possibility is that, $f(u_i) = x-1, f(u_j) = x+1$ for some i,j.

This implies $e_f(0) - e_f(1) \geq n-2-2 > 1$, a contradiction.

Theorem: 2.8

K_n is difference cordial iff $n \leq 4$.

Proof:

Suppose $K_n$is difference cordial.

Then $\frac{n(n-1)}{2} \leq 2n-1$.

This implies $n \leq 4$.

K_1, K_2 are difference cordial by theorem 2.5.

By corollary 2.6, K_3 is difference cordial.

A difference cordial labeling of K_4 is given in figure 2.2.

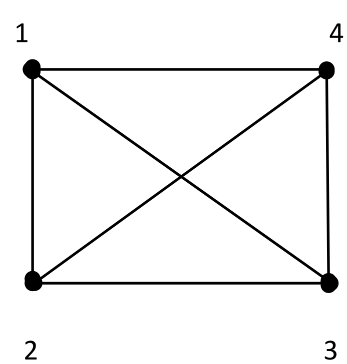

Figure 2.2

Now we look into the complete bipartite graph $K_{m,n.}$

Theorem: 2.9

If $m \geq 4$ and $n \geq 4$, then $K_{m,n}$ is not difference cordial.

Proof:

Suppose $K_{m,n}$ is difference cordial.

By theorem 2.3, $mn \leq 2(m+n) - 1$.

This implies $mn - 2m - 2n + 1 \leq 0$, a contradiction to $m \geq 4$ and $n \geq 4$.

Theorem: 2.10

$K_{2,n}$ is difference cordial iff $n \leq 4$.

Proof:

Let $V(\ K_{2,n}\ ) = V_1 \cup V_2$where $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_i: 1 \leq i \leq n\}$.

$K_{1,2}$, $K_{2,2}$ are difference cordial by theorem 2.5 and corollary 2.6 respectively.

A difference cordial labeling of $K_{2,3}$ and $K_{2,4}$ are given in figure 2.3.

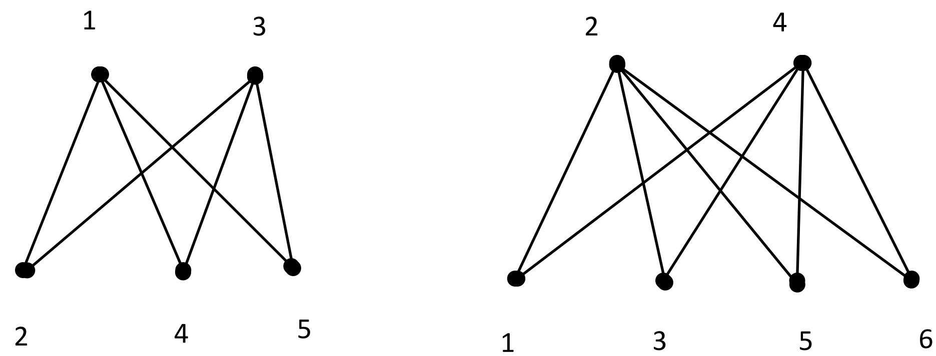

Figure 2.3

Now, we assume $n \geq 5$.

Suppose f is a difference cordial labeling.

Let $f(u_1) = r_1, f(u_2) = r_2$.

Then $K_{2,n}$ has at most 4 edges with label 1.

The maximum value is attained if the vertices in the set V_2receive the labels $r_1 - 1,\ r_1 + 1, r_2 - 1,\ r_2 + 1$.

Therefore $e_f(1) \leq 4,\ e_f(0) \geq 2n - 4$.

Hence $e_f(0) - e_f(1) \geq 2n - 8 > 2$, is contradiction.

Theorem: 2.11

$K_{3,n}$ is difference cordial iff $n \leq 4$.

Proof:

$K_{3,1}$, $K_{3,2}$ are difference cordial graphs by theorem 2.7 and theorem 2.10.

A difference cordial labeling of $K_{3,3}$, $K_{3,4}$ are given in figure 2.4.

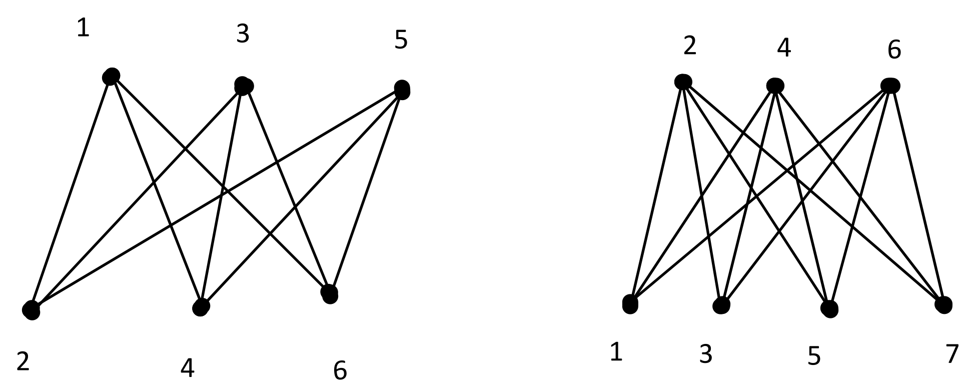

Figure 2.4

For any injective map f on $V(K_{3,5})$, $e_f(1) \leq 6$.

Therefore, $K_{3,5}$ is not difference cordial.

Assume $n \geq 6$.

Suppose $K_{3,n}$ is difference cordial, then by theorem 2.3, $3n \leq 2(n+3) - 1$, a contradiction.

Theorem: 2.12

If $m+n \geq 9$ then $B_{m,n}$ is not difference cordial.

Proof:

Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and

$E(B_{m,n}) = \{uu_i, vv_j, uv : 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

Assume $f(u) = x$ and $f(v) = y$.

Case (i): $y \neq x-1$ and $y \neq x+1$.

To get the edge label 1, $u_i$and u_j must receive the labels $x-1, x+1$ respectively for some i, j and v_i, v_j must receive the labels $y-1, y+1$ respectively for some i, j.

Hence $e_f(1) \leq 4$.

Case (ii): $y = x-1$ or $x+1$.

Obviously, in this case $e_f(1) \leq 3$.

Thus, by case (i), (ii), $e_f(1) \leq 4$.

Therefore, $e_f(0) \geq q-4 \geq m+n-3$.

Then $e_f(0) - e_f(1) \geq m+n-3-4 \geq 2$, a contradiction.

Theorem: 2.13

$B_{1,n}$ is difference cordial iff $n \leq 5$.

Proof:

Let $V(B_{1,n}) = \{u, v, u_1, v_i: 1 \leq i \leq m\}$ and $E(B_{1,n}) = \{uu_1, vv_i, uv: 1 \leq j \leq n\}$.

Case (i): $n \leq 5$.

$B_{1,1}$ is difference cordial by theorem 2.4.

A difference cordial labeling of $B_{1,2}$ is in figure 2.5.

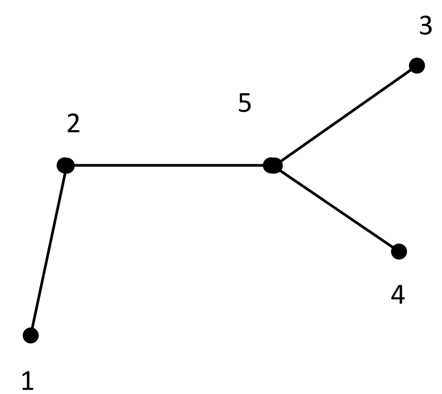

Figure 2.5

For $3 \leq n \leq 5$, Define,

$$f(u) = 2, f(v) = 4, f(u_1) = 1, f(v_1) = 3, f(v_i) = 3 + i, \qquad 2 \leq i \leq n.$$

Clearly this f is a difference cordial labeling.

Case (ii): $n > 5$.

When $n \geq 8$, the result follows from theorem 2.12.

When $n = 6$ or 7, $e_f(1) \leq 3$.

Therefore, $e_f(1) \leq 3$

This is a contradiction.

Theorem: 2.14

$B_{2,n}$ is difference cordial iff $n \leq 6$.

Proof:

Let $V(B_{2,n}) = \{u, v, u_1, u_2, v_i : 1 \leq i \leq n\}$ and

$E(B_{2,n}) = \{uu_1, uu_2vv_i, uv : 1 \leq i \leq n\}$.

Case (i): $n = 2,3$.

When $n = 2$,define $f(u) = 4, f(v) = 2, f(u_1) = 6, f(u_2) = 5, f(v_1) = 1, f(v_2) = 3$.

When $n = 3$, define $f(u) = 5, f(v) = 2, f(u_1) = 6, f(u_2) = 7, f(v_1) = 1, f(v_2) = 3, f(v_3) = 4$.

Clearly, f is a difference cordial labeling.

Case (ii): $n = 4,5,6$.

Define $f(v) = 2, f(v_1) = 1, f(v_i) = 1 + i, \quad 2 \leq i \leq n, f(u_1) = n + 2,$

$f(u) = n + 3, f(u_2) = n + 4$

In this case, $e_f(0) = n - 1$ and $e_f(1) = 4$.

Therefore, f is a difference cordial labeling.

Case (iii): $n \geq 7$.

Proof follows from theorem 2.12.

Theorem: 2.15

$B_{3,n}$ is difference cordial iff $n \leq 5$.

Proof:

Let and $V(B_{3,n}) = \{u, v, u_i, v_j: 1 \leq i \leq 3, 1 \leq j \leq n\}$ and

$$E(B_{m,n}) = \{uu_i, vv_j, uv: 1 \leq i \leq 3, 1 \leq j \leq n\}.$$

$B_{3,1}, B_{3,2}$ are difference cordial graphs by theorems 2.13 and 2.14 respectively.

For $3 \leq n \leq 5$, define, $f(v) = 2, f(v_1) = 1, f(v_i) = 1 + i,\ 2 \leq i \leq n, f(u_1) = n + 2, f(u) = n + 3, f(u_2) = n + 4, f(u_3) = n + 5$.

In this case, $e_f(0) = n$ and $e_f(1) = 4$.

Therefore, f is a difference cordial labeling.

For $n \geq 6$, the result follows from the theorem 2.12.

Remark: 2.16

$B_{4,4}$ is difference cordial.

Theorem: 2.17

The wheel W_n is difference cordial.

Proof:

Let $W_n = C_n + K_1$ where C_n is the cycle $u_1u_2 \ldots u_nu_1$ and $V(K_1) = \{u\}$.

Define a map $f: V(W_n) \rightarrow \{1,2, \ldots, n+1\}$ by $f(u) = 1, f(v_i) = i + 1, 1 \leq i \leq n$.

Then $e_f(0) = n$ and $e_f(1) = n$.

Theorem: 2.18

The fan F_n is difference cordial for all n.

Proof:

Let $F_n = P_n + K_1$ where P_n is the path $u_1u_2 \dots u_n$ and $V(K_1) = \{u\}$.

The function f given in theorem 2.17 is also a difference cordial labeling.

Since $e_f(0) = n - 1$ and $e_f(1) = n$

CHAPTER - 3

DIFFERENCE CORDIAL LABELING OF SUBDIVISION OF SNAKE GRAPHS

Theorem: 3.1

$S(T_n)$ is difference cordial for all $n > 2$.

Proof:

Let P_n be the path $u_1 u_2 \dots u_n$.

Let $V(T_n) = V(P_n) \cup \{v_i : \leq i \leq n-1\}$.

Let $V(S(T_n)) = \{x_i, y_i, w_i : 1 \leq i \leq n-1\} \cup$ V(Tn) and $E(S(T_n)) =$

$\{u_i x_i, x_i v_i, y_i v_i, y_i u_{i+1}, u_i w_i, w_i u_{i+1}, : 1 \leq i \leq n-1\}$.

Define an injective map $f: V(S(T_n)) \to \{1, 2, \dots, 5n-4\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \leq i \leq n$$

$$f(w_i) = 2i \qquad 1 \leq i \leq n-1$$

$$f(x_i) = 2n + 2i - 2 \qquad 1 \leq i \leq n-1$$

$$f(y_i) = 4n - 3 + i \qquad 1 \leq i \leq n-1$$

$$f(v_i) = 2n + 2i - 1 \qquad 1 \leq i \leq n-1$$

Since $e_f(0) = e_f(1) = 3n - 3$, f is a difference cordial labeling of $S(T_n)$.

Theorem: 3.2

$S(Q_n)$ is difference cordial.

Proof:

Let $P_n$be the path $u_1u_2 \dots u_n$.

Let $V(Q_n) = V(P_n) \cup \{v_i, w_i : \leq i \leq n-1\} \cup V(P_n)$.

Let $V(S(Q_n)) = \{x_i, u_i', z_i, y_i : 1 \leq i \leq n-1\} \cup \mathrm{V}(Q_n)$ and

$E(S(Q_n)) = \{u_iu_i', u_i'u_{i+1}, y_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_ix_i, x_iv_i, v_iz_{i,}, z_iw_{i,}, w_iy_i : 1 \leq i \leq n-1\}$

Define a by map $f: V(S(Q_n)) \to \{1, 2, \dots, 7n-6\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \leq i \leq n$$

$$f(u_i') = 2i \qquad 1 \leq i \leq n-1$$

$$f(v_i) = 2n + 3i - 3 \qquad 1 \leq i \leq n-1$$

$$f(z_i) = 2n + 3i - 2 \qquad 1 \leq i \leq n-1$$

$$f(w_i) = 2n + 3i - 1 \qquad 1 \leq i \leq n-1$$

$$f(x_i) = 5n - 4 + i \qquad 1 \leq i \leq n-1$$

$$f(y_i) = 6n - 5 + i \qquad 1 \leq i \leq n-1$$

Since, $e_f(0) = e_f(1) = 4n - 4$ f is a difference cordial labeling of $S(Q_n)$.

Theorem: 3.3

$S(DT_n)$ is difference cordial.

Proof:

Let $V(S(DT_n)) = \{u_i : 1 \le i \le n\} \cup \{x_i, y_i, v_i, x_i', y_i', w_i, z_i : 1 \le i \le n-1\}$ and $E(S(DT_n)) = u_i z_i, z_i u_{i+1}, u_i x_i, y_i u_{i+1}, x_i v_i, u_i x_i' : 1 \le i \le n-1\} \cup \{v_i y_i, w_i y_i', x_i' w_i, y_i' u_{i+1} : 1 \le i \le n-1\}$

Define a by map $f: V(S(DT_n)) \to \{1, 2, \dots, 8n-7\}$ by

$$f(u_i) = 4i - 3 \qquad 1 \le i \le n$$

$$f(v_i) = 7n - 6 + i \qquad 1 \le i \le n-1$$

$$f(x_i) = 4i = 2 \qquad 1 \le i \le n-1$$

$$f(v_i) = 4i - 1 \qquad 1 \le i \le n-1$$

$$f(y_i) = 4i \qquad 1 \le i \le n-1$$

$$f(x_i') = 4n + 2i - 4 \qquad 1 \le i \le n-1$$

$$f(w_i) = 4n + 2i - 3 \qquad 1 \le i \le n-1$$

$$f(y_i') = 6n - 5 + i \qquad 1 \le i \le n-1.$$

Obviously the above vertex labeling is a difference cordial labeling of $S(DT_n)$.

Theorem: 3.4

$S(DQ_n)$ is difference cordial.

Proof:

Let $V(S(DQ_n)) = V(DQ_n) \cup \{u_i', v_i', w_i', z_i', x_i', y_i', z_i : 1 \le i \le n-1\}$ and

$E(S(DQ_n)$=$\{u_iu_i', u_ix_i', v_i'v_i, x_i'x_i, v_iz_i', x_iz_i, z_i'w_i: 1 \leq i \leq n-1\} \cup$

$\{z_iy_i, w_iw_i', y_iy_i', w_i'u_{i+1}, y_i'u_{i+1}: 1 \leq i \leq n-1\}$.

Define a by map$f: V(S(DQ_n)) \rightarrow \{1, 2, \dots, 12n-11\}$ by

$$f(u_i) = 6i-5 \qquad 1 \leq i \leq n$$

$$f(u_i') = 6i-4 \qquad 1 \leq i \leq n-1$$

$$f(v_i) = 6i-3 \qquad 1 \leq i \leq n-1$$

$$f(z_i') = 6i-2 \qquad 1 \leq i \leq n-1$$

$$f(w_i) = 6i-1 \qquad 1 \leq i \leq n-1$$

$$f(w_i') = 6i \qquad 1 \leq i \leq n-1$$

$$f(x_i') = 6n+2i-6 \qquad 1 \leq i \leq n-1$$

$$f(x_i) = 6n+2i-5 \qquad 1 \leq i \leq n-1$$

$$f(z_i) = 8n+i-7 \qquad 1 \leq i \leq n-1$$

$$f(y_i) = 9n+i-8 \qquad 1 \leq i \leq n-1$$

$$f(y_i') = 10n+i-9 \qquad 1 \leq i \leq n-1$$

$$f(u_i') = 11n+i-10 \qquad 1 \leq i \leq n-1.$$

Since, $e_f(0) = e_f(1) = 7n-7$ f is a difference cordial labeling of $S(DQ_n)$.

Theorem: 3.5

$S(A(T_n))$ is difference cordial.

Proof:

Let the edges u_iu_{i+1}, u_iv_i and v_iu_{i+1} be subdivided by u_i', x_i and y_i respectively.

Case (i): Let the triangle be start from u_1 and ends with u_n.

In this case $S(A(T_n))$ consists of $\frac{7n-2}{2}$ vertices and 4n – 2 edges.

Define a map $f: V(S(A(T_n))) \to \{1, 2, \dots, \frac{7n-2}{2}\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \le i \le n$$

$$f(u_i') = 2i \qquad 1 \le i \le n - 1$$

$$f(x_i) = 2n - 1 + i \qquad 1 \le i \le \frac{n}{2}$$

$$f(y_i) = \frac{5n-2}{2} + i \qquad 1 \le i \le \frac{n}{2}$$

$$f\left(v_{\frac{n}{2}-i+1}\right) = 3n - 1 + i \quad 1 \le i \le \frac{n}{2}$$

Since, $e_f(0) = e_f(1) = 2n - 1$, f is a difference cordial labeling of $S(A(T_n))$.

Case (ii): Let the triangle be start from u_2 and ends with u_{n-1}.

In this case $S(A(T_n))$ has $\frac{7n-8}{2}$ vertices and 4n – 6 edges.

Define a injective map $f: V(S(A(T_n))) \to \{1, 2, \dots ., 7n - 8/2\}$ by

$$f(u_i) = 2i - 2 \qquad 1 \le i \le n$$

$$f(u_i') = 2i - 1 \qquad 1 \le i \le n - 1$$

$$f(x_i) = 2n - 2 + i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(y_i) = \frac{5n-6}{2} + i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(v_i) = 3n - 3 + i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(u_1) = 3n - 3.$$

Obviously, f is a difference cordial labeling of $S(A(T_n))$.

Case (iii): Let the triangle be start from $u_1$and ends with u_n.

In this case, the order and size of $S(A(T_n))$ has $\frac{7n-5}{2}$ vertices and 4n – 4 edges.

The difference cordial labeling of $S(A(T_3))$ is given in figure 3.1.

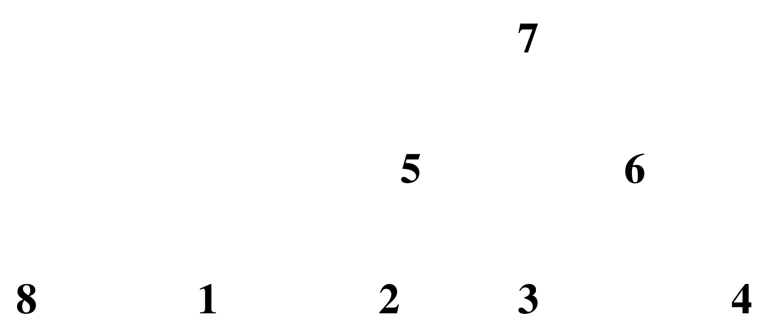

Figure 3.1

For $n > 3$, Define a injective map$f: V(S(A(T_n))) \rightarrow \{1, 2, \ldots, 7n - 8/2\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \leq i \leq n$$

$$f(u_i') = 2i \qquad 1 \leq i \leq n - 1$$

$$f(x_i) = 2n - 1 + i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(y_i) = \frac{5n-3}{2} + i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(v_i) = 3n - 2 + i \qquad 1 \le i \le \frac{n-1}{2}$$

Since $e_f(0) = e_f(1) = 2n - 2$, f is a difference cordial labeling of $S(A(T_n))$.

Theorem: 3.6

$S(DA(T_n))$ is difference cordial.

Proof:

Let the edges $u_iu_{i+1}, u_iv_i, v_iu_{i+1}, w_i, w_iu_{i+1}$, be subdivided by $u_i', x_i, y_i, x_i', y_i'$ respectively.

Case (i): Let the triangles be start from u_1 and ends with u_n.

Here, the number of vertices and edges in $S(DA(T_n))$ are 5n – 1 vertices and 6n – 2 edges respectively.

Define a map $f: V(S(DA(T_n))) \to \{1, 2, \dots, 5n-1\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \le i \le n$$

$$f(u_i') = 2i \qquad 1 \le i \le n-1$$

$$f(x_i) = 2n - 2 + 2i \qquad 1 \le i \le \frac{n}{2}$$

$$f(v_i) = 2n - 1 + 2i \qquad 1 \leq i \leq \frac{n}{2}$$

$$f\left(y_{\frac{n}{2}-i+1}\right) = 3n - 1 + i \qquad 1 \leq i \leq \frac{n}{2}$$

$$f(x_i') = \frac{7n-2}{2} + i \qquad 1 \leq i \leq \frac{n}{2}$$

$$f(y_i') = 4n - 1 + i \qquad 1 \leq i \leq \frac{n}{2}$$

$$f(w_i) = \frac{9n-2}{2} + i \qquad 1 \leq i \leq \frac{n}{2}$$

Since $e_f(0) = e_f(1) = 3n - 1$, f is a difference cordial labeling of $S(DA(T_n))$.

Case (ii): Let the two triangles be start from $u_2$and ends withu_{n-1}.

In this case, the order and size of$S(DA(T_n))$are 5n – 7 an d 6n – 10respectively.

Define a function$f: V(S(DA(T_n))) \rightarrow \{1, 2, \ldots, 5n - 7\}$ by $f(u_1) = 2n - 1$,

$$f(u_i) = 2i - 2 \qquad 2 \leq i \leq n$$

$$f(u_i') = 2i - 1 \qquad 1 \leq i \leq n - 1$$

$$f(x_i) = 2n - 2 + 2i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(v_i) = 2n - 1 + 2i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(y_i) = 4n - 5 + i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(x_i') = 3n + 2i - 4 \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(y_i') = \frac{9n-12}{2} + i \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(w_i) = 3n + 2i - 3 \qquad 1 \le i \le \frac{n-2}{2}$$

Since $e_f(0) = e_f(1) = 3n - 5$, f is a difference cordial labeling of $S(DA(T_n))$.

Case (iii): Let the triangles be start from u_2 and ends with u_n.

In this case, the order and size of $S(DA(T_n))$ consist of 5n – 4 vertices and 6n – 6 edges respectively.

Define a function$f: V(S(DA(T_n))) \to \{1, 2, \dots, 5n - 4\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \le i \le n$$

$$f(u_i') = 2i \qquad 1 \le i \le n - 1$$

$$f(x_i) = 2n + 2i - 2 \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(v_i) = 2n + 2i - 1 \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(x_i') = 3n + 2i - 3 \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(w_i) = 3n + 2i - 2 \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(y_i) = 4n - 3 + i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(y_i') = \frac{9n-7}{2} + i \qquad 1 \le i \le \frac{n-1}{2}$$

Since $e_f(0) = e_f(1) = 3n - 3$, f is a difference cordial labeling of $S(DA(T_n))$.

Theorem: 3.7

$S(AQ_n)$ is difference cordial.

Proof:

Let the edges $u_iu_{i+1}, u_iv_i, v_iw_i, w_iu_{i+1}$ be subdivided by u_i', v_i', z_i, w_i' respectively.

Case (i): Let the squares be starts from u_1 and ends with u_n.

In this case the order and size of $S(AQ_n)$ are $\frac{9n-2}{2}$ and 5n – 2 respectively.

Define a map $f: V(S(AQ_n)) \to \{1, 2, \dots, \frac{9n-2}{2}\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \le i \le n$$

$$f(u_i') = 2i \qquad 1 \le i \le n-1$$

$$f(v_i') = 2n + 2i - 2 \qquad 1 \le i \le \frac{n}{2}$$

$$f(v_i) = 2n + 2i - 1 \qquad 1 \le i \le \frac{n}{2}$$

$$f(w_i') = 3n - 1 + i \qquad 1 \le i \le \frac{n}{2}$$

$$f\left(w_{\frac{n}{2}-i+1}\right) = 4n - 1 + i \qquad 1 \le i \le \frac{n}{2}$$

Since $e_f(0) = e_f(1) = \frac{5n-2}{2}$, f is a difference cordial labeling of $S(AQ_n)$.

Case (ii): Let the squares be starts from u_2 and ends with u_{n-1}.

The difference cordial labeling of $S(AQ_4)$ is given in figure 3.2.

10 12 11

8 9

7 1 2 3 4 5 6

Figure 3.2

For $n > 4$, Define a map $f: V(S(AQ_n)) \to \{1, 2, \dots, \frac{9n-12}{2}\}$ by

$$f(u_i) = 2i - 2 \qquad 2 \leq i \leq n$$

$$f(u_i') = 2i - 1 \qquad 1 \leq i \leq n - 1$$

$$f(v_i') = 2n + 2i - 3 \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(v_i) = 2n + 2i - 2 \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(w_i') = 3n - 4 + i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(z_i) = \frac{7n-10}{2} + i \qquad 1 \leq i \leq \frac{n-2}{2}$$

$$f(w_i) = 4n - 6 + i \qquad 1 \leq i \leq \frac{n-2}{2}$$

and $f(u_1) = \frac{9n-12}{2}$

Since $e_f(0) = e_f(1) = 5n - 8$, f is a difference cordial labeling of $S(AQ_n)$.

Case (iii): Let the squares be start from u_2 and ends with u_n.

The difference cordial labeling of $S(AQ_n)$ is given in figure 3.3.

8 9 7

6 10

1 2 3 4 5

Figure 3.3

For $n > 3$, Define a map $f: V(S(AQ_n)) \rightarrow \{1, 2, \dots, \frac{9n-7}{2}\}$ by

$$f(u_i) = 2i - 1 \qquad 2 \leq i \leq n$$

$$f(u_i') = 2i \qquad 1 \leq i \leq n - 1$$

$$f(v_i') = 2n + 2i - 2 \qquad 1 \leq i \leq \frac{n-1}{2}$$

$$f(v_i) = 2n + 2i - 1 \qquad 1 \leq i \leq \frac{n-1}{2}$$

$$f(z_i) = 3n - 2 + i \qquad 1 \leq i \leq \frac{n-1}{2}$$

$$f(w_i) = \frac{7n-5}{2} + i \qquad 1 \leq i \leq \frac{n-1}{2}$$

$$f(w_i') = 4n - 3 + i \qquad 1 \leq i \leq \frac{n-1}{2}$$

Since $e_f(0) = e_f(1) = \frac{5n-5}{2}$, f is a difference cordial labeling of $S(AQ_n)$.

Illustration: 3.8

The difference cordial labeling of $DA(Q_8)$ is given in figure 3.4.

17 28 35 19 29 34 21 30 33 23 31 32

16 24 18 25 20 26 22 27

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Figure 3.4

The difference cordial labeling of $DA(Q_8)$ is given in figure 3.5.

16 24 27 18 25 28 20 26 29

15 21 17 22 19 23

30 1 2 3 4 5 6 7 8 9 10 11 12 13 14

Figure 3.5

The difference cordial labeling of $DA(Q_7)$ is given in figure 3.6.

15 20 23 17 21 24 19 22 25

14 26 16 27 18 28

1 2 3 4 5 6 7 8 9 10 11 12 13

Figure 3.6

Theorem: 3.9

$S(DAQ_n)$ is difference cordial.

Proof:

Let the edges $u_iu_{i+1}, u_iv_i, v_iw_i, w_iu_{i+1}, u_ix_i, x_iy_i, y_iu_{i+1}$ be subdivided by $u_i', v_i', z_i, w_i', x_i', z_i', y_i$ respectively.

Case (i): Let the squares be start from $u_1$and ends with u_n.

The order and size of $S(DAQ_n)$ are 7n – 1 and 8n – 2 respectively.

Define a map$f: V(S(DAQ_n)) \rightarrow \{1, 2, \ldots, 7n - 1\}$ by

$$f(u_i) = 2i - 1 \qquad 1 \le i \le n$$

$$f(u_i') = 2i \qquad 1 \le i \le n - 1$$

$$f(v_i) = 2n + 3i - 3 \qquad 1 \le i \le \frac{n}{2}$$

$$f(z_i) = 2n + 3i - 2 \qquad 1 \le i \le \frac{n}{2}$$

$$f(w_i) = 2n + 3i - 1 \qquad 1 \le i \le \frac{n}{2}$$

$$f(x_i) = \frac{7n-6}{2} + 3i \qquad 1 \le i \le \frac{n}{2}$$

$$f(z_i') = \frac{7n-4}{2} + 3i \qquad 1 \le i \le \frac{n}{2}$$

$$f(y_i) = \frac{7n-2}{2} + 3i \qquad 1 \le i \le \frac{n}{2}$$

$$f(x_i') = \frac{11n-2}{2} + i \qquad 1 \le i \le \frac{n}{2}$$

$$f(w_i') = \frac{13n-2}{2} + i \qquad 1 \le i \le \frac{n}{2}$$

$$f\left(y'_{\frac{n}{2}-i+1}\right) = 5n - 1 + i \qquad 1 \le i \le \frac{n}{2}$$

$$f(v_i') = 6n - 1 + i \qquad 1 \le i \le \frac{n}{2}$$

Since $e_f(0) = e_f(1) = 4n - 1$, f is a difference cordial labeling of $S(DAQ_n)$.

Case (ii): Let the squares be start from $u_2$and ends with u_{n-1}.

The order and size of $S(DAQ_n)$ are 7n – 11 and 8n – 14 respectively.

Define a map$f: V(S(DAQ_n)) \to \{1, 2, \dots, 7n - 11\}$ by

$$f(u_i) = 2i - 2 \qquad 2 \le i \le n$$

$$f(u_i') = 2i - 1 \qquad 1 \le i \le n - 1$$

$$f(v_i) = 2n + 3i - 4 \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(z_i) = 2n + 3i - 3 \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(w_i) = 2n + 3i - 2 \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(x_i) = \frac{7n-14}{2} + 3i \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(z_i') = \frac{7n-12}{2} + 3i \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(y_i) = \frac{7n-10}{2} + 3i \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(w_i') = \frac{11n-18}{2} + i \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(y_i') = \frac{13n-22}{2} + i \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(v_i') = 5n - 8 + i \qquad 1 \le i \le \frac{n-2}{2}$$

$$f(x_i') = 6n - 10 + i \qquad 1 \le i \le \frac{n-2}{2}$$

and $f(u_1) = 7n - 11$.

Since $e_f(0) = e_f(1) = 4n - 7$, f is a difference cordial labeling of $S(DAQ_n)$.

Case (iii): Let the squares be start from $u_2$and ends with u_n.

Assign the labels to the vertices u_i $(1 \le i \le n)$,u_i' $(1 \le i \le n-1)$,v_i , z_i , w_i $(1 \le i \le \frac{n-1}{2})$ as in case 1.

$$f(x_i) = \frac{7n-9}{2} + 3i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(z_i') = \frac{7n-7}{2} + 3i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(y_i) = \frac{7n-5}{2} + 3i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(w_i') = \frac{11n-9}{2} + i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(y_i') = \frac{13n-11}{2} + i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(v_i') = 5n - 4 + i \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(x_i') = 6n - 5 + i \qquad 1 \le i \le \frac{n-1}{2}$$

Since $e_f(0) = e_f(1) = 4n - 4$, f is a difference cordial labeling of $S(DAQ_n)$.

CHAPTER -4

4-DIFFERENCE CORDIAL LABELING OF CYCLE AND WHEEL RELATED GRAPHS

Theorem: 4.1

Cycle C_n is a 4 – difference cordial graph.

Proof:

Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$.

We define labeling function $f: V(C_n) \to \{1,2,3,4\}$ as follows.

Case (i): n is odd.

$$f(v_{4i+1}) = 1; \qquad 0 \le i \le \left\lfloor \frac{n-1}{4} \right\rfloor$$

$$f(v_{4i+2}) = 2; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor$$

$$f(v_{4i}) = 3; \qquad 1 \le i \le \left\lfloor \frac{n-1}{4} \right\rfloor$$

$$f(v_{4i+3}) = 4; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor.$$

Case (ii): n is even.

Subcase (i): $n \equiv 0 (mod\ 4)$.

$$f(v_{4i}) = 1; \qquad 1 \le i \le \frac{n}{4}$$

$$f(v_{4i+3}) = 2; \qquad 0 \le i \le \frac{n-4}{4}$$

$$f(v_{4i+1}) = 3; \qquad 0 \le i \le \frac{n-4}{4}$$

$$f(v_{4i+2}) = 4; \qquad 0 \le i \le \frac{n-4}{4}$$

Subcase (ii): $n \equiv 2(mod\ 4)$.

$$f(v_1) = 2;$$

$$f(v_2) = 1;$$

$$f(v_{4i+1}) = 1; \qquad 1 \leq i \leq \frac{n-2}{4}$$

$$f(v_{4i+2}) = 2; \qquad 1 \leq i \leq \frac{n-2}{4}$$

$$f(v_{4i}) = 3; \qquad 1 \leq i \leq \frac{n-2}{4}$$

$$f(v_{4i+3}) = 4; \qquad 0 \leq i \leq \frac{n-6}{4}$$

In each case cycle C_n satisfies the conditions for 4 – difference cordial labeling.

Hence, C_n is a 4 – difference cordial graph.

Example: 4.2

The 4 – difference cordial labeling of C_{18} is shown in Figure 4.1

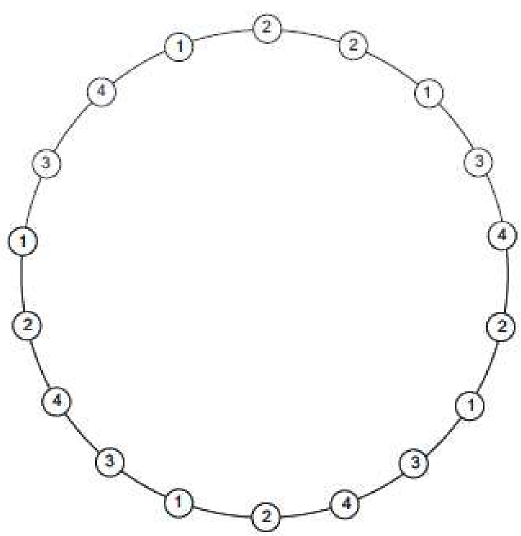

Figure 4.1

Theorem: 4.3

W_n is a 4 – difference cordial graph.

Proof:

Let v_0 be the apex vertex and $v_1, v_2, \dots, v_n$ be the rim vertices of W_n.

We define labeling function $f: V(W_n) \to \{1,2,3,4\}$ as follows.

Case (i): n is odd.

$$f(v_{4i}) = 1; \qquad 1 \le i \le \left\lfloor \frac{n-1}{4} \right\rfloor$$

$$f(v_{4i+1}) = 2; \qquad 0 \le i \le \left\lfloor \frac{n-1}{4} \right\rfloor$$

$$f(v_{4i+2}) = 3; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor$$

$$f(v_{4i+3}) = 4; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor.$$

Case (ii): n is even.

$$f(v_1) = 2,$$

$$f(v_2) = 3,$$

$$f(v_3) = 4,$$

$$f(v_{4i+3}) = 1; \qquad 1 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor$$

$$f(v_{4i}) = 2; \qquad 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor$$

$$f(v_{4i+1}) = 3; \qquad 1 \le i \le \left\lfloor \frac{n-2}{4} \right\rfloor$$

$$f(v_{4i+2}) = 4; \qquad 1 \le i \le \left\lfloor \frac{n-2}{4} \right\rfloor$$

In each case wheel graph W_n satisfies the conditions of 4 – difference cordial labeling.

Hence, $W_n$is 4 – difference cordial graph.

Example: 4.4

4 – difference cordial labeling of W_{11} is shown in Figure 4.2.

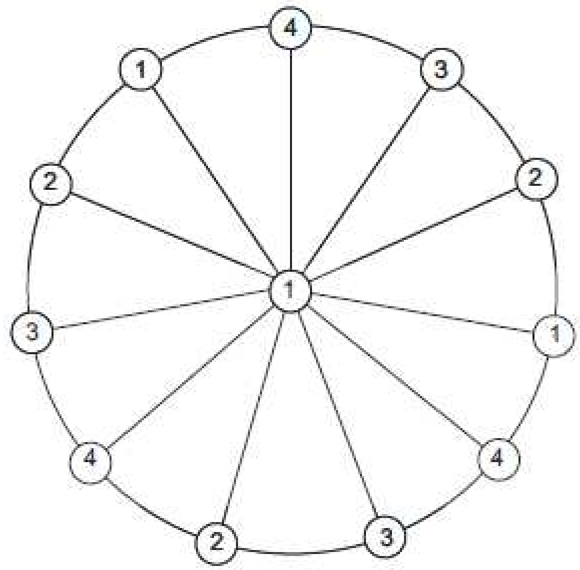

Figure 4.2

Theorem: 4.5

Crown $C_n \odot K_1$ is a 4 – difference cordial graph.

Proof:

Let $V(C_n \odot VK_1) = \{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$, where $v_1, v_2, \dots, v_n$ are rim vertices and $v'_1, v'_2, \dots, v'_n$ are pendant vertices.

We define labeling function $f: V(C_n \odot K_1) \to \{1,2,3,4\}$ as follows.

Case (i): n is odd.

$$f(v_{2i+1}) = 1; \qquad 0 \le i \le \frac{n-1}{2}$$

$$f(v_{2i}) = 2; \qquad 1 \le i \le \frac{n-1}{2}$$

$$f(v'_{2i+1}) = 3; \qquad 0 \leq i \leq \frac{n-1}{2}$$

$$f(v'_{2i}) = 4; \qquad 1 \leq i \leq \frac{n-1}{2}.$$

Case (ii): n is even.

$$f(v_{2i+1}) = 1; \qquad 0 \leq i \leq \frac{n-2}{2}$$

$$f(v_{2i}) = 2; \qquad 1 \leq i \leq \frac{n}{2}$$

$$f(v'_{2i+1}) = 3; \qquad 0 \leq i \leq \frac{n-2}{2}$$

$$f(v'_{2i}) = 4; \qquad 1 \leq i \leq \frac{n}{2}$$

In each case crown graph $C_n \odot K_1$ satisfies the conditions of 4 – difference cordial labeling. Hence, $C_n \odot K_1$is 4 – difference cordial graph.

Example: 4.6

4 – difference cordial labeling of crown $C_9 \odot K_1$ is shown in Figure 4.3.

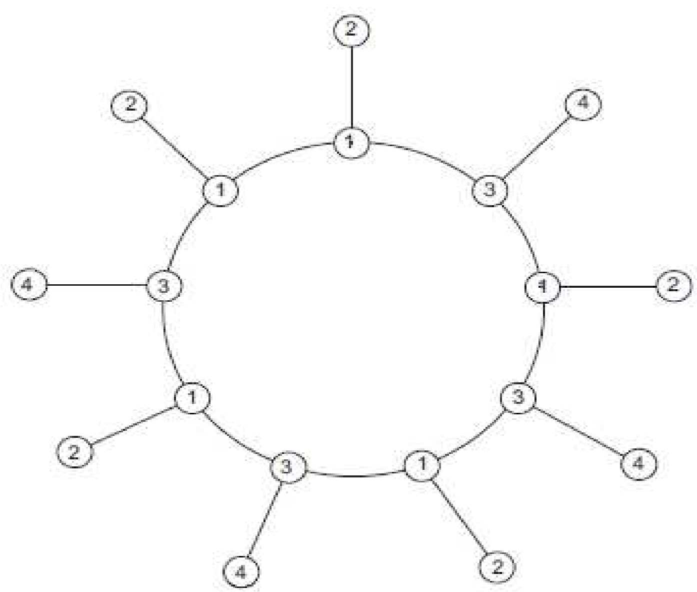

Figure 4.3

Theorem: 4.7

H_n is a 4 – difference cordial graph.

Proof:

Let $V(H_n) = \{v_0, v_1, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$, wherev$_0$ is apex vertex, $\{v_1, v_2, \dots, v_n\}$ are rim vertices and $\{v'_1, v'_2, \dots, v'_n\}$ are pendant vertices.

We define labeling function $f: V(H_n) \to \{1,2,3,4\}$ as follows.

Case (i): n is odd.

$$f(v_{4i}) = 1; \qquad 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$f(v_{4i+1}) = 2; \qquad 0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$f(v_{4i+2}) = 3; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor$$

$$f(v_{4i+3}) = 4; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor$$

$$f(v'_{4i+3}) = 1; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor$$

$$f(v'_{4i+2}) = 2; \qquad 0 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor$$

$$f(v'_{4i+1}) = 3; \qquad 0 \le i \le \left\lfloor \frac{n-1}{4} \right\rfloor$$

$$f(v'_{4i}) = 4; \qquad 1 \le i \le \left\lfloor \frac{n-1}{4} \right\rfloor$$

Case (ii): n is even.

$$f(v_{2i+1}) = 2; \qquad 0 \le i \le \frac{n-2}{2}$$

$$f(v_{2i}) = 4; \qquad 1 \le i \le \frac{n}{2}$$

$$f(v'_{2i+1}) = 1; \qquad 0 \le i \le \frac{n-2}{2}$$

$$f(v'_{2i}) = 3; \qquad 1 \le i \le \frac{n}{2}$$

In each case helm graph H_n satisfies the conditions of 4 – difference cordial labeling.

Hence, H_n 4 – difference cordial graph.

Example: 4.8

4 – difference cordial labeling of helm $H_9$is shown in Figure 4.4.

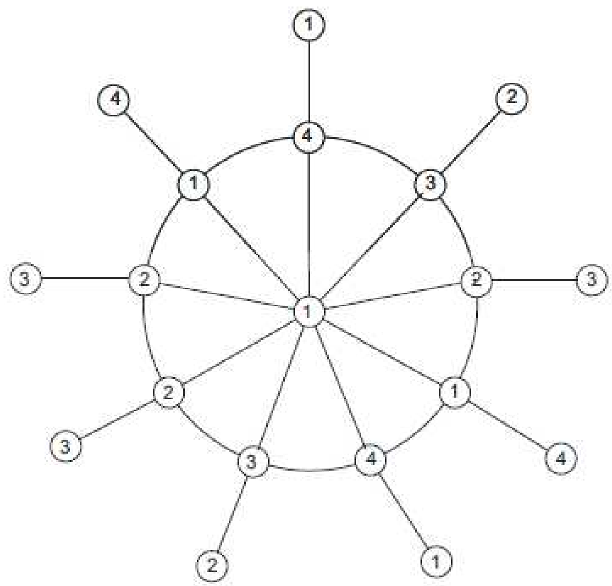

Figure 4.4

Theorem: 4.9

Gear G_n is a 4 – difference cordial graph.

Proof:

Let $V(G_n) = \{v_0, v_1, \ldots, v_{2n}\}$, wherev$_0$ is apex vertex, $\{v_1, v_3, \ldots, v_{2n-1}\}$ are the vertices of degree 3 and$\{v_2, v_4, \ldots, v_{2n}\}$ are the vertices of degree 2.

We define labeling function $f: V(G_n) \to \{1,2,3,4\}$ as follows.

Case (i): n is odd.

$$v_0 = 3.$$

$$f(v_{4i+1}) = 1; \qquad 0 \le i \le \frac{n-1}{2}$$

$$f(v_{4i+2}) = 2; \qquad 0 \le i \le \frac{n-1}{2}$$

$$f(v_{4i+3}) = 3; \qquad 0 \le i \le \frac{n-3}{2}$$

$$f(v_{4i+4}) = 4; \qquad 0 \le i \le \frac{n-3}{2}$$

Case (ii): n is even.

$$v_0 = 1.$$

$$f(v_{4i+1}) = 1; \qquad 0 \le i \le \frac{n}{2} - 1$$

$$f(v_{4i+2}) = 2; \qquad 0 \le i \le \frac{n}{2} - 1$$

$$f(v_{4i+3}) = 3; \qquad 0 \le i \le \frac{n}{2} - 1$$

$$f(v_{4i+4}) = 4; \qquad 0 \le i \le \frac{n}{2} - 1$$

In each case the gear graph G_n satisfies the conditions of 4 – difference cordial labeling.

Hence, G_n is 4 – difference cordial graph.

Example: 4.10

4 – difference cordial labeling of G_5 is shown in Figure 4.5

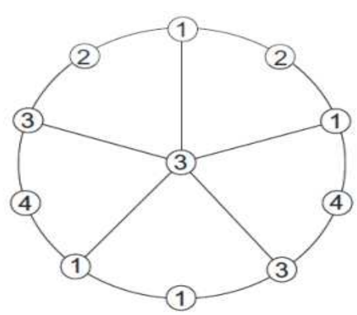

Figure 4.5

A STUDY ON RADIO LABELING

A project submitted to

ST. MARY'S COLLEGE (Autonomus), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

T. KANTHIMATHI

Reg. No: 19SPMT12

Under the guidance of

Ms. I. ANBU RAJAMMAL M.Sc., M.Phil., B.Ed., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON RADIO LABELING"** is submitted to **St.Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **T. KANTHIMATHI (Reg. No: 19SPMT12)**

J. Anne

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous,
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled "**A STUDY ON RADIO LABELING**" submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. I. Anbu Rajammal M.Sc., M.Phil., B.Ed., S.E.T.,** Assistant Professor, Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi**.

T. Karthiga...

Station: Thoothukudi **Signature of the Student**

Date: 19. 04. 2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER - 1

PRELIMINARIES

Definition: 1.1

A graph G is an ordered triple $(V(G), E(G), \psi(G))$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$ of edges and an incidence function ψ_G that associates with each edge of G and unordered pair of vertices of G. i.e., If e is an edge and u and v are vertices, then $\psi_G(e) = uv$, u and v are called the ends of e.

Definition: 1.2

A graph with no loops or multiple edges is called a simple graph we specify a simple graph by its set of vertices and set of edges , treating the edge set as a set of unordered pairs of vertices and write $e = uv$ or $e = vu$ for an edge e with endpoints u and v.

Definition: 1.3

A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G of $E(H)$.

Definition: 1.4

A walk consists of an alternating sequence of vertices and edges consecutive elements of which are incident that begins and ends with a vertex.

Definition: 1.5

If all the of edges of a walk are different then the walk is called a trail. If in addition all the vertices are difficult then the trail is called path.

Definition: 1.6

A bipartite graph is one whose vertex set can be partition into two subsets x and y. So that each edge has one end in x and one end in y. Let $X=\{v_2, v_4, v_8\}$; $Y=\{v_1, v_3, v_7\}$. Such a partition (X,Y) is called a bipartition of the graph.

Definition: 1.7

A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge.

Definition: 1.8

If G is a connected graph, the spanning tree in G is a sub graph of G which includes every vertex of G and is also a tree.

Definition: 1.9.

A graph $G = (V, E)$ is directed if the edge set is composed of ordered vertex (node) pairs A graph is undirected if the edge set is composed of unordered vertex pair.

Definition: 1.10

A graph that is in one piece is said to be connected whereas one which splits in to several pieces is disconnected.

Definition: 1.11

A graph G is regular if each vertex has same degree. i.e., $d(v) = k$, for all $v \in V$.

Definition: 1.12

A simple graph in which each pair of distinct vertices is joined by and edge is called a complete graph. It is denoted by K_n, where n is the number of vertices.

Definition: 1.13

A cycle graph is a graph consisting of a single cycle .The cycle graph with n vertices is denoted by C_n.

Definition: 1.14

A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then corresponding edge e_2 in G_2 has its end points the vertices u_2 and v_2 in G_2 which correspond to u_1 and v_1 respectively .Such a pair of correspondences is called a graph isomorphism.

Definition: 1.15

Let G be a connected graph. For any two vertices u and v; the distance between u and v denoted by $d_G(u, v)$ or $d(u, v)$ is the length of a shortest (u, v) path in G. A distance-two labeling (or λ-labeling) with span k is a function $c: V(G) \to \{1,2, \dots, k\}$ having the maximum value k such that the following relations are satisfied for any two distinct vertices u and v :

$$|c(u) - c(v)| \geq \begin{cases} 2, & if\ d(u, v) = 1 \\ 1, & if\ d\ (u, v) = 2. \end{cases}$$

The λ-number of G is the smallest k such that G admits a distance-two labeling with span k.

CHAPTER-2

RADIO LABELING

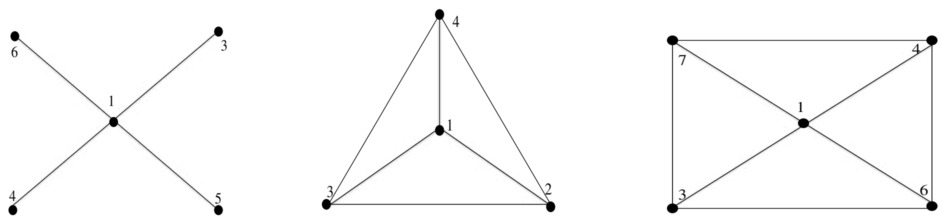
Definition: 2.1

A radio labeling or multi-level distance labeling with span k for a graph G is a function $c : V(G) \to \{1, 2, \dots, k\}$ having the maximum value k such that the following condition holds for any two distinct vertices u and v:

$$d(u,v) + |c(u) - c(v)| \geq 1 + diam(G)$$

This condition is referred to as radio condition.

We denote by $S(G,c)$ the set of consecutive integers $\{m, m+1, \dots, M\}$,where $m = min_{u \in V(G)} c(u)$ and $M = max_{u \in V(G)} c(u)$ is the span of c, denoted $span(c)$.



S_4 W_3 W_4

Figure: 2.1 Radio labeling on different kinds of graphs.

Definition: 2.2

The radio number of G, denoted by $rn(G)$, is the minimum span of a radio labeling for G. A radio labeling c of G with $span(c) = rn(G)$ will be called optimal radio labeling for G.

Note: 2.3

If $diam(G) = 2$, then radio labeling and λ -labeling become identical.

Definition: 2.4

A graph G with n vertices is called radio graceful if $rn(G) = n$.

Proposition: 2.5

`For a complete graph $rn(K_n) = n$.

Proof.

Let G be the complete graph K_n. (Figure 2.2) for $n \in N$.

For any $u, v \in G$ the distance will be 1 and the diameter of G is 1.

Hence the radio condition shows:

$$|c(u) - c(v)| \geq diam(G) + 1 - d(u, v)$$

$$|c(u) - c(v)| \geq 1 + 1 - 1$$

$$|c(u) - c(v)| \geq 1.$$

Hence the minimal radio labeling on K_n will be n

$rn(K_n) = n.$

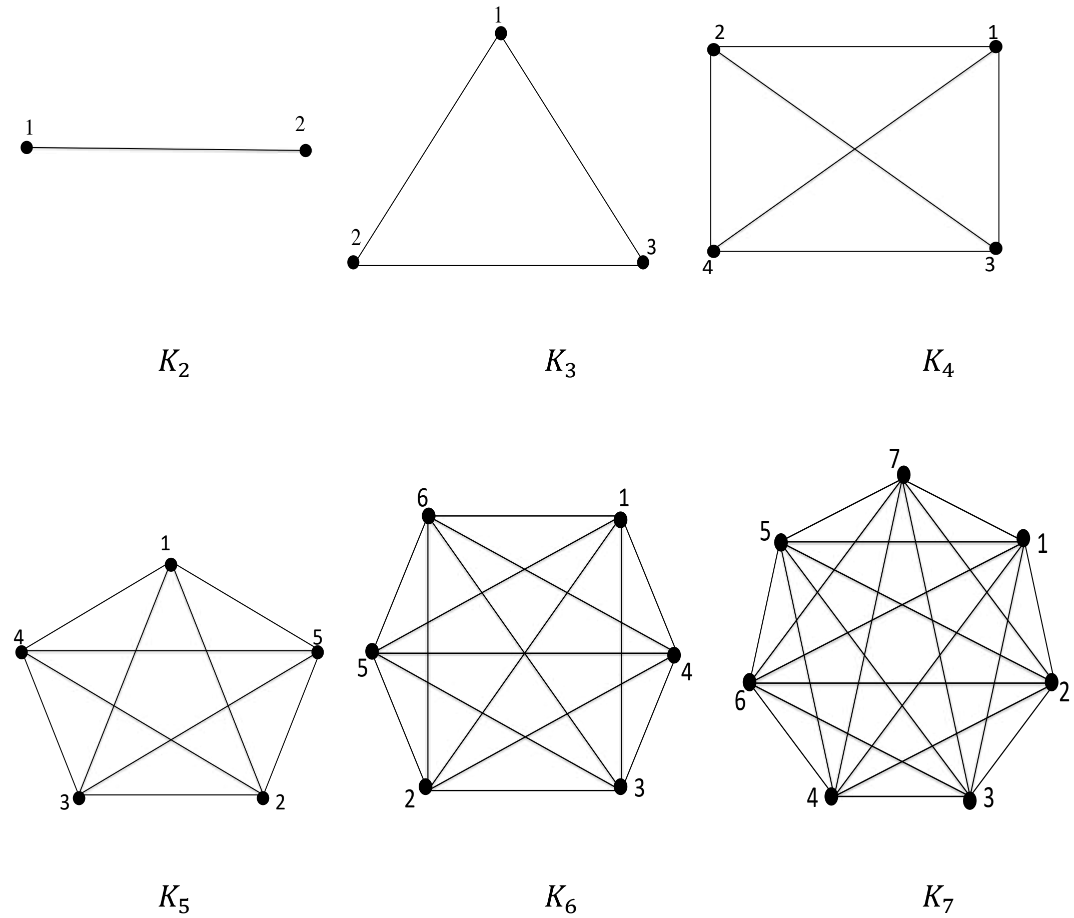


Figure: 2.2 Complete graph

Proposition: 2.6

Let G be the star graph S_n (Figure 2.3), then $rn(S_n) = n + 2$.

Proof.

For any $u, v \in G$ the distance will be 1 for neighboring vertices and two for non-neighboring vertices.

The diameter of G is 2.

Hence the radio condition shows:

For neighboring vertices:

$$|c(u) - c(v)| \geq 2 + 1 - 1$$

$$|c(u) - c(v)| \geq 2,$$

Or For non-neighboring vertices:

$$|c(u) - c(v)| \geq 2 + 1 - 2$$

$$|c(u) - c(v)| \geq 1.$$

Let $n \geq 2,$

the star graph S_n with the vertices $z, u_1, u_2, \dots, u_n$ where z is the center of the graph. Because $diam(S_n) = 2$ and $d(z, u_i) = 1$ for any $1 \leq i \leq n$ the radio condition for the vertices z and u_i becomes

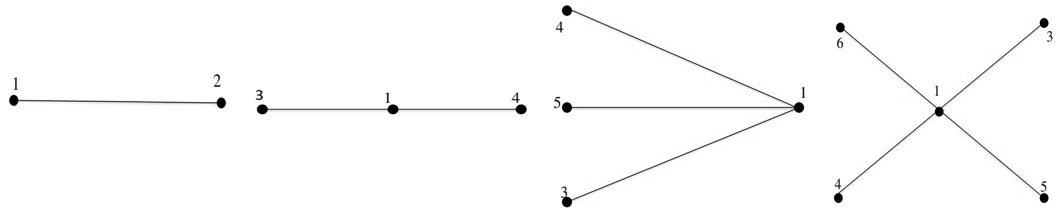
$$|c(z) - c(u_i)| \geq 2.$$

So for any labeling c of S_n,

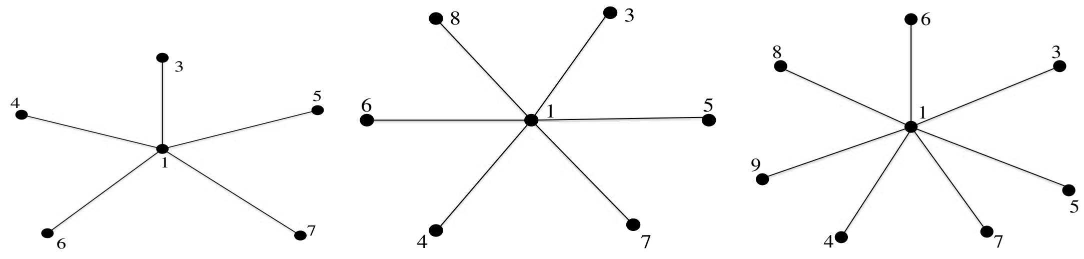
$c(z) + 1$ cannot be the label of an edge, then $rn(S_n) = n + 2.$

Then we can construct a label $f(z) = 1$, $f(u_i) = 2 + i$ for $1 \leq i \leq n$.

Hence $rn(S_n) = n + 2.$



S_1 S_2 S_3 S_4



S_5 S_6 S_7

Figure: 2.3 Star graph

Proposition: 2.7

The radio number of a wheel graph (Figure 2.4) with n vertices (W_n) is:

$$rn(W_n)=\begin{cases} 4, & if\ n = 3 \\ 7, & if\ n = 4 \\ n+2, & if\ n \geq 5 \end{cases}$$

Proof.

The $diam(W_n) = 2$. If $n = 3$ the graph is isomorphic to K_4.

Hence $rn\big((W_n)\big) = 4$.

For $n \geq 4$,

let z be the center of the graph and $u_1, u_2, \dots, u_n$ the other vertices of the cycle.

If $n = 4$, assume that $c(u_1) < c(u_3)$, $c(u_2) < c(u_4)$ and

$c(u_1) = \min\left\{\frac{c(u_i)}{1} \geq i \geq 4\right\}$.

Consider the following cases:

$$c(u_1) < c(u_2) < c(u_3) < c(u_4),$$

$$c(u_1) < c(u_2) < c(u_4) < c(u_3),$$

$$c(u_1) < c(u_3) < c(u_2) < c(u_4)$$

We get that $span(f) = 7$ such that: $(z) = 1, c(u_1) = 3, c(u_3) = 4,$

$c(u_2) = 6, c(u_4) = 7.$

Hence $rn(W_n) = 7.$

If $n \geq 5,$ as in the case of the S_n graph, because

$diam(W_n) = 2$ and $(z, u_i) = 1$ for any $1 \leq i \leq n,$

the radio condition for z and u_i is:

$$|c(u) - c(v)| \geq 2$$

Hence if c is the radio labeling of W_n then $c(z) + 1$ cannot be the radio labeling of any vertex,

so $rn(W_n) = n + 2.$

More over, we can construct a radio labeling c for W_n with $span(c) = n + 2$ as follows:

For n even, $c(z) = 1,$

$c(u_{2i+1}) = 3 + i$ for $0 \leq i \leq \frac{n}{2} - 1,$

$c(u_{2i}) = 2 + \frac{n}{2} + i$ for $1 \leq i \leq \frac{n}{2} - 1,$

and For n odd,

$c(z) = 1,$

$c(u_{2i+1}) = 3 + i$ for $0 \leq i \leq \frac{n-1}{2} - 1,$

$c(u_{2i}) = 3 + \frac{n-1}{2} + i$ for $1 \leq i \leq \frac{n-1}{2}$.

We can check that c is for both cases the radio labeling for W_n, because:

$$|c(v_i) - c(v_{i+1})| \geq \begin{cases} 2 + \dfrac{n}{2} - 3, & if\ n\ is\ evn \\ 3 + \dfrac{n-1}{2} - 3, & if\ n\ is\ odd. \end{cases}$$

More so, in both cases $span(c) = n + 2,$ hence $rn(W_n) = n + 2.$

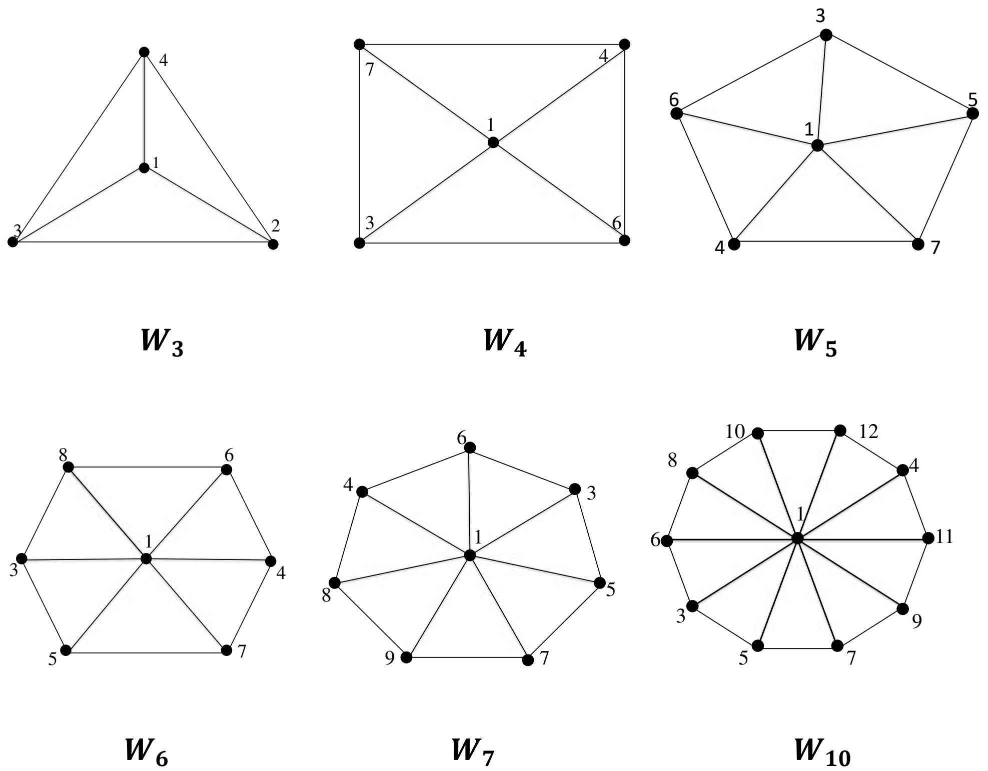


Figure: 2.4 Wheel graph

Proposition: 2.8

In a complete bipartite graph K_{m+n} (Figure 2.5) the radio number $rn(K_{m+n})$ is

$m + n + 1.$

Proof.

Let K_{m+n} be the complete bipartite graph with $m+n$ vertices, that means m vertices on the left hand side and n in the right hand side or vice-versa.

So $V(K_{m+n}) = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$, and there exists an edge between any a_i and b_j. So,

$$\left|c(a_i) - c\left(b_j\right)\right| \geq diam - d\left(a_i, b_j\right) + 1 = 2 - 1 + 1 = 2.$$

Let $A = \left\{c(a_i) | i \in \{1, m\}\right\}$, $B = \{c\left(b_j\right) | i \in \{1, n\}\}$

let $m_a = \max(A),\ m_b = \max(B)$.

Suppose without loss of generality that $m_a < m_b$.

Let $l = \min\{c(b_i) | c(b_i) \geq m_a\}$.

Note that $l > m_a$.

Suppose for the sake of contradiction that $l - 1 \in B$.

Then there exists $b_k$such that $c(b_k) = l - 1 \geq m_a$.

So $\min\{c(b_i) | c(b_i) \geq m_a\} = l - 1 \notin B$.

Suppose for the sake of contradiction that $l - 1 \in A$.

Then there exists , $a_k b_k$ such that

$c(a_k) = l - 1$ and $c(b_k) = l$ but $|c(a_k) - c(b_k)| = 1$.

This means that $l - 1 \notin A$.

Thus $l - 1 \notin A \cup B$.

So $\{1,2, \dots, m+n\} \not\subseteq \{c(a_1), \dots, c(a_m), c(b_1), \dots, c(b_n)\}$, and then

$\{1,2,\ldots,m+n\} \neq \{c(a_1),\ldots,c(a_m),c(b_1),\ldots,c(b_n)\}$.

Hence there exists $u \in \mathbb{N}$, $u \geq m+n+1$ such that $u \in A \cup B$.

So $rn(K_{m,n}) \geq m+n+1$.

Note that the following is a valid radio labeling

$c(b_1) = m+2$, $c(b_2) = m+3$, $c(b_n) = m+n+1$.

Thus $rn(K_{m,n}) = m+n+1$.

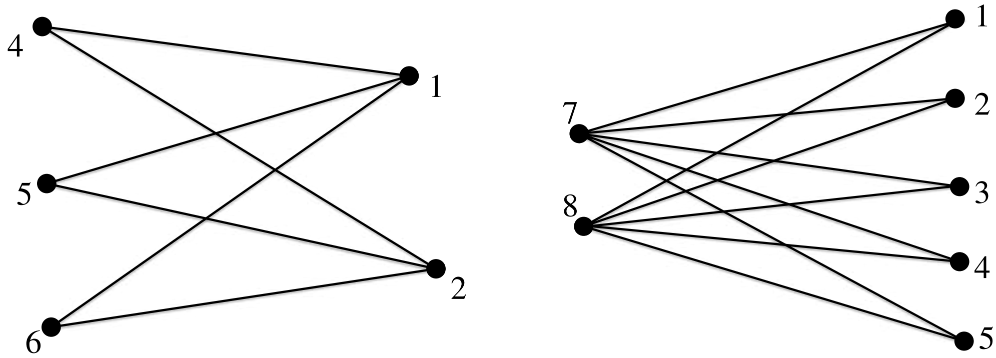


Figure: 2.5 Bipartite graph

CHAPTER- 3

RADIO NUMBER FOR MONGOLIAN TENT GRAPH

Definition: 3.1

The ladder graph, denoted by L_n, is the graph with vertex set

$V\ (L_n) = \{u_i, v_i\ :\ 1 \leq i \leq n\}$ and edge set

$E(L_n) = \{u_i v_{i+1};\ v_i v_{i+1}: 1 \leq i \leq n-1\} \cup\ u_i v_i\ : 1 \leq i \leq n.$

Definition: 3.2

Mongolian tent, denoted by Mt_n, is the graph obtained from the ladder graph L_n by adding a new vertex z and joining each vertex v_i; $1 \leq i \leq n$ with z .

Example: 3.3

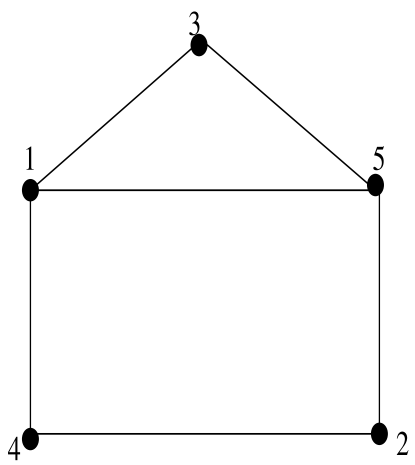


Figure 3.1: Mt_2

The following remark will be useful in our proofs.

Remark: 3.4

Let c be an optimum radio labeling of graph G.

We can associate to c an ordering of the vertices of G, increasing by their labels. Denote by $a_1, \ldots, a_n$ the vertices of G in this order:

$c(a_1) < c(a_2) < \cdots < c(a_n)$.

We have

- $c(a_1) = 1$
- $rn(G) = span(c) = 1 + \sum_{i=1}^{n-1}(c(a_{i+1}) - c(a_i))$
- If $c(a_{i+1}) - c(a_i) = 1$, then we have $d(a_i, a_{i+1}) = diam(G)$.

In order to find a lower bound for $rn(G)$, for graphs with small diameter is sometimes useful to determine how many pairs (a_i, a_{i+1}) with $c(a_{i+1}) - c(a_i) = 1$ we can have.

If there can be at most x such pairs, then we have:

$$rn(G) \geq 1 + x + 2(n - 1 - x).$$

Next, we introduce the notion of forbidden values associated to a vertex v for a radio labeling c.

Let c be a radio labeling of graph G.

Since vertex v has label $c(v)$ then, by radio condition, some values from $S(G, c)$ that are close to $c(v)$ cannot be labels for other vertices.

We will call these values forbidden values associated to vertex v.

Theorem: 3.5

a. Mongolian tent Mt_2 is radio graceful.

b. The radio number of Mongolian tent Mt_3 is 11.

c. The radio number of Mongolian tent $Mt_4$is 12.

Proof.

In order to prove that the values stated in the Theorem are lower bounds for the radio number, we will use the idea from Remark 3..3. Consider c an optimal radio labeling and denote by $a_1, a_2, \dots, a_m$ the vertices of the graph in increasing order of their labels

. We investigate the maximum number of pairs (a_i, a_{i+1}) with

$$c(a_{i+1}) - c(a_i) = 1$$

By radio condition, these pairs must have the property that

$$d(a_i, a_{i+1}) = diam(G).$$

For proving that the claimed values are upper bounds for the radio numbers of considered graphs,

we will provide radio labeling having spans equal to these values.

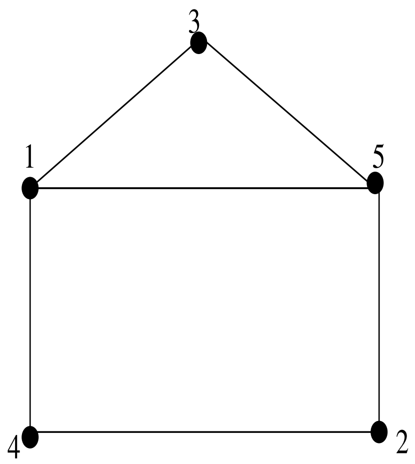
a) The Mongolian tent Mt_2, is a planar graph with 5 vertices, 6 edges and diameter 2.

We have $rn(Mt_2) \geq |V\ (Mt_2)| = 5$.

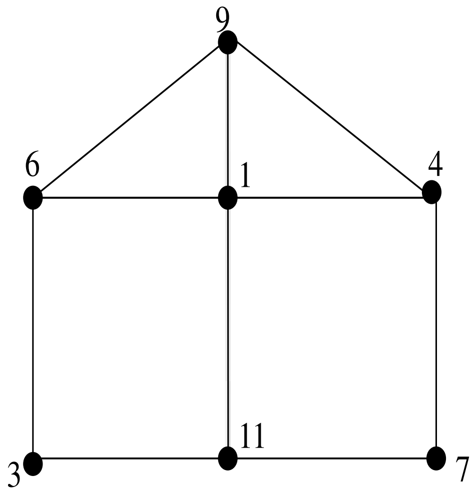
The radio labeling c of Mt_2 represented in Fig. 3.1(a), shows that $rn(Mt_2) \leq 5$.

It implies that $rn(Mt_2) = 5$.

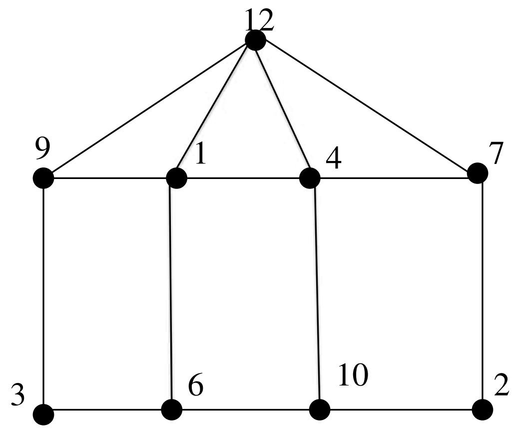
Therefore Mt_2 is radio graceful



a)Mt_2



b)Mt_3



c) Mt_4

Figure: 3.2 Radio labeling for Mongolian tent graphs

b) Mt_3 has $m = 7$ vertices and $diam(Mt_3) = 3$.

There are only two pairs of vertices at distance 3 in Mt_3, hence we have

$$rn(Mt_3) \geq 1 + 2 \cdot 1 + (m - 1 - 2) \cdot 2 = 1 + 2 + 8 = 11.$$

The radio labeling of M t_3 illustrated in Fig. 1 (b) shows that

$$rn(Mt_3) = 11.$$

We conclude that $rn(Mt_3) = 11$.

c) Mongolian tent Mt_4 has $m = 9$ vertices and $diam(Mt_4) = 3$.

There are 7 pairs of vertices at distance 3 in Mt_4.

In order to easily observe these pairs, consider the distance-3 graph associated to Mt_4, that is the graph having the same vertices as $Mt_4$and the edge set consisting of the pairs of vertices that are at distance 3 in Mt_4, shown in Fig.3.2

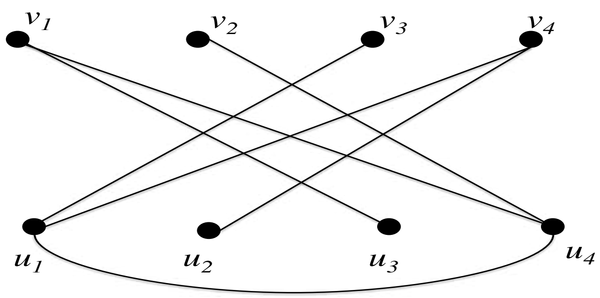


Figure: 3.3 Distance-3 graph of Mt_4.

In order to obtain a more precise estimation of the maximum number of pairs $(a_i, a_{i+1})$with $c(a_{i+1}) - c(a_i) = 1$,

we study how many triplets (a_i, a_{i+1}, a_{i+2}) may have consecutive labels:

$c(a_i), c(a_i) + 1, c(a_i) + 2.$

By radio condition we must have $d(a_i, a_{i+2}) \geq 2.$

Such a triplet corresponds to a path of length 2 in the distance-3 graph associated to Mt_4 ,whose extremities are at distance at least 2 in Mt_4.

It is easy to see that there are only two such path: $[v_2, u_4, u_1]$ and $[u_4, u_1, v_3]$, which have 2 vertices in common.

It follows that we can have at most 5 pairs (a_i, a_{i+1}) with consecutive labels (otherwise more triplets with consecutive labels will occur),

hence

$$rn(Mt_4) \geq 1 + 5 \cdot 1 + (m - 1 - 5) \cdot 2 = 1 + 5 + 6 = 12$$

The radio labeling of Mt_4 represented in Fig. 3.1 (c) shows that $rn(Mt_4) \leq 12$, hence $rn(Mt_4) = 12$.

Theorem: 3.6

For $n \geq 5$, $rn(Mt_n) \geq 4n + 2$.

Proof.

Assume $n \geq 5$. Then $diam(Mt_n) = 4$, so any radio labeling c of Mt_n must satisfy the radio condition $d(u, v) + |c(u) - c(v)| \geq 5$ for all distinct vertices $u, v \in V(Mt_n)$.

Let c be an optimal radio labeling for Mt_n.

We count the number of values needed for labels and add the minimum number of forbidden values for c.

Thus, since $d(z, r) \leq 2$ for all vertices $r \neq z$; the values

$$\{c(z) - 2;\ c(z) - 1;\ c(z) + 1;\ c(z) + 2\} \cap S(Mt_n, c)$$

are forbidden.

Similarly, as $d(v_i, r) \leq 3$ for all v_i and for any $r \neq v_i$; the values

$$\{c(v_i) - 1, c(v_i) + 1\} \cap S(Mt_n, c)$$

are forbidden, for every $i \in \{1,2, \dots, n\}$.

However, as $d(u_i, r) = 4$ for some vertex r;

it is possible to use consecutive labels on u_i and r. (i.e. there are no forbidden values associated with the vertices $\{u_1, u_2, \dots, u_n\}$.)

Remark that the number of forbidden values associated to z is $|c(z) - 2, c(z) - 1;\ c(z) + 1;\ c(z) + 2\} \cap S(Mt_n, c)| \geq 2$, with equality only if $c(z) \in \{1, span(c)\}$.

Also, $|\{c(v_i) - 1, c(v_i) + 1\} \cap S(Mt_n, c)| \geq 1$, with equality only if $c(v_i) \in \{1, span(c)\}$.

Moreover, these forbidden values are distinct, since by radio condition we must have $|c(z) - c(v_i)| \geq 3$ and $|c(v_i) - c(v_j)| \geq 2$ for every $i \neq j$.

The minimum number of forbidden values for c is then obtained in two situations (when there exists i such that $\{c(v_i), c(z)\} = \{1;\ span(c)\}$ or there exists $i \neq j$ such that $\{c(v_i), c(v_j)\} = \{1, span(c)\}$) and this number is

$$3 + 2n - 2 = 2n + 1.$$

Adding in the $2n + 1$ values needed to label the $2n + 1$ vertices provides a total of $4n + 2$ labels, hence $rn(Mt_n) \geq 4n + 2$; for $n \geq 5$.

Theorem: 3.7

For $n \geq 5;\ rn(Mt_n) \leq 4n + 2$.

Proof.

We shall propose a radio labeling of Mt_n with span $4n + 2$; which implies $(Mt_n) \leq 4n + 2$.

Let $n \geq 5$.

The radio labeling $c : V(Mt_n) \to \mathbb{Z}^+$ is defined as follows:

$c(z) = 4n + 2$

$$c(u_i) = \begin{cases} 4i, & if\ 1 \le i \le n-1 \\ 3\,, & if \quad i = n \end{cases}$$

Case A- n is odd:

$$c(v_i) = \begin{cases} 2n+4i, & if\ 1 \le i \le \dfrac{n+1}{2} - 1 \\ 1\ , & if\ i = \dfrac{n+1}{2} \\ 2(2i-n), & if\ \dfrac{n+1}{2} + 1 \le i \le n \end{cases}$$

Case B- n is even:

$$c(v_i) = \begin{cases} 2(n+2i+1), & if\ 1 \le i \le \frac{n}{2} - 1 \\ 1\ , & if\ i = \frac{n}{2} \\ 2(2i+1-n), & if\ \frac{n}{2} + 1 \le i \le n \end{cases}$$

In both cases the span of c is equal to $4n+2$ and it is reached for $c(z)$.

Claim: The labeling c is a valid radio labeling.

We must show that the radio condition d$(u, v)\ + |c(u) - c(v)|\ diam(Mt_4) + 1 =\ 5$ holds for all pairs of vertices (u, v) (where $u \neq\ v$).

1. Consider the pair (z, r) (for any $r \neq z$). As $d(z, r) \ge 1$ and $c(r) \le 4n-2$,

we have $d(z, r) + |c(z) - c(r)| \ge 1\ + |4n\ +\ 2\ (4n\ 2)| \ge 5$ for any $r \neq z$.

The radio condition is satisfied.

2. Consider the pairs $(v_i\,, v_j)$ (with $\neq j$). Note that $d(v_i\,, v_j) \ge 1$ for $i \neq j$. $|c(v_i) - c(v_j)| \ge\ 4$ for all $v_i \neq v_j$: Hence, again, the radio condition is satisfied.

3. Consider the pairs (u_i, u_j) (with $i \neq j$).

We have $d(u_1, u_n) = 4$,and the labels diffence for this pair is

$$|c(u_1) - c(u_n)| = 1;$$

4. so the radio condition for (u_1, u_n) is satisfied.

Note that $d(u_i, u_j) \geq 1$ for $i \neq j$ and the label difference for each pair is $|c(u_i) - c(u_j)| \geq 4$, except the pair (u_1, u_n).

The radio condition is then satisfied for all distinct u_i.

5. Finally, consider the pairs (u, v) where $u \in \{u_1, u_2, \dots, u_n\}$ and $v \in \{v_1, v_2, \dots, v_n\}$.

We have $c(u) \in \{3,4,8,12, \dots, 4(n-1)\}$ If $d(u, v) = 2$ then by the way c was defined,

$$|c(u) - \&c(v)| \geq 2n - 3 \geq 7 \text{ for } n \geq 5.$$

If $d(u, v) = 2$, then $|c(u) - \&c(v)| \geq 2n - 7 \geq 3$ for $n \geq 5$.

When $d(u, v) = 3$,

$$|c(u) - \&c(v)| \geq 2.$$

It follows that the radio condition is satisfied for these pairs.

These four cases establish the claim that c is a radio labeling of Mt_n.

Thus $rn(Mt_n) \leq span(c) \leq 4n + 2$.

Theorem: 3.8

The radio number of Mongolian tent Mt_n is $4n + 2$ when $n \geq 5$.

Proof.

Theorem 3.6 shows $rn(Mt_n) \geq 4n + 2$ for $n \geq 5$;

and Theorem 3.7 shows $rn(Mt_n) \leq 4n + 2$ for $n \geq 5$.

Therefore,

$rn(Mt_n) = 4n + 2.$

CHAPTER-4

RADIO NUMBER FOR DIAMOND GRAPH

Definition: 4.1

Diamond graph, denoted by d_n, is the graph obtained from the Mongolian tent graph Mt_n by adding a new vertex z_1 and joining each vertex u_i, $1 \leq i \leq n$ with z_1.

Example: 4.2

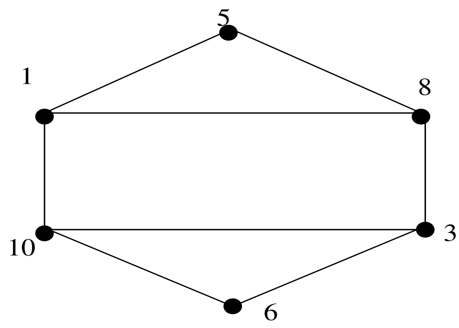


Figure:4.1 d_n graph

Theorem:4.3.

For diamond graphs the following relations hold:

a) $rn(d_2) = 10$

b) $rn(d_3) = 12$

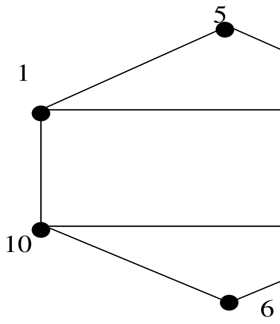
c) $rn(d_4) = 14$

d) $rn(d_5) = 15$

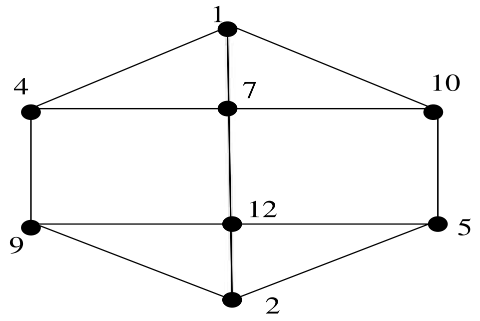
Proof.

In Fig. 4.2 are shown radio labelings having spans equal to the values stated in the Theorem,

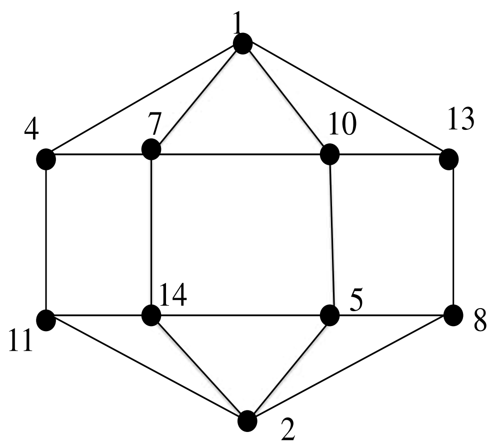
hence these values are upper bounds for the radio numbers of considered graphs.



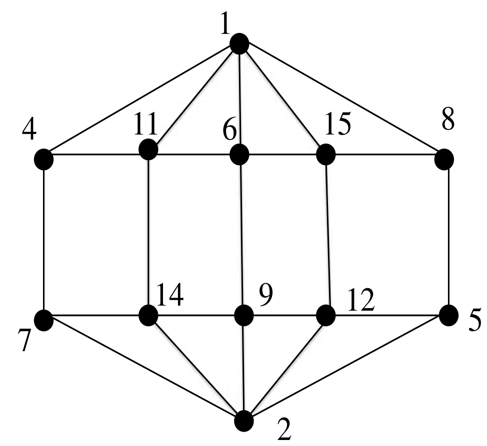
a) d_2



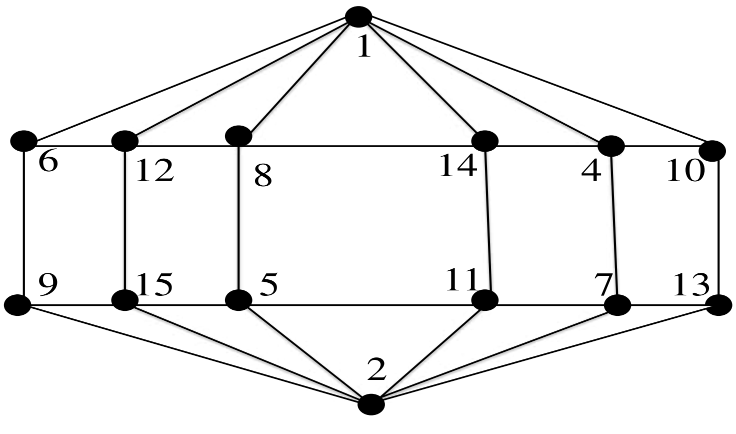
b) d_3



c) d_4



d) d_5



e) d_6

Figure: 4.2 Radio labeling for Diamond graphs

In order to prove that they are also lower bounds, we will use the same arguments as in Theorem 3.4 , based on Remark 3.4.

Consider c an optimal radio labeling and denote by $a_1, a_2, \dots, a_m$ the vertices of the graph in increasing order of their labels.

a) We have $m = |V\ (d_2)| = 6$ and $diam(d_2) = 3$.

There is only one pair of vertices at distance 3 in d_2 (that is (z, z_1)),

hence we have

$$rn(d_2) \geq 1 + 1 \cdot 1 + (m-1-1) \cdot 2 = 1 + 1 + 8 = 10$$

b) We have $m = |V\ (d_3)| = 8$ and $diam(d_3) = 3$.

There are three pairs of vertices at distance 3 in d_2: $(z, z_1), (v_1, u_3)$ and (v_3, u_1),

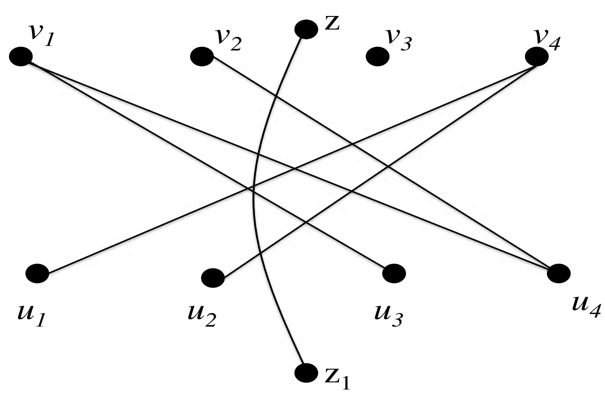
hence we have

$$rn(d_3) \geq 1 + 3 \cdot 1 + (m-1-3) \cdot 2 = 1 + 3 + 8 = 12$$

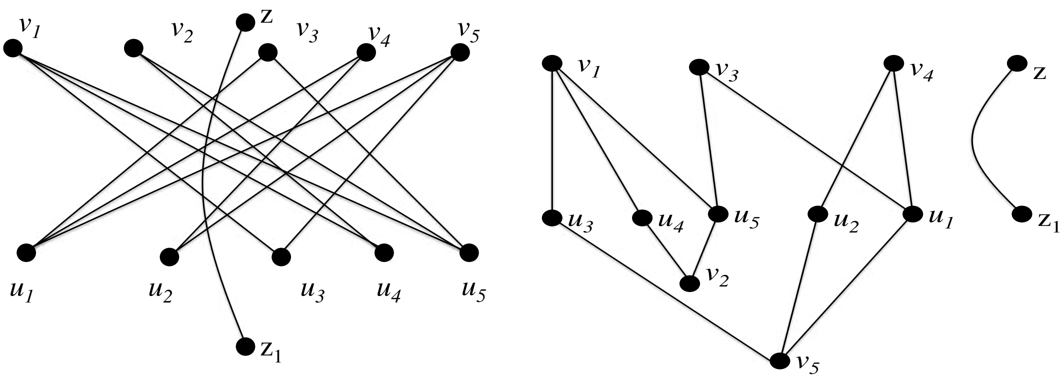
c) d_4 has $m = 10$ vertices and $diam(d_4) = 3$.

Consider the distance-3 graph associated to d_4, shown in Fig.4.2 (a).

As in proof of Theorem 3.5, we observe that there is no path of length 2 in the distance-3 graph associated to d_4 whose extremities are at distance at least 2 in d_4, hence there are no triplets (a_i, a_{i+1}, a_{i+2}) having consecutive labels.



a)



b)**Figure 4.2 Distance-3 graph for d_4 and d_5.**

It follows that we can have at most $\frac{m}{2} = 5$ pairs of vertices $(a_i, a_{i+1})$with consecutive labels, hence $rn(d_4) \geq 1 + 5 \cdot 1 + (m - 1 - 5) \cdot 2 = 1 + 5 + 8 = 14$.

d)We have $|V(d_5)| = 12$ and $diam(d_5) = 3$.

We consider again paths of length 2 in the distance-3 graph associated to d_5, shown in Fig. 4.2(b).

There are 3 paths of length 2 in the distance 3 graph associated to d_4 joining vertices at distance at least 2 in d_4: $[u_5, v_1, u_3]$, $[u_1, v_5, u_3]$, $[u_5, v_3, u_1]$.

These paths contain 6 of the vertices of the graph, so there are no triplets of vertices with consecutive labels containing some of the other 6 vertices.It follows that there are atmost $(6 - 1) + \frac{6}{2} = 8$ pairs of vertices with consecutive labels, hence $rn(d_5)1 + 8 \cdot 1 + (m - 1 - 8) \cdot 2 = 1 + 8 + 6 = 15$.

Theorem: 4.4

For $n \geq 6$, the radio number of diamond graph d_n is $2n + 3$.

Proof.

Recall the vertex set and edge set of diamond graph as follows:

$$V(d_n) = \{v_i, u_i: 1 \leq i \leq n\} \cup \{z, z_1\}$$

$E(d_n) = \{u_i, u_{i+1}, v_i, v_{i+1}: 1 \leq i \leq n - 1\} \cup \{u_i, v_i, zv_i, z_1u_i: 1 \leq i \leq n\}$.

For $n \geq 6$; $diam(d_n) = 3$:

The diamond graph contains $2n + 2$ vertices and $5n - 2$ edges.

First we will prove that $rn(d_n) \geq 2n + 3$.

For that, let c be a radio labeling ford_n.

We will prove that c has at least one forbidden value, associated to one of the vertices z and z_1.

By symmetry we can assume $c(z) < c(z_1)$.

Denote $a = c(z)$.

As z_1 is the only vertex at distance 3 of z, $a - 1$ and $a + 1$ can be used as label only for z_1.

Assume $c(z_1) = a + 1 = b$.

As $d(z_1, r) \leq 2$ for all $r \in \{z, z_1\}$, if $b + 1 = a + 2$ is assigned to any other vertices, then the condition (1) is not satisfed.

It follows that if $c(z_1) = a + 1$ then either $c(z) - 1$ is a forbidden value associated to z (if $c(z) > 1$), or $c(z) + 2$ (if $c(z) = 1$).If $c(z_1)$ is not labeled with $a + 1$ then, since $a = c(z) < c(z_1)$ $span(c)$, value $a + 1$ is forbidden.

Therefore $rn(d_n)$ must be greater or equal to $|V(d_n)| + 1 = 2n + 3$. To prove $rn(d_n) \leq 2n + 3$, we define a labeling $c : V(d_n) \to \{1,2, \dots, 2n + 3\}$ as follows such that radio condition is satisfied.For $n = 6$ such a labeling is shown in Fig. 4.1 (e).Let $n \geq 7$.

Case A- n is even

$c(z) = 1, c(z_1) = 2$

$$c(v_i) = \begin{cases} n+6-i, & if\ i \equiv 0(mod\ 2) \\ 4\,, & if\ i = 1 \\ 2n+5-i, & if\ i \geq 3\ and\ i \equiv 1(mod\ 2) \end{cases}$$

$$c(u_i) = \begin{cases} n+9-i\,, & if\ \ i \equiv 0(mod\ 2) \\ 8-i\,, & if\ \ i = 1{,}3 \\ 2n+8-i, & if\ \ i \geq 5\ and\ i \equiv 1(mod\ 2) \end{cases}$$

Case B- n is odd

We divide this case into two subcases.

B.1: $n \equiv 3\ (mod\ 4)$. Then we define $c(z) = 2n+3, c(z) = 2n+2$

$$c(u_i) = \begin{cases} \dfrac{3n+i+1}{2}, & if\ i \equiv 0(mod\ 4) \\ \dfrac{i+2}{2}, & if\ i \equiv 1(mod\ 4) \\ \dfrac{n+1+i}{2}, & if\ i \equiv 2(mod\ 4) \\ \dfrac{2n+1+i}{2}, & if\ i \equiv 3(mod\ 4) \end{cases}$$

$$c(v_i) = \begin{cases} \dfrac{n+1+i}{2}, & if\ i \equiv 0(mod\ 4) \\ \dfrac{2n+1+i}{2}, & if\ i \equiv 1(mod\ 4) \\ \dfrac{3n+i+1}{2}, & if\ i \equiv 2(mod\ 4) \\ \dfrac{i+1}{2}, & if\ i \equiv 3(mod\ 4) \end{cases}$$

B.2: $n \equiv 1\ (mod\ 4)$. Then we define $c(z) = 2n+2, c(z) = 2n+3$

$$c(u_i) = \begin{cases} n+1+\frac{i}{2}, & if\ i \equiv 0(mod\ 4) \\ \frac{n+2+i}{2}, & if\ i \equiv 1(mod\ 4) \\ 2n, & if\ i = 2 \\ \frac{3n+i-1}{2}, & if\ i > 2\ and\ i \equiv 2(mod\ 4) \\ \frac{i+1}{2}, & if\ i \equiv 3(mod\ 4) \end{cases}$$

$$c(v_i) = \begin{cases} \frac{3n+i+1}{2}, & if\ i \equiv 0(mod\ 4) \\ \frac{i+1}{2}, & if\ i \equiv 1(mod\ 4) \\ \frac{2n+2+i}{2}, & if\ i \equiv 2(mod\ 4) \\ \frac{n+1+i}{2}, & if\ i \equiv 3(mod\ 4) \end{cases}$$

Claim: The labeling c is a valid radio labeling.

We must show that the radio condition

$$d(u,v)\ + |c(u)\ c(v)| \geq\ 1\ +\ diam(d_n)\ =\ 4$$

holds for all pairs of vertices (u,v) (where $u \neq\ v$).

Case A: Assume n is even.

We consider all types of pairs of vertices.

1: Consider the pairs (z,r) for any vertex r$\notin \{z,z_1\}$. As 1$\leq d(z,r) \leq 2$, r $\notin \{z,z_1\}$, $c(z) = 1,\ c(r) \geq 4, |c(z) - c(r)| \geq 3$, it follows that

$$d(z,r) + |c(z)c(r)| \geq 1 + 3 = 4.$$

2: For pair (z,z_1), as $d(z,z_1) = 3$ and $|c(z) - c(z_1)| =\ 1$, the radio condition is satisfied.

3: Consider the pairs (z_1,r) for any vertex $r \notin \{z,v_1\}$.

As 1$\leq d(z_1,r) \leq 2$, r $\notin \{z,v_1\}$, $c(z_1) = 2,\ c(r) \geq 5,\ |c(z_1) - c(r)| \geq 3$, and

$d(z_1, v_1) = 2, |c(z_1) - c(v_1)| = 2,$ the radio condition (2) is satisfied.

4: Consider the pairs (v_i, v_j) (with $i \neq j$) If $d(v_i, v_j) = 1$ we have

$d(v_i, v_j) + |c(v_i) - c(v_j)| \geq 1 + |n-2| \geq 6$, otherwise $|c(v_i) - c(v_j)| \geq 2$.

Therefore the radio condition is satisfied for such pairs.

5: Consider the pairs (u_i, u_j) (with $i \neq j$) Similar as Case 3.

6: Consider the pairs (u_i, v_j).

We examine the label difference for each pair, when distance between vertices is one, two, three. As $1 \leq d(u_i, v_j) \leq 3$, so

- if $d(u_i, v_j) = 1$ then $i = j$ and $|c(u_i) - c(v_j)| \geq 3$
- if $d(u_i, v_j) = 2$ then $i = j \pm 1$ and $|c(u_i) - c(v_j)| \geq |n-5| \geq 2$ for $n \geq 7$
- if $d(u_i, v_j) = 3$ then $|c(u_i) - c(v_j)| \geq 1$.

Hence the radio condition is satisfied.

Case B: n is odd.

B 1 If $n \equiv 3 \ (mod\ 4)$ we have the following cases

1: Consider the pair (z, r) for any vertex $r \notin \{z, z_1\}$.

As $1 \leq d(z, r) \leq 2,\ r \notin \{z, z_1\},\ c(z) = 2n + 3,\ c(r) \leq 2n,\ |c(z) - c(r)| \geq 3$.

Hence $d(z, r) + |c(z) - c(r)| \geq 1 + 3 = 4$.

2: As $d(z, z_1) = 3$ and $|c(z) - c(z_1)| = 1$ the radio condition is satisfied.

3: Consider the pairs (z_1, r) for any vertex $r \notin \{z, z_1\}$. As $1 \leq d(z_1, r) \leq 2,\ r \notin \{z, z_1\}$, when $d(z_1, r) = 1$ we have $|c(z_1) - c(r)| \geq 3$ and when $d(z_1, r) = 2$, then $|c(z_1) - c(r)| \geq 2$: It follows that the radio condition (2) is satisfied.

4: Consider the pairs (v_i, v_j) (with $i \neq j$). As $d(v_i, v_j) \leq 2$ for $i \neq j$.

- if $d(v_i, v_j) = 1$ then $i = j\ \ 1$ and $|c(v_i) - c(v_j)| \geq \frac{n+1}{2} \geq 4$ for $n \geq 7$

- if $d(v_i, v_j) = 2$ then $|c(v_i) - c(v_j)| \geq 2$.

5: Consider the pairs (u_i, u_j) (with $i \neq j$). As $d(u_i, u_j) \leq 2$ for $i \neq j$,

- if $d(u_i, u_j) = 1$ then $i = j \pm 1$ and $|c(u_i) - c(u_j)| \geq \frac{n+1}{2} \geq 4$ for $n \geq 7$
- if $d(u_i, u_j) = 2$ then $|c(u_i) - c(u_j)| \geq 2n \geq 7$.

Hence the radio condition is also satisfied for these pairs.

6: Consider the pairs (u_i, v_j). We examine the labels difference for each pair, when distance between vertices is one, two, three. As $1 \leq d(u_i, v_j) \leq 3$, so

- if $d(u_i, v_j) = 1$ then $i = j$ and $|c(u_i) - c(v_j)| \geq 3$
- if $d(u_i, v_j) = 2$ then $i = j \pm 1$ and $|c(u_i) - c(v_j)| \geq |n - 5| \geq 2$
- if $d(u_i, v_j) = 3$ then $|c(u_i) - c(v_j)| \geq 1$

.Hence the radio condition (2) is satisfied.

The situation when $n \equiv 1\ (mod\ 4)$ is similar as $n \equiv 3\ (mod\ 4)$.

For all cases we establish the claim that c is a radio labeling of d_n.

Thus $rn(d_n) \leq 2n + 3$. Hence $rn(d_n) = 2n + 3$.

CHAPTER-5

RADIO LABELING OF SOME CYCLE RELATED GRAPHS

Definition: 5.1

A Chord of a cycle C is an edge not in C whose end vertices lie in C.

Definition: 5.2

The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Definition: 5.3

For a graph G the split graph is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The new graph is denoted as $spl(G)$.

Definition: 5.4

A petal graph is a connected graph G with $\Delta(G) = 3$ and $\delta(G) = 2$ in which the set of vertices of degree three induces a 2-regular graph and the set of vertices of degree two induces an empty graph.

In a petal graph G if w is a vertex of G with degree two, having neighbours v_1, v_2 then the path $P_w = v_1 w v_2$ is called petal of G. We name w the center of the petal and v_1, v_2 the basepoints.

If $dv_1, v_2 = k$, we say that the size of the petal is k. If the size of each petal is k then it is called a k −petal graph.

Theorem: 5.5

Let G be a cycle with chords. Then,

$$rn(G) = \begin{cases} (k+2)(2k-1)+1+\sum_i (d_i - d_i'), & n \equiv 0(mod\ 4) \\ 2k(k+2)+1+\sum_i (d_i - d_i'), & n \equiv 2(mod\ 4) \\ 2k(k+1)+\sum_i (d_i - d_i'), & n \equiv 1(mod\ 4) \\ (k+2)(2k+1)+\sum_i (d_i - d_i'), & n \equiv 3(mod\ 4) \end{cases}$$

Proof.

Let C_n denote the cycle on n vertices and $V(C_n) = \{v_0, v_1, \dots, v_{n-1}\}$ be such that v_i is adjacent to v_{i+1} and v_{n-1} is adjacent to v_0.

We denote $d = diam(C_n)$.

The labels are assigned with the help of the following two sequences.

- the distance gap sequence $D = (d_0, d_1, \dots, d_{n-2})$
- the color gap sequence $F = (f_0, f_1, \dots, f_{n-2})$

The distance gap sequence in which each $d_i \leq d$ is a positive integer is used to generate an ordering of the vertices of C_n.

Let $\tau : \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\}$ be defined as $\tau(0) = 0$ and $\tau(i+1) = \tau(i) + d_i\ (mod\ n)$.

Here τ is a corresponding permutation.

Let $x_i = v_{\tau(i)}$ for $i = 0, 1, 2, \dots, n-1$.

Then $\{x_0, x_1, \dots, x_{n-1}\}$ is an ordering of the vertices of C_n.

Let us denote $d(x_i, x_{i+1}) = d_i$.

The color gap sequence is used to assign labels to the vertices of C_n.

Let f be the labeling defined by $f(x_0) = 0$ and for $i \geq 1$, $f(x_i + 1) = f(x_i) + f_i$. By the definition of radio labeling, $f_i \geq d - d_i + 1$ for all i.

We adopt the scheme for distance gap sequence and color gap sequence proceed as follows.

Case 1: $n = 4k$.

In this case, $diam(G) = 2k$.

Using the sequences given below we can generate the radio labeling of cycle C_n for $n \equiv 0(mod\ 4)$ with minimum span.

The distance gap sequence is given by

$$d_i = \begin{cases} 2k, & if\ i\ is\ even \\ k, & if\ i \equiv 1(mod\ 4) \\ k+1 & if\ i \equiv 3(mod\ 4) \end{cases}$$

And the color gap sequence is given by

$$f_i = \begin{cases} 1, & if\ i\ is\ even \\ k+1, & if\ i\ is\ odd \end{cases}$$

Then, for $i = 0, 1, 2, \ldots k-1$ we have the following permutation,

$$\tau(4i) = 2ik + i(mod\ n)$$

$$\tau(4i+1) = (2i+2)k + i(mod\ n)$$

$$\tau(4i+2) = (2i+3)\,k + i(mod\ n)$$

$$\tau(4i+3) = (2i+1)k + i(mod\ n)$$

Now we add chords in cycle C_n such that diameter of the cycle remains unchanged. Label the vertices of this newly obtained graph using the above permutation.

Suppose the new distance between x_i and x_{i+1} is $d_i'(x_i, x_{i+1})$, then due to chords in the cycle it is obvious that $d_i \geq d_i'$

We define the color gap sequence as $f_i' = f_i + (d_i - d_i'), 0 \leq i \leq n-2$. So that span f' for cycle with chord is

$$f_0' + f_1' + \cdots + f_{n-2}' = f_0 + f_1 + f_2 + \cdots + f_{n-2} + \sum (d_i - d_i')$$

$= rn(C_n) + \sum(d_i - d_i')$

$= (k+2)(2k-1) + 1 + \sum(d_i - d_i'), 0 \le i \le n-2$

Which is an upper bound for the radio number for the cycle with arbitrary number of chords when $n = 4k$.

Case 2: $n = 4k + 2$.

In this case $diam(G) = 2k + 1$.

Using the sequences given below we can generate radio labeling of the cycle C_n for $n \equiv 2(mod\ 4)$ with minimum span.

The distance gap sequence is given by

$$d_i = \begin{cases} 2k+1, & if\ i\ is\ even \\ k+1, & if\ i\ is\ odd \end{cases}$$

and the color gap sequence is given by

$$f_i = \begin{cases} 1 & if\ i\ is\ even \\ k+1, & if\ i\ is\ odd \end{cases}$$

Hence for $i = 0, 1, \ldots 2k$, we have the following permutation,

$\tau(2i) = i(3k+2)(mod\ n)$

$\tau(2i+1) = i(3k+2) + 2k + 1(mod\ n)$

Now we add chords in the cycle C_n such that the diameter of the cycle remains unchanged.

Label the vertices of this newly obtained graph by using the above permutation.

So, that span f' for cycle with chord is

$$f_0' + f_1' + \cdots + f_{n-2}' = f_0 + f_1 + f_2 + \cdots + f_{n-2} + \sum(d_i - d_i')$$

$= rn(C_n) + \sum(d_i - d_i')$

$= 2k(k+2) + 1 + \sum(d_i - d_i'), 0 \le i \le n-2$

which is an upper bound for the radio number for cycle with arbitrary number of chords when $n = 4k + 2$.

Case 3: $n = 4k + 1$.

In this case $diam(G) = 2k$.

Using the sequences given below we can generate radio labeling of cycle C_n for $n \equiv 1(mod\ 4)$ with minimum span.

The distance gap sequence is given by

$d_{4i} = d_{4i+2} = 2k - i$

$d_{4i+1} = d_{4i+3} = k + 1 + i$

and the color gap sequence is given by $f_i = 2k - d_i + 1$

Then we have,

$\tau\,(2i) = i(3k + 1)\ (mod\ n), 0 \le i \le 2k$

$\tau\,(4i + 1) = 2(i + 1)k\ (mod\ n), 0 \le i \le k - 1$

$\tau\,(4i + 3) = 2(i + 1)k\ (mod\ n), 0 \le i \le k - 1$

Label the vertices of this newly obtained graph by using the above permutation.

So, that span of f' for cycle with chords is

$$f_0' + f_1' + \cdots + f_{n-2}' = f_0 + f_1 + f_2 + \cdots + f_{n-2} + \sum (d_i - d_i{}')$$

$$= rn(C_n) + \sum(d_i - d_i{}')$$

$$= 2k(k + 2) + \sum(d_i - d_i{}')\,, 0 \le i \le n - 2$$

which is an upper bound for the radio number for cycle with arbitrary number of chords when $n = 4k + 1$.

Case 4: $n = 4k + 3$.

In this case $diam(G) = 2k + 1$.

Using the sequences below we can give radio labelling of the cycle C_n for

$n \equiv 3(mod4)$ with minimum span.

The distance gap sequence is given by

$d_{4i} = d_{4i+2} = 2k + 1 - i$

$d_{4i+1} = k + 1 + i$

$d_{4i+3} = k + 2 + i$

And the color gap sequence is given by

$$f_i = \begin{cases} 2k - d_i + 2, & if\ i \not\equiv 3(mod\ 4) \\ 2k - d_i + 3, & otherwise \end{cases}$$

Then we have the following permutation,

$\tau(4i) = 2i(k + 1)(mod\ n)$, $0 \le i \le k$

$\tau(4i + 1) = (i + 1)(2k + 1)(mod\ n)$, $0 \le i \le k$

$\tau(4i + 2) = (2i - 1)(k + 1)(mod\ n)$, $0 \le i \le k$

$\tau(4i + 3) = i(2k + 1) + k(mod\ n)$, $0 \le i \le k - 1$, $1 \le i \le n - 2$

Label the vertices of this newly obtained graph by using the above permutation. So, that span of f' for cycle with chords is

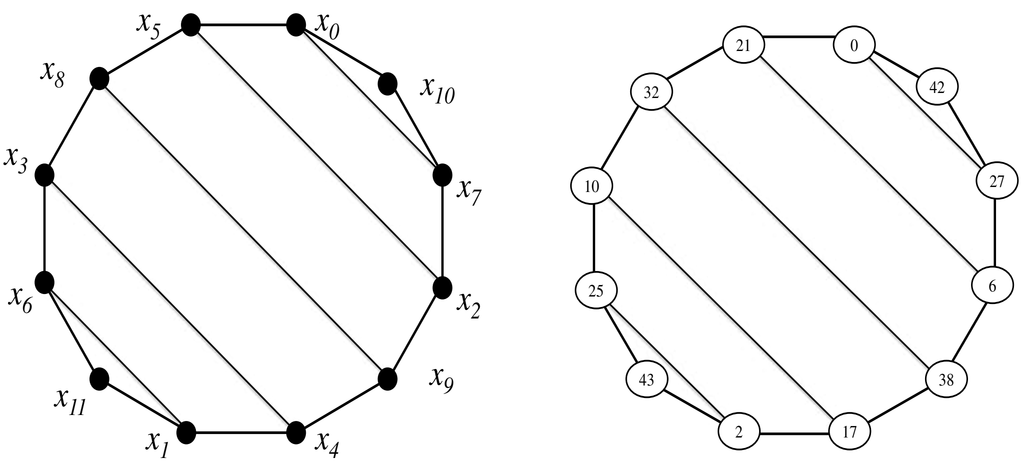
$$\begin{aligned} f_0' + f_1' + \cdots + f_{n-2}' &= f_0 + f_1 + f_2 + \cdots + f_{n-2} + \sum (d_i - d_i') \\ &= rn(C_n) + \sum (d_i - d_i') \\ &= (k + 2)(2k + 1) + \sum (d_i - d_i') \end{aligned}$$

which is an upper bound for the radio number for cycle with arbitrary number of chords for the cycle with chords.

Thus, in all possibilities we have the upper boundsof the radio numbers.

Illustartion: 5.6

Consider the graph $C_{12}$with 5 chords. The radio labeling is shown in Figure 5.1



Ordinary labelling for cycle with chords Radio labelling for cycle with chords for $\boldsymbol{C_{12}}$

Figure: 5.1 Ordinary and Radio labeling for cycle with chords for $\boldsymbol{C_{12}}$

Theorem: 5.7

Let G be an $n/2$-petal graph constructed from an even cycle C_n. Then

$$rn(G) \leq \begin{cases} \dfrac{3p}{2} + n\left\lfloor\dfrac{n}{4}\right\rfloor - \left\lfloor\dfrac{n}{8}\right\rfloor + 2n - 2 \\ (p-1) + n\left\lfloor\dfrac{n}{4}\right\rfloor - \left\lfloor\dfrac{n}{8}\right\rfloor + 2n \end{cases}$$

Proof.

Let G be an $n/2$- petal graph with vertices $v_0, v_1, \dots, v_{n-1}$ of degree and $v_1', v_2', \dots, v_p{}'$ of degree 2. Here v_i is adjacent to v_{i+1} and v_{n-1} is adjacent to v_0.

Case 1: $n \equiv 0(mod\ 4)$ and $diam(G) = \left\lfloor\frac{n}{4}\right\rfloor + 2$.

First we label the vertices of degree 2.

Let $v_1', v_2', \dots, v_p'$ be the vertices on the petals satisfying the order defined by the following distance sequence.

$$d_i' = \begin{cases} \left\lfloor\dfrac{n}{4}\right\rfloor + 2, & if\ i\ is\ even \\ \left\lfloor\dfrac{n}{4}\right\rfloor + 1, & if\ i\ is\ odd \end{cases}$$

The color gap sequence for vertices on the petal is defined as

$$f_i' = \begin{cases} 1, & if\ i\ is\ even \\ 2, & if\ i\ is\ odd \end{cases}$$

Let v_1 be the vertex on the cycle C_n such that $d(v_p', v_1) = \left\lfloor \frac{n}{8} \right\rfloor + 1 = d(v_{p-1}', v_1)$

Label v_1 as $f(v_1) = f(v_p') = diam(G) - \left\lfloor \frac{n}{8} \right\rfloor$.

For the remaining vertices of degree 3, we use the permutation defined for the cycle $C_n$in case 1 of Theorem 4.5.

The color gap sequence for the same vertices is defined as

$f_i = \left\lfloor \frac{n}{4} \right\rfloor + 2,\ 0 \le i \le n-2.$

The span of $f = \frac{3p}{2} + n\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor + 2n - 2$ which is an upper bound for the radio number of the $\frac{n}{2}$- petal graph when $n \equiv 0(mod\ 4)$.

Case 2: $n \equiv 2(mod\ 4)$ and $diam(G) = \left\lfloor \frac{n}{4} \right\rfloor + 2$.

First we label the vertices of degree 2. Let $v_1', v_2', \ldots, v_p'$ be the vertices on the petals satisfying the order defined by the following distance sequence.

$d(v_i', v_{i+1}') = \left\lfloor \frac{n}{4} \right\rfloor + 2.$

The color gap sequence for the same vertices is defined as

$f_i' = 1, 1 \le i \le p.$

Let v_1 be the vertex on the cycle C_n such that $d(v_p', v_1) = \left\lfloor \frac{n}{8} \right\rfloor + 1 = d(v_{p-1}', v_1)$.

Label v_1 as $f(v_1) = f(v_p') + diam(G) - \left\lfloor \frac{n}{8} \right\rfloor$.

For the remaining vertices of degree 3, we use the permutation defined for the cycle

C_n in case 2 of Theorem 5.5

The color gap sequence for the same vertices is defined as

$f_i = \left\lfloor \frac{n}{4} \right\rfloor + 2, 0 \leq i \leq n-2.$

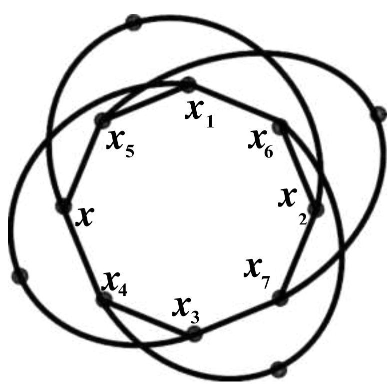
The span of $f = p - 1 + n\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor + 2n$

Which is an upper bound for the radio number of the $\frac{n}{2}$- petal graph when $n \equiv 2(mod\ 4)$

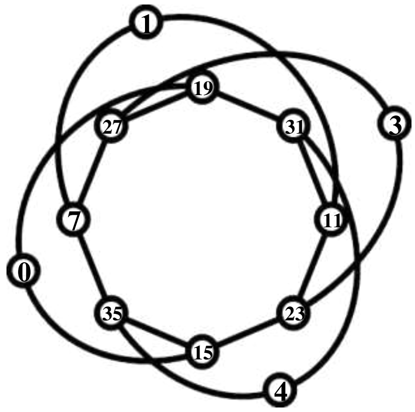
Illustration: 5.8

Consider the $n/2$- petal graph of C_8.

The radio labelling is shown in the Figure 5.2



Ordianry labelling for $n/2$- petal graph of C_8



Radio labelling for $n/2$- petal graph of C_8

Figure: 5.2 Ordinary and radio labelling for $n/2$- petal graph of C_8.

Theorem: 5.9

For any cycle C_n,

$$rn(spl(C_n)) = \begin{cases} 2[(k+2)(2k-1)+1]+k+1, & n \equiv 0(mod\ 4) \\ 2[2k(k+2)+1]+k+1, & n \equiv 2(mod\ 4) \\ 2[2k(k+1)]+k, & n \equiv 1(mod\ 4) \\ 2[(k+2)(2k+1)]+k+1, & n \equiv 3(mod\ 4) \end{cases}$$

Proof.

Let G be the split graph of C_n Let $v_1', v_2', \dots, v_n'$ be the duplicated vertices corresponding to $v_1, v_2, \dots, v_n$.

We initiate the labelling by assingning the labels to vertices of cycle and then to their duplicated vertices because $d(v_i, v_j) = d(v_i, v_j')$.

Inorder to obtain the labelling with minimum span we employ twice the distance gap sequence, the color gap sequence and the permutation scheme used by Liu and Zhu. This labeling procedure will generate exact radio number as optimality of the permutation is established .

Case 1: $n \equiv 4k (n > 4)$.

Then, $diam(G) = 2k$.

We first label the vertices $v_1, v_2, \dots, v_n$ as in Case 1 of Theorem 5.5 and then we label the vertices $v_1', v_2', \dots, v_n'$ as follows:

Define $f(v_j') = f(v_k) + k + 1$ where v_k is the last labelled vertex in the cycle and v_j' is the vertex such that $d(v_k, v_j') = k$.

Now using the permutation used in Case 1 of Theorem 4.5 for cycle C_n with $n \equiv 0(mod\ 4)$, label the duplicated vertices starting from v_j'.

Then $rn(spl(C_n)) = f_0 + f_1 + f_2 + \cdots + f_{n-2} + 2k + 1 + f_0' + f_1' + \cdots f_{n-2}'$

As $f_i = f_i'$ for $i = 0,1,2, \dots, n-2$, we have

$$rn(spl(C_n)) = 2f_0 + 2f_1 + 2f_2 + \cdots 2f_{n-2} + k + 1$$

$$= 2[(k + 2)(2k - 1) + 1] + k + 1.$$

Case 2: $n \equiv 4k + 2 (n > 6)$.

Then $diam(G) = 2k + 1$.

We first label the vertices $v_1, v_2, \dots, v_n$ as in Case 2 of Theorem 4.5 and then we label the vertices $v_1{}', v_2{}', \dots, v_n{}'$ as follows:

Define $f(v_j') = f(v_k) + k + 1$ where v_k is the last labelled vertex in the cycle and v_j' is the vertex such that $d(v_k, v_j') = k + 1$

Using the permutation for cycle C_n with $n \equiv 2(mod\, 4)$ in Case 2 of Theorem 4.5, label the duplicated vertices starting from v_j'.

Then

$$rn(spl(C_n)) = f_0 + f_1 + f_2 + \cdots + f_{n-2} + 2k + 1 - k - 1 + 1 + f_0' + f_1' + \cdots f_{n-2}{}'$$

As $f_i = f_i{}'$ for $i = 0,1,2, \dots, n - 2$, we have

$$rn(spl(C_n)) = 2f_0 + 2f_1 + 2f_2 + \cdots 2f_{n-2} + k + 1$$

$$= 2[2k(k + 2) + 1] + k + 1.$$

Case 3: $n \equiv 4k + 1 (n > 5)$.

Then $diam(G) = 2k$.

We first label the vertices $v_1, v_2, \dots, v_n$ as in Case 3 of Theorem 5.5 and then we label the vertices $v_1{}', v_2{}', \dots, v_n{}'$ as follows:

Define $f(v_j') = f(v_k) + k$ where v_k is the last labelled vertex in the cycle and v_j' is the vertex such that $d(v_k, v_j') = k + 1$

Using the permutation for cycle C_n with $n \equiv 1(mod\, 4)$ in Case 3 of Theorem 5.5, label the duplicated vertices starting from v_j'.Then

$$rn(spl(C_n)) = f_0 + f_1 + f_2 + \cdots + f_{n-2} + 2k - k - 1 + 1 + f_0' + f_1' + \cdots f_{n-2}'$$

As $f_i = f_i{}'$ for $i = 0,1,2, \dots, n-2$, we have

$$rn(spl(C_n)) = 2f_0 + 2f_1 + 2f_2 + \cdots 2f_{n-2} + k$$

$$= 2[2k(k+1)] + k.$$

Case 4: $n \equiv 4k + 3 (n > 3)$.

Then $diam(G) = 2k + 1$.

We first label the vertices $v_1, v_2, \dots, v_n$ as in Case 4 of Theorem 5.5 and then we label the vertices $v_1{}', v_2{}', \dots, v_n{}'$ as follows:

Let v_j' be the vertex such that $d(v_k, v_j') = k + 1$

Define $f(v_j') = f(v_{n-1}) + k + 1$ where v_k is the last labelled vertex in the cycle and v_j' is the vertex such that $d(v_k, v_j') = k + 1$.

Using the permutation for cycle C_n with $n \equiv 3(mod\, 4)$ in Case 4 of Theorem 4.5, label the duplicated vertices starting from v_j'.Then

$$rn(spl(C_n)) = f_0 + f_1 + f_2 + \cdots + f_{n-2} + 2k + 1 - k - 1 + 1 + f_0' + f_1' + \cdots f_{n-2}'$$

As $f_i = f_i{}'$ for $i = 0,1,2, \dots, n-2$, we have

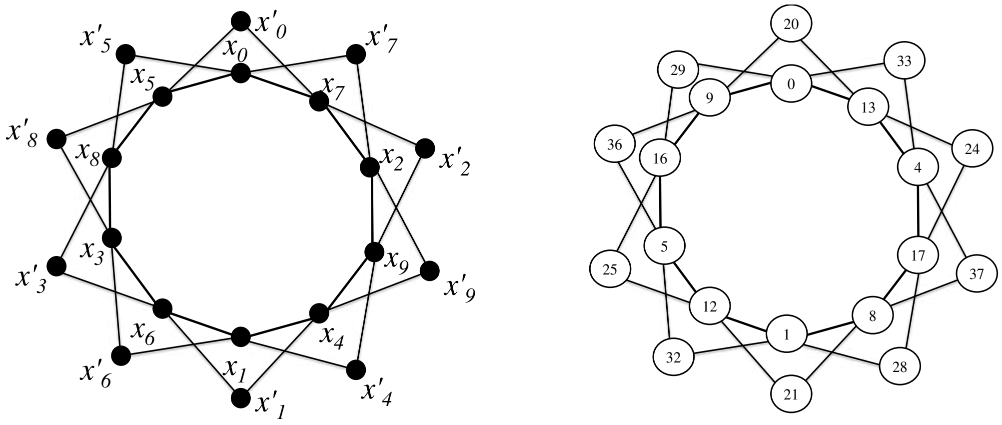
$$rn(spl(C_n)) = 2f_0 + 2f_1 + 2f_2 + \cdots 2f_{n-2} + k + 1$$

$$= 2[(k+1)(2k+1)] + k + 1.$$

Thus in all the four cases we have determined the radio number of G.

Illustartion: 5.10

Consider the graph $spl(C_{10})$. The radio labeling is shown in figure 5.3



Ordinary labeling for $spl(C_{10})$ Radio labeling for $spl(C_{10})$

Figure 5.3: Ordinary and Radio labeling for $spl(C_{10})$

Application: 5.11

Above result can be applied for the purpose of expansion of existing circular network of radio transmitters. By applying the concept of duplication of vertex the number of radio transmitters are doubled and separation of the channels assigned to the stations is such that interference can be avoided. Thus our result can play a vital role for the expansion of radio transmitter network without disturbing the existing one. In the expanded network the distance between any two transmitters is large enough to avoid the interference.

Theorem: 5.12

For any cycle C_n,

$$rn(M(C_n)) = \begin{cases} 2(k+2)(2k-1)+n+3 & n \equiv 0(mod\ 4) \\ 4k(k+2)+k+n+3, & n \equiv 2(mod\ 4) \\ 4k((k+1))+k+n, & n \equiv 1(mod\ 4) \\ 2(k+2)(2k+1)+k+n+1, & n \equiv 3(mod\ 4) \end{cases}$$

Proof.

Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle $C_n$and $u'_1, u'_2, \ldots, u'_n$ be the newly insertedvertices corresponding to the edges of C_n to obtain $M(C_n)$. In

$M(C_n)$ the diameter is increased by 1. Here $d(u_i, u_j) \geq d(u_i, u'_j)$ for $n \equiv 0,2(mod\ 4)$ and $d(u_i, u_j) = d(u_i, u'_j)$ for $n \equiv 1,3(mod\ 4)$.

Throughout the discussion, first we label the vertices $u_1, u_2, \dots, u_n$ and then the newly inserted vertices $u'_1, u'_2, \dots, u'_n$.

For this purpose we will employ twice the permutation scheme for respective cycles as in Theorem 5.5.

Case 1: $n \equiv 4k$.

In this case $diam(M(C_n)) = 2k + 1$.

Since $f_i + f_{i+1} \leq f_i{}' + f_{i+1}{}'$ for all i, the diatance gap sequence to order the vertices of the original cycle C_n is defined as follows:

$$d_i = \begin{cases} 2k+1, & if\ i\ is\ even \\ k+1, & if\ i \equiv 1(mod\ 4) \\ k+2, & if\ i \equiv 3(mod\ 4) \end{cases}$$

The color gap sequence is defined as follows:

$$f_i = \begin{cases} 1, & if\ i\ is\ even \\ k+1, & if\ i\ is\ odd \end{cases}$$

Let u'_1 be the vertex on the inscribed cycle such that $d(u_k, u'_1) = k + 1$ and $f = k + 1$, where u_k is the last labelled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

$$d_i = \begin{cases} 2k, & if\ i\ is\ even \\ k, & if\ i \equiv 1(mod\ 4) \\ k+1, & if\ i \equiv 3(mod\ 4) \end{cases}$$

The color gap sequence is defined as follows:

$$f_i' = \begin{cases} 2, & if\ i\ is\ even \\ k+2, & if\ i \equiv 1(mod\ 4) \\ k+1, & if\ i \equiv 3(mod\ 4) \end{cases}$$

Then $rn\big(M(C_n)\big) = 2(k+2)(2k-1) + n + 3$.

Case 2: $n \equiv 4k+2$.

In this case $diam\big(M(C_n)\big) = 2k+2$.

Since $f_i + f_{i+1} \leq f_i' + f_{i+1}'$ for all i, the diatance gap sequence to order the vertices of the original cycle C_n is defined as follows:

$$d_i = \begin{cases} 2k+2, & if\ i\ is\ even \\ k+3, & if\ i\ is\ odd \end{cases}$$

The color gap sequence is defined as follows:

$$f_i = \begin{cases} 1, & if\ i\ is\ even \\ k+1, & if\ i\ is\ odd \end{cases}$$

Let u_1' be the vertex on the inscribed cycle such that $d(u_k, u_1') = k+1$ and $f = k+2$, where u_k is the last labelled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

$$d_i = \begin{cases} 2k+1, & if\ i\ is\ even \\ k+1, & if\ i\ is\ odd \end{cases}$$

The color gap sequence is defined as follows:

$$f_i' = \begin{cases} 2, & if\ i\ is\ even \\ k+2, & if\ i\ is\ odd \end{cases}$$

Then $rn\big(M(C_n)\big) = 4k(k+2) + k + n + 3$.

Case 3: $n \equiv 4k+1$.

In this case $diam\big(M(C_n)\big) = 2k+1$.

Since $f_i + f_{i+1} \leq f_i' + f_{i+1}'$ for all i, the diatance gap sequence to order the vertices of the original cycle C_n is defined as follows:

$d_{4i} = d_{4i+2} = 2k + 1 - i$

$d_{4i+1} = d_{4i+3} = k + 2 + i$

And the color gap sequence is given by

$f_i = (2k + 1) - d_i + 1$

Let u_1' be the vertex on the inscribed cycle such that $d(u_k, u_1') = k + 1$ and $f = k + 1$, where u_k is the last labelled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

$d_{4i} = d_{4i+2} = 2k - i$

$d_{4i+1} = d_{4i+3} = k + 1 + i$

And the color gap sequence is given by

$f_i' = 2k - d_i + 2$

Then $rn(M(C_n)) = 4k(k + 1) + k + n$.

Case 4: $n \equiv 4k + 3$.

In this case $diam(M(C_n)) = 2k + 2$.

Since $f_i + f_{i+1} \leq f_i' + f_{i+1}'$ for all i, the diatance gap sequence to order the vertices of the original cycle C_n is defined as follows:

$d_{4i} = d_{4i+2} = 2k + 2 - i$

$d_{4i+1} = d_{4i+3} = k + 2 + i$

And the color gap sequence is given by

$f_i = 2k - d_i + 3$

Let u_1' be the vertex on the inscribed cycle such that $d(u_k, u_1') = k + 1$ and $f = k + 1$, where u_k is the last labelled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

$d_{4i} = d_{4i+2} = 2k + 1 - i$

$d_{4i+1} = d_{4i+3} = k + 1 + i$

$d_{4i+3} = k + 2 + i$

and the color gap sequence is given by

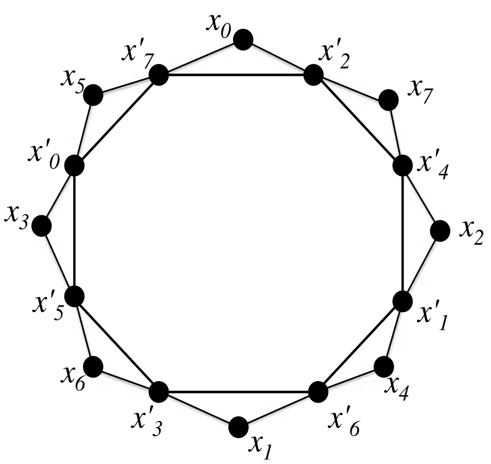
$f_i' = 2k - d_i + 3$

Then $rn\big(M(C_n)\big) = 2(k + 2)(2k + 1) + n$.

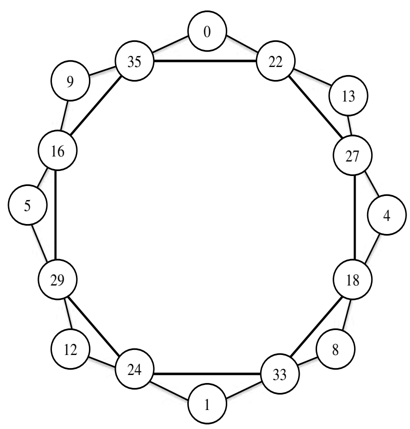
Thus the radio number is completely determined for the graph $M(C_n)$.

Illustration: 5.13

Consider the graph $M(C_8)$. The radio labeling is shown in Figure 5.4



Ordinary labeling for $M(C_8)$



Radio labeling for $M(C_8)$

Figure: 5.4 Ordinary and radio labeling of $M(C_8)$.

Application: 5.14

Above result is useful for the expansion of an existing radio transmitters network. In the expanded network two newly installed nearby transmitters are connected and interference is also avoided between them. Thus the radio labeling described in Theorem 5.13 is rigorously applicable to accomplish the task of channel assignment for the feasible network.

The comparison between radio number of $C_n, spl(C_n), M(C_n)$ is tabulated in the following Table 1.

$\boldsymbol{n}$	**Radio number of $\boldsymbol{C_n}$**	**Radio number of $\boldsymbol{spl(C_n)}$**	**Radio number of $\boldsymbol{M(C_n)}$**
$0(mod\ 4)$	$(k+2)(2k-1)+1$	$2[(k+2)(2k-1)+1]+k+1$	$2(k+2)(2k-1)\ +n+3$
$2(mod\ 4)$	$2k(k+2)+1$	$2[2k(k+2)+1]+k+1$	$4k(k+2)+k+n+3$
$1(mod\ 4)$	$2k(k+1)$	$2[2k(k+1)]+k$	$4k(k+1)+k+n$
$3(mod\ 4)$	$(k+2)(2k+1)$	$2[(k+2)(2k+1)]+k+1$	$2(k+2)(2k+1)+n$

Table: 1 Comparisonof radio numbers of $\boldsymbol{C_n, spl(C_n), M(C_n)}$.

A STUDY ON CONCEPT OF INTUITIONISTIC FUZZY

IN Γ – NEAR RING

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

K. KARBAGASELVI

Reg. No: 19SPMT13

Under the guidance of

Dr. P. ANBARASI RODRIGO M.Sc., B.Ed., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON CONCEPT OF INTUITIONISTIC FUZZY IN Γ – NEAR RINGS "** submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **K. KARBAGASELVI (Reg. No: 19SPMT13)**

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON CONCEPT OF INTUITIONISTIC FUZZY IN Γ - NEAR RING"** submitted for the degree of Master of Science is my work carried out under the guidance of **Dr. P. Anbarasi Rodrigo M.Sc., B.Ed., Ph.D.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

K. Karbagavalli

Station: Thoothukudi **Signature of the Student**

Date: 10.06.21

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Place: Thoothukudi

Date: 10.04.21

CONTENT

CHAPTER I

Preliminaries

Definition 1.1

A non-empty set R together with two binary operations denoted by " + " and "$\cdot$" called addition and multiplication which satisfy the following axioms is called a ring.

(i) (R, +) is an abelian group

(ii) "$\cdot$" is an associative binary operation on R

(iii) $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(a+b)\cdot c = a\cdot c + b\cdot c$, for all $a,b,c \in R$

Definition 1.2

A non-empty set M with two binary operations " + " and "$\cdot$" is called a near-ring if it satisfies the following conditions:

(i) (M,+) is a group (not necessarily abelian)

(ii) (M,$\cdot$) is a semi group

(iii) $(x+y)z = xz + yz$, for all $x,y,z \in M$.

Precisely speaking, it is a right near-ring and we will use the word near-ring to mean right near-ring. We denote xy instead of $x\cdot y$.

Definition 1.3

A subgroup B of (N,+) is said to be a bi-ideal of N if $BNB \cap (BN) * B \subseteq B$. In the case of zero symmetric near-ring subgroup B of (N,+) is a bi-ideal $BNB \subseteq B$.

Definition 1.4

A Γ- near-ring is a triple $(M,+,\Gamma)$ if

(i) $(M,+)$ is a group

(ii) Γ is a non-empty set of binary operators on M such that, for each $\alpha \in \Gamma$, $(M,+,\alpha)$ is a near-ring

(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$.

Definition 1.5

A Γ-near-ring M is said to be zero-symmetric if $m\gamma 0 = 0$ for all $m \in M$ and for all $\gamma \in \Gamma$. We assume that M is a zero-symmetric Γ-near-ring.

Definition 1.6

Let G be a set with binary operation " * " defined on it. Let $S \subseteq G$. If for each $a,b \in S$, $a * b$ (computed in G) is in S, we say that S is closed with respect to the binary operation " * ".

Definition 1.7

A subset A of a Γ-near-ring M is called a left (resp. right) ideal of M if

(i) $(A,+)$ is a normal divisor of $(M,+)$

(ii) $u\alpha(x+v) - u\alpha v \in A$ (resp. $x\alpha u \in A$), for all $x \in A$, $\alpha \in \Gamma$ and $u,v \in M$.

Definition 1.8

Let M be a Γ-near-ring. A subgroup A of M is called a bi-ideal of M if $(A\Gamma M\Gamma A) \cap (A\Gamma M)\Gamma * A \subseteq A$. where the operation " * " is defined by,

$A\Gamma * B = \{a\gamma(a'+b) - a\gamma a') \ / \ a, a' \in A, \gamma \in \Gamma, b \in B\}$.

Definition 1.9

Let M be a Γ-near-ring. A subgroup Q of M is called a quasi-ideal of M if $(Q\Gamma M) \cap (M\Gamma Q) \cap (M\Gamma) * Q \subseteq Q$.

Definition 1.10

A subgroup B of $(M,+)$ is bi- ideal if and only if $B\Gamma M\Gamma B \subseteq B$. A bi- ideal B of M is said to be k- bi- ideal if $y \in B$ and if $x + y \in B$ (or) $y + x \in B$ then $x \in B$.

Definition 1.11

Let M and N be Γ- near- rings. A mapping $f : M \to N$ is said to be a homomorphism if $f(a\,\alpha\, b) = f(a)\,\alpha\, f(b)$, for all $a,b \in M$ and $\alpha \in \Gamma$.

Definition 1.12

Let M and N be any two Γ- near- rings. Then the mapping $f : M \to N$ is said to be a Γ- near- ring homomorphism if $f(x+y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$, for all $x,y \in M$ and $\gamma \in \Gamma$.

Definition 1.13

A fuzzy set in a set M is a function $\mu : M \to [0,1]$.

Definition 1.14

The complement of a fuzzy set μ, denoted by $\mu' = 1 - \mu(x)$, for all $x \in M$.

Definition 1.15

A fuzzy set μ in a Γ- near- ring M is called a fuzzy left (resp. right) ideal of M if,

(i) $\mu(x- y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(y + x - y) \geq \mu(x)$, for all $x,y \in M$

(iii) $\mu(u\alpha(x+v) - u\alpha v) \geq \mu(x)$, (resp. $\mu(x\alpha u) \geq \mu(x)$), for all $x,u,v \in M$ and $\alpha \in \Gamma$.

Definition 1.16

A fuzzy set μ in M is called a fuzzy bi- ideal of M if

(i) $\mu(x - y) \geq \min\{\mu(x),\mu(y)\}$, for all $x,y \in M$

(ii) $\mu(x\alpha y\beta z) \geq \min\{\mu(x),\mu(z)\}$, for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$.

Definition 1.17

A fuzzy bi- ideal μ in M is called a fuzzy k- bi- ideal of M if, for all $x,y \in M$,

$\mu(x) \geq \min\{ \max\{\mu(x+y),\mu(y+x)\},\mu(y)\}$.

Definition 1.18

Let x be a non- empty fixed set. An intuitionistic fuzzy set IFS A in X is an object having the form $A = \{x,\mu_A(x),\nu_A(x) >/ \ x \in X\}$, where the functions $\mu_A : X\rightarrow[0,1]$ and $\nu_A : X\rightarrow[0,1]$ degree of membership and degree of non- membership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 1.19

An intuitionistic fuzzy set $A = \{(x;\mu_A(x);\gamma_A(x))/x \in X\}$ in X can be identified to an ordered pair (μ_A,γ_A) in $I^X \times I^X$.

Notation 1.20

For the sake of simplicity, we shall use the symbol $A =< \mu_A, \nu_A >$ for the IFS, $A = \{ < x,\mu_A(x),\nu_A(x) >/ \ x \in X\}$.

Definition 1.21

Let X be a nonempty set and let $A = <\mu_A, \nu_A>$ and $B = <\mu_B, \nu_B>$ be IFSs in X. Then:

(i) $A \subset B$ if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$

(ii) $A = B$ if $A \subset B$ and $B \subset A$

(iii) $A^c = <\nu_A, \mu_A>$

(iv) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$

(v) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$

Definition 1.22

Let A be an IFS in a Γ-near-ring M. For each pair $< \tau, \sigma > \in [0,1]$ with $t + s \leq 1$, the set $A_{<t,s>} = \{x \in X / \mu_A(x) \geq t$ and $\nu_A(x) \leq s\}$ is called a $< t, s >$ level subset of A.

Definition 1.23

Let $A = <\mu_A, \nu_A>$ be an IFS in M and let $t \in [0,1]$. Then the set $U(\mu_A; t) = \{x \in M: \mu_A(x) \geq t\}$ and $L(\nu_A; t) = \{x \in M: \nu_A(x) \leq t\}$ are called upper level set and lower level set of A, respectively.

Definition 1.24

A system $(R, +, \cdot)$ is a Boolean semi-ring if and only if the following properties hold

(i) $(R,+)$ is an additive (abelian) group (whose 'zero' will be denoted by '0')

(ii) $(R,\cdot)$ is a semi group of idempotents in the sense $aa = a$, for all $a \in R$

(iii) $a(b+c) = ab + ac$ and

(iv) $abc = bac$, for all $a,b,c \in R$.

Example 1.25

Let $(G,+)$ be any abelian group defined by $ab = b$ for all $a,b \in G$. Then $(G,+)$ is a boolean semi-ring.

Definition 1.26

A non-empty set R together with two binary operations " + " and " $\cdot$ " satisfying the following conditions is called Boolean like semi-ring.

(i) $(R,+)$ is an abelian group

(ii) $(R,\cdot)$ is a semi group

(iii) $a\cdot(b+c) = a\cdot b + a\cdot c$, for all $a,b,c \in R$

(iv) $a + a = 0$, for all a in R

(v) $ab(ab+ab) = ab$, for all $a,b \in R$.

Definition 1.27

Let μ be a fuzzy set defined on R then μ is said to be a fuzzy ideal of R if,

(i) $\mu(x - y) \geq \min\{\mu(x),\mu(y)\}$, for all $x,y \in R$

(ii) $\mu(ra) \geq \mu(a)$, for all $r,a \in R$

(iii) $\mu((r+a)s + ra) \geq \mu(a)$, for all $r,a,s \in R$

Definition 1.28

Let μ be a fuzzy set defined on R. Then μ is said to be a fuzzy bi-ideal of R if,

(i) $\mu(x - y) \geq \min\{\mu(x),\mu(y)\}$, for all $x,y \in R$

(ii) $\mu(xyz) \geq \min\{\mu(x),\mu(z)\}$, for all $x,y,z \in R$

Definition 1.29

By μ_t we denote a level subset of μ, for $\{x \in M / \mu(x) \geq t\}$ where $t \in [0,1]$.

Definition 1.30

If μ is a fuzzy set M and f is a function defined on M, then the fuzzy set ϑ in f(M) defined by, $\vartheta(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(M)$ is called the image of ϑ under f.

Definition 1.31

If ϑ is a fuzzy set f(M), then the fuzzy set $\mu = \vartheta \circ f$ in M. i.e., the fuzzy set defined by $\mu(x) = \vartheta(f(x))$ for all x in M is called the pre image of ϑ under f.

Definition 1.32

A fuzzy set μ in M is said to have the sup property if for any subset T of M there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$.

Definition 1.33

Let A and B be sets such that $A \subseteq B$. Define $\chi_A : B \to [0,1]$ by $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$. Then χ_A is called as characteristic function of A.

Definition 1.34

μ is said to be fuzzy normal divisor with respect to the addition it μ satisfies

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(y + x - y) \geq \mu(x)$, for all $x, y \in M$.

CHAPTER II

Intuitionistic Fuzzy Ideals

Definition 2.1

An IFS A= < μ_A; ν_A > in M is called an intuitionistic fuzzy left (resp. right) ideal of a Γ – near- ring M if

(i) $\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$

(ii) $\mu_A(y+x- y) \geq \mu_A(x)$

(iii) $\mu_A(u\alpha(x+v)- u\alpha v) \geq \mu_A(x)$ (resp. $\mu_A(x\alpha u) \geq \mu_A(x)$)

(iv) $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\}$

(v) $\nu_A(y+x- y) \leq \nu_A(x)$

(vi) $\nu_A(u\alpha(x+v)- u\alpha v) \leq \nu_A(x)$ (resp. $\nu_A(x\alpha u) \leq \nu_A(x)$), for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$

Example 2.2

Let R be the set of all integers then R is a ring.

Take $M = \Gamma = R$. Let $a, b \in M$, $\alpha \in \Gamma$, suppose $a\alpha b$ is the product of $a, \alpha, b \in R$. Then M is a Γ- near- ring.

Define an IFS $A = < \mu_A; \nu_A >$ in R as follows.

$\mu_A(0) = 1$ and $\mu_A(\pm1) = \mu_A(\pm2) = \mu_A(\pm3) = \ldots = t$ and

$\nu_A(0) = 1$ and $\nu_A(\pm1) = \nu_A(\pm2) = \nu_A(\pm3) = \ldots = s$

where $t \in [0,1], s \in [0,1]$ and $t + s \leq 1$.

By routine calculations,

Clearly, A is an intuitionistic fuzzy bi- ideal of a Γ- near- ring R.

Theorem 2.3

A is an ideal of a Γ−near- ring M if and only if $\tilde{A} = < \mu_A , \nu_A >$ where

$\mu_A(x) = \begin{cases} 1 & x \in A, \\ 0 & \text{otherwise} \end{cases}$ $\quad \nu_A(x) = \begin{cases} 0 & x \in A, \\ 1 & \text{otherwise} \end{cases}$

Is an intuitionistic fuzzy left (resp. right) ideal of M.

Proof

($\Rightarrow$): Let A be a left (resp.right) ideal of M. Let x, y, u, v $\in$ M and $\alpha \in \Gamma$.

If x, y $\in$ A, then x - y $\in$ A, y + x - y $\in$ A and $(u\alpha(x + v) - u\alpha v) \in A$.

Therefore $\mu_A(x-y) = 1 \geq \{\mu_A(x) \wedge \mu_A(y)\}$, $\mu_A(y+x-y) = 1 \geq \mu_A(x)$ and $\mu_A(u\alpha(x+v)-u\alpha v) = 1 = \mu_A(x)$ (resp. $\mu_A(x\alpha u) = \mu_A(x) =1$), $\nu_A(x-y) = 0 \leq \{\nu_A(x) \vee \nu_A(y)\}$, $\nu_A(y+x-y) = 0 \leq \nu_A(x)$ and $\nu_A(u\alpha(x+v)-u\alpha v) = 0 = \nu_A(x)$ (resp. $\nu_A(x\alpha u) = \nu_A(x) = 0$).

If $x \notin A$ (or) $y \notin A$ then $\mu_A(x) = 0$ (or) $\mu_A(y) = 0$

$\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$, $\mu_A(y+x-y) \geq \mu_A(x)$ and $\mu_A(u\alpha(x+v)-u\alpha v) \geq \mu_A(x)$ (resp. $\mu_A(x\alpha u) \geq \mu_A(x)$), $\nu_A(x-y) \leq \{\nu_A(x) \vee \nu_A(y)\}$, $\nu_A(y+x-y) \leq \nu_A(x)$ and $\nu_A(u\alpha(x+v)-u\alpha v) \leq \nu_A(x)$ (resp. $\nu_A(x\alpha u) \leq \nu_A(x)$).

Hence $\tilde{A}$ is an intuitionistic fuzzy left (resp.right) ideal of M.

($\Leftarrow$) : Let $\tilde{A}$ be an intuitionistic fuzzy left (resp.right) ideal of M. Let x, y $\in$ M and $\alpha \in \Gamma$.

If x, y, u, v $\in$ A, then

$\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\} = 1$

$\nu_A(x-y) \leq \{\nu_A(x) \vee \nu_A(y)\} = 0$

So, x - y $\in$ A.

$\mu_A(y+x-y) \geq \mu_A(x) = 1$

$\nu_A(y+x-y) \leq \nu_A(x) = 0$

So, $(y + x - y) \in A$.

Also,

$\mu_A(u\alpha(x+v)-u\alpha v) \geq \mu_A(x) = 1$ (resp. $\mu_A(x\alpha u) = \mu_A(x) = 1$)

$\nu_A(u\alpha(x+v)-u\alpha v) \leq \nu_A(x) = 0$ (resp. $\nu_A(x\alpha u) = \nu_A(x) = 0$)

So, $(u\alpha(x+v)-u\alpha v) \in A$.

Hence A is a left (resp.right) ideal of M.

Theorem 2.4

Let A be an intuitionistic fuzzy left (resp.right) ideal of M and $t \in [0,1]$ then

(i) $U(\mu_A;t)$ is either empty or an ideal of M.

(ii) $L(\nu_A;t)$ is either empty or an ideal of M.

Proof

(i) Let $x,y \in U(\mu_A;t)$.

Then $\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\} \geq t$,

Hence $x-y \in L(\nu_A;t)$

$\mu_A(y+x-y) \geq \mu_A(x) \geq t$

Hence $(y+x-y) \in U(\mu_A;t)$

Let $x \in M$, $\alpha \in \Gamma$ and $u,v \in U(\mu_A;t)$.

Then $\mu_A(u\alpha(x+v)-u\alpha v) \geq \mu_A(x) \geq t$ and so $(u\alpha(x+v)-u\alpha v) \in U(\mu_A;t)$.

Hence $U(\mu_A;t)$ is an ideal of M.

(ii) Let $x, y \in L(\nu_A;t)$

Then $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\} \leq t$.

Hence $x - y \in L(\nu_A;t)$

$\nu_A(y + x - y) \geq \nu_A(x) \geq t$

Hence $(y + x - y) \in U(\mu_A;t)$

Let $x \in M$, $\alpha \in \Gamma$ and $u,v \in L(\nu_A;t)$.

Then $\nu_A(u\alpha(x+v) - u\alpha v) \geq \nu_A(x) \geq t$ and so $(u\alpha(x+v) - u\alpha v) \in L(\nu_A;t)$

Hence $L(\nu_A;t)$ is an ideal of M.

Theorem 2.5

Let I be the left (resp.right) ideal of M. If the intuitionistic fuzzy set $A = <\mu_A; \nu_A>$ in M is defined by

$$\mu_A(x) = \begin{cases} p & \text{if } x \in I, \\ s & \text{0therwise} \end{cases} \quad \text{and} \quad \nu_A = \begin{cases} u & \text{if } x \in I, \\ v & \text{otherwise} \end{cases}$$

for all $x \in M$ and $\alpha \in \Gamma$, where $0 \leq s < p$, $0 \leq v < u$ and $p + u \leq 1$, $s + v \leq 1$, then A is an intuitionistic fuzzy left (resp.right) ideal of M and $U(\mu_A; p) = I = L(\nu_A; u)$

Proof

Let $x, y \in M$ and $\alpha \in \Gamma$.

If at least one of x and y does not belong to I, then

$\mu_A(x - y) \geq s = \{\mu_A(x) \wedge \mu_A(y)\}$,

$\nu_A(x - y) \leq v = \{\nu_A(x) \vee \nu_A(y)\}$.

If $x, y \in I$, then

$x - y \in I$ and so $\mu_A(x - y) = p = \{\mu_A(x) \wedge \mu_A(y)\}$ and

$\nu_A(x - y) = u = \{\nu_A(x) \vee \nu_A(y)\}$.

$\mu_A(y+x-y) \geq s = \mu_A(x)$ ν_A
$(y+x-y) \leq v = \nu_A(x)$

If $x, y \in I$, then
$(y + x - y) \in I$ and so $\mu_A(y+x-y) = p = \mu_A(x)$ and $\nu_A(y+x-y) = u = \nu_A(x)$.

If $u, v \in I$, $x \in M$ and $\alpha \in \Gamma$ then $(u\alpha(x+v)-u\alpha v) \in I$,

$\mu_A(u\alpha(x+v)-u\alpha v) = p = \mu_A(x)$ and $\nu_A(u\alpha(x+v)-u\alpha v) = u = \nu_A(x)$. (resp. $\mu_A(x\alpha u) = p = \mu_A(x)$ and $\nu_A(x\alpha u) = u = \nu_A(x)$)

If $y \notin I$, then $\mu_A(u\alpha(x+v)-u\alpha v) = s = \mu_A(x)$, $\nu_A(u\alpha(x+v)-u\alpha v) = v = \nu_A(x)$. (resp. $\mu_A(x\alpha u) = s = \mu_A(x)$ and $\nu_A(x\alpha u) = v = \nu_A(x)$)

Therefore A is an intuitionistic fuzzy left (resp.right) ideal.

Definition 2.6

Let f be a mapping from a Γ−near-ring M onto a Γ−near-ring N. Let A be an intuitionistic fuzzy ideal of M. Now A is said to be f−invariant if $f(x) = f(y)$ implies $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.

Definition 2.7

A function $f: M \to N$, where M and N are Γ−near-rings, is said to be a Γ−homomorphism if $f(a + b) = f(a) + f(b)$, $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 2.8

Let $f: X \to Y$ be a mapping of a Γ−near-ring and A be an intuitionistic fuzzy set of Y. Then the map $f^{-1}(A)$ is the pre-image of A under f, if $\mu_{f^{-1}(A)}(x) = \mu_A(f(x))$ and $\nu_{f^{-1}(A)}(x) = \nu_A(f(x))$, for all $x \in X$.

Definition 2.9

Let f be a mapping from a set X to the set Y. If A= < μ_A; ν_A > and B = < μ_B; ν_B > are intuitionistic fuzzy subsets in X and Y respectively, then a) the image of A under f is the intuitionistic fuzzy set f(A) =< $\mu_{f(A)}$, $\nu_{f(A)}$ > defined by

$$\mu_{f^{-1}(A)}(x) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \varphi, \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{f^{-1}(A)}(x) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \varphi, \\ 0 & \text{otherwise,} \end{cases}$$

for all y ∈ Y.

b) the pre image of A under f is the intuitionistic fuzzy set $f^{-1}(B)$ =< $\mu_{f^{-1}(B)}$, $\nu_{f^{-1}(B)}$ > defined by

$$\mu_{f^{-1}(B)}(x) = \begin{cases} \bigvee_{y \in f^{-1}(x)} \mu_B(x) & \text{if } f^{-1}(y) \neq \varphi, \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{f^{-1}(B)}(x) = \begin{cases} \bigwedge_{y \in f^{-1}(x)} \nu_B(x) & \text{if } f^{-1}(y) \neq \varphi, \\ 0 & \text{otherwise,} \end{cases}$$

for all x ∈ X, where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$

Theorem 2.10

Let M and N be two Γ- near- rings and θ : M→N be a Γ- epimorphism and let B = < μ_B; ν_B > be an intuitionistic fuzzy set of N. If $\theta^{-1}(B)$ =< $\mu_{\theta^{-1}(B)}(x)$, $\nu_{\theta^{-1}(B)}(x)$ > is an intuitionistic fuzzy left (resp. right) ideal of M, then B is an intuitionistic fuzzy left (resp.right) ideal of N.

Proof

Let x, y, u, v ∈ N and ∈ Γ, then there exists a,b,c, d ∈ M such that $\theta(a) = x$, $\theta(b) = y, \theta(c) = u$, $\theta(d) = v$.

It follows that

$$\mu_B(x-y) = \mu_B(\theta(a)-\theta(b))$$

$$= \mu_B(\theta(a-b))$$

$$= \mu^{-1}_{\theta\ (B)}(a-b) \geq \{\mu^{-1}_{\theta\ (B)}(a) \wedge \mu^{-1}_{\theta\ (B)}(b)\}$$

$$= \{\mu_B(\theta(a)) \wedge \mu_B(\theta(b))\}$$

$$= \{\mu_B(x) \wedge \mu_B(y)\}$$

$$\nu_B(x-y) = \nu_B(\theta(a)-\theta(b))$$

$$= \nu_B(\theta(a-b))$$

$$= \nu^{-1}_{\theta\ (B)}(a-b) \leq \{\nu^{-1}_{\theta\ (B)}(a) \wedge \nu^{-1}_{\theta\ (B)}(b)\}$$

$$= \{\nu_B(\theta(a)) \vee \nu_B(\theta(b))\}$$

$$= \{\nu_B(x) \vee \nu_B(y)\}$$

$$\mu_B(y+x-y) = \mu_B(\theta(b)+\theta(a)-\theta(b))$$

$$= \mu_B(\theta(b+a-b))$$

$$= \mu^{-1}_{\theta\ (B)}(b+a-b) \geq \mu^{-1}_{\theta\ (B)}(a)$$

$$= \mu_B(\theta(a))$$

$$= \mu_B(x)$$

$$\nu_B(y+x-y) = \nu_B(\theta(b)+\theta(a)-\theta(b))$$

$$= \nu_B(\theta(b+a-b))$$

$$= \nu_{\theta^{-1}(B)}(b+a-b) \le \nu_{\theta^{-1}(B)}(a)$$

$$= \nu_B(\theta(a))$$

$$= \nu_B(x)$$

Also,

$$\mu_B(u\alpha(x+v)-u\alpha v) = \mu_B(\theta(c)\alpha(\theta(a)+\theta(d)) - \theta(c)\alpha\theta(d))$$

$$= \mu_B(\theta(c\alpha(a+d)-c\alpha d))$$

$$= \mu_{\theta^{-1}(B)}(\theta(c\alpha(a+d)-c\alpha d))$$

$$\ge \mu_{\theta^{-1}(B)}(a)$$

$$= \mu_B(\theta(a))$$

$$= \mu_B(x).$$

$$\nu_B(u\alpha(x+v)-u\alpha v) = \nu_B(\theta(c)\alpha(\theta(a)+\theta(d)) - \theta(c)\alpha\theta(d))$$

$$= \nu_B(\theta(c\alpha(a+d)-c\alpha d))$$

$$= \nu_{\theta^{-1}(B)}(\theta(c\alpha(a+d)-c\alpha d))$$

$$\ge \nu_{\theta^{-1}(B)}(a)$$

$$= \nu_B(\theta(a))$$

$$= \nu_B(x).$$

Similarly, $\mu_B(x\alpha u) \ge \mu_B(x)$ and $\nu_B(x\alpha u) \le \nu_B(x)$.

Hence B is an intuitionistic fuzzy left (resp.right) ideal of N.

Theorem 2.11

An intuitionistic fuzzy set $A = <\mu_A; \nu_A>$ in a Γ- near- ring M is an intuitionistic fuzzy left (resp.right) ideal if and only if $A_{<t,s>} = \{x \in M | \mu_A(x) \geq t, \nu_A(x) \leq s\}$ is a left (resp. right) ideal of M for $\mu_A(0) \geq t$, $\nu_A(0) \leq s$.

Proof

(⇒): Suppose that $A = <\mu_A; \nu_A>$ is an intuitionistic fuzzy left (resp.right) ideal of M and let $\mu_A(0) \geq t$, $\nu_A(0) \leq s$. Let $x, y, u, v \in A_{<t,s>}$ and $\alpha \in \Gamma$.

Then $\mu_A(x) \geq t$, $\nu_A(x) \leq s$ and $\mu_A(y) \geq t$, $\nu_A(y) \leq s$.

Hence $\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\} \geq t$,

$\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\} \leq s$.

$\mu_A(y+x-y) \geq \mu_A(x) \geq t$

$\nu_A(y + x - y) \leq \nu_A(x) \leq s$.

$\mu_A(u\alpha(x+v)-u\alpha v) \geq \mu_A(x) \geq t$ and $\nu_A(u\alpha(x+v)-u\alpha v) \leq \nu_A(x) \leq s$

(resp. $\mu_A(x\alpha u) \geq \mu_A(x) \geq t$ and $\nu_A(x\alpha u) \leq \nu_A(x) \leq s$).

Therefore $x - y \in A_{<t,s>}$, $(y + x - y) \in A_{<t,s>}$ and $(u\alpha(x+v)-u\alpha v) \in A_{<t,s>}$, for all x, $y \in A_{<t,s>}$ and $\alpha \in \Gamma$.

So $A_{<t,s>}$ is a left (resp. right) ideal of M.

:(⇐) Suppose that $A_{<t,s>}$ is an intuitionistic fuzzy left (resp.right) ideal of M for $\mu_A(0) \geq t$, $\nu_A(0) \leq s$.

Let $x, y \in M$ be such that $\mu_A(x) = t_1$, $\nu_A(x) = s_1$, $\mu_A(y) = t_2$, $\nu_A(y) = s_2$.

Then $x \in A_{<t_1, s_1>}$ and $y \in A_{<t_2, s_2>}$.

We may assume that $t_2 \leq t_1$ and $s_2 \geq s_1$ without loss of generality.

It follows that $A_{<t_2, s_2>} \subseteq A_{<t_1, s_1>}$ so that $x, y \in A_{<t_1, s_1>}$

Since $A_{<t_1, s_1>}$ is an ideal of M, we have $x - y \in A_{<t_1, s_1>}$, $(y + x - y) \in A_{<t_1, s_1>}$ and

$(u\alpha(x+v) - u\alpha v) \in A_{<t_1, s_1>}$, for all $\alpha \in \Gamma$.

$\mu_A(x-y) \geq t_1 \geq t_2 = \{\mu_A(x) \wedge \mu_A(y)\}$,

$\nu_A(x-y) \leq s_1 \leq s_2 = \{\nu_A(x) \vee \nu_A(y)\}$.

$\mu_A(y+x-y) \geq t_1 \geq t_2 = \mu_A(x)$,

$\nu_A(y+x-y) \leq s_1 \leq s_2 = \nu_A(x)$.

$\mu_A\,(u\alpha(x+v) - u\alpha v) \geq t_1 \geq t_2 = \mu_A(x)$ and

$\nu_A\,(u\alpha(x+v) - u\alpha v) \leq s_1 \leq s_2 = \nu_A(x)$.

Therefore A is an intuitoinistic fuzzy left (resp. right) ideal of M.

Theorem 2.12

If the IFS $A = < \mu_A; \nu_A >$ is an intuitionistic fuzzy left (resp. right) ideal of a Γ-near-ring M, the sets $M\mu_A = \{x \in M / \mu_A(x) = \mu_A(0)\}$ and $M\nu_A = \{x \in M / \nu_A(x) = \nu_A(0)\}$ are left (resp. right) ideals.

Proof

Let $x, y, u, v \in M\mu_A$ and $\alpha \in \Gamma$.

Then, $\mu_A(x) = \mu_A(0)$, $\mu_A(y) = \mu_A(0)$.

Since A is an intuitionistic fuzzy left (resp. right) ideal of a Γ- near-ring M, we get

$\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\} = \mu_A(0)$.

But $\mu_A(0) \geq \mu_A(x-y)$. So, $x - y \in M\mu_A$.

$\mu_A(y+x-y) \geq \mu_A(x) = \mu_A(0)$.

But $\mu_A(0) \geq \mu_A(y+x-y)$. So, $y + x - y \in M\mu_A$.

$\mu_A (u\alpha(x+v)- u\alpha v) \geq \mu_A(x) = \mu_A(0)$ (resp. $\mu_A(x\alpha u) \geq \mu_A(x) = \mu_A(0)$).

Hence $(u\alpha(x+v)- u\alpha v) \in M\mu_A$.

Therefore $M\mu_A$ is a left (resp. right) ideal of M.

Similarly, let $x, y, u, v \in M\mu_A$ and $\alpha \in \Gamma$. Then $\nu_A(x) = \nu_A(0)$, $\nu_A(y) = \nu_A(0)$.

Since A is an intuitionistic fuzzy left (resp. right) ideal of a Γ- near-ring M,

$\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\} = \nu_A(0)$.

But $\nu_A(0) \leq \nu_A(x-y)$. So, $x - y \in M\nu_A$.

$\nu_A(y+x-y) \leq \nu_A(x) = \nu_A(0)$.

But $\nu_A(0) \leq \nu_A(y+x-y)$. So, $y + x - y \in M\nu_A$.

$\mu_A (u\alpha(x+v)- u\alpha v) \geq \mu_A(x) = \mu_A(0)$ (resp.$\mu_A(x\alpha u) \geq \mu_A(x) = \mu_A(0)$).

$\nu_A (u\alpha(x+v)- u\alpha v) \leq \nu_A(x) = \nu_A(0)$ (resp. $\nu_A(x\alpha u) \leq \nu_A(x) = \nu_A(0)$).

Hence $(u\alpha(x+v)- u\alpha v) \in M\nu_A$.

Therefore $M\nu_A$ is a left (resp. right) ideal of M.

Definition 2.13

A Γ- near- ring M is said to be together if for each $a \in M$ there exists an $x \in M$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.14

Let $A = <\mu_A; \nu_A>$ and $B = <\mu_B; \nu_B>$ be two intuitionistic fuzzy subsets of a Γ-near- ring M. The product AΓB is defined by

Theorem 2.15

If $A = <\mu_A; \nu_A>$ and $B = <\mu_B; \nu_B>$ are two intuitionistic fuzzy left (resp. right) ideal of M, then $A\cap B$ is an intuitionistic fuzzy left (resp. right) ideal of M. If A is an intuitionistic fuzzy right ideal and B is an intuitionistic fuzzy left ideal, then $A\Gamma B \subseteq A\cap B$.

Proof

Suppose A and B are intuitionistic fuzzy ideals of M and let $x, y, z, z' \in M$ and $\alpha \in \Gamma$.

Then, $\mu_{A\cap B}(x-y) = \mu_A(x-y) \wedge \mu_A(x-y)$

$$\geq [\mu_A(x) \wedge \mu_A(y)] \wedge [\mu_B(x) \wedge \mu_B(y)]$$

$$= [\mu_A(x) \wedge \mu_B(x)] \wedge [\mu_A(y) \wedge \mu_B(y)]$$

$$= \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y),$$

$\nu_{A\cap B}(x-y) = \nu_A(x-y) \vee \nu_A(x-y)$

$$\leq [\nu_A(x) \vee \nu_A(y)] \vee [\nu_B(x) \vee \nu_B(y)]$$

$$= [\nu_A(x) \vee \nu_B(x)] \vee [\nu_A(y) \vee \nu_B(y)]$$

$$= \nu_{A\cap B}(x) \vee \nu_{A\cap B}(y).$$

$$\mu_{A\cap B}(y+x-y) = \mu_A(y+x-y) \wedge \mu_B(y+x-y)$$

$$\geq [\mu_A(x)] \wedge [\mu_B(x)]$$

$$=\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y),$$

$$\nu_{A\cap B}(y+x-y) = \nu_A(y+x-y) \vee \nu_A(y+x-y)$$

$$\leq [\nu_A(x)] \vee [\nu_B(x)]$$

$$= \nu_{A\cap B}(x) \vee \nu_{A\cap B}(y).$$

Since A and B are intuitionistic fuzzy ideals of M, we have

$\mu_A(x\alpha(y+z)-x\alpha z) \geq \mu_A(x)$, $\nu_A(x\alpha(y+z)-x\alpha z) \leq \nu_A(x)$ and $\mu_B(y\alpha x) \geq \mu_B(x)$, $\nu_B(y\alpha x) \leq \nu_B(x)$.

Now $\mu_{A\cap B}(x\alpha(y+z)-x\alpha z) = \mu_A(x\alpha(y+z)-x\alpha z) \wedge \mu_B(x\alpha(y+z)-x\alpha z)$

$$\geq [\mu_A(x)] \wedge [\mu_B(x)]$$

$$=\mu_{A\cap B}(x) \ \ (\text{resp. } \mu_{A\cap B}(y\alpha x) \geq \mu_{A\cap B}(x)),$$

$$\nu_{A\cap B}(x\alpha(y+z)-x\alpha z) = \nu_A(x\alpha(y+z)-x\alpha z) \vee \nu_B(x\alpha(y+z)-x\alpha z)$$

$$\leq [\nu_A(x)] \vee [\nu_B(x)]$$

$$= \nu_{A\cap B}(x) \ \ (\text{resp. } \nu_{A\cap B}(y\alpha x) \leq \nu_{A\cap B}(x)).$$

Hence $A\cap B$ is an intuitionistic fuzzy left ideal of M.

To prove the second part if $\mu_{A\Gamma B}(x) = 0$ and $\nu_{A\Gamma B}(x) = 1$, there is nothing to show.

From the definition of $A\Gamma B$, $\mu_A(x) = \mu_A(y\alpha(z+z')-y\alpha z') \geq \mu_A(z)$,

$$\nu_A(x) = \nu_A(y\alpha(z+z')-y\alpha z') \leq \nu_A(z).$$

Since A is an intuitionistic fuzzy right ideal and B is an intuitionistic fuzzy left ideal, we have $\mu_A(x) = \mu_A(z\alpha y) \geq \mu_A(z)$, $\nu_A(x) = \nu_A(z\alpha y) \leq \nu_A(z)$,

$\mu_B(x) = \mu_B(z\alpha y) \geq \mu_B(z)$, $\nu_B(x) = \nu_B(z\alpha y) \leq \nu_B(z)$.

Hence by theorem 3.5,

$$\mu_{A\Gamma B}(x) = \bigvee_{x=y\alpha z}\{\mu_A(y) \wedge \mu_B(z)\}$$

$$\leq \mu_A(x) \wedge \mu_B(x)$$

$$= \mu_{A\cap B}(x) \quad \text{and}$$

$$\nu_{A\Gamma B}(x) = \bigwedge_{x=y\alpha z}\{\nu_A(y) \vee \nu_B(z)\}$$

$$\geq \nu_A(x) \vee \nu_B(x)$$

$$= \nu_{A\cap B}(x)$$

Which means that $A\Gamma B \subseteq A\cap B$.

Theorem 2.16

A Γ- near- ring M is regular if and only if for each intuitionistic fuzzy right ideal A and each intuitionistic fuzzy left ideal B of M, $A\Gamma B = A\cap B$.

Proof

($\Rightarrow$): Suppose R is regular. $A\Gamma B \subseteq A\cap B$. Thus it is sufficient to show that $A\cap B \subseteq A\Gamma B$. Let $a \in M$ and $\alpha, \beta \in \Gamma$.

Then, by hypothesis, there exists an $x \in M$ such that $a = a\alpha x\beta a$.

Thus, $\mu_A(a) = \mu_A(a\alpha x\beta a) \geq \mu_A(a\alpha x) \geq \mu_A(a)$,

$\nu_A(a) = \nu_A(a\alpha x\beta a) \leq \nu_A(a\alpha x) \leq \nu_A(a)$.

So $\mu_A(a\alpha x) = \mu_A(a)$ and $\nu_A(a\alpha x) = \nu_A(a)$.

On the other hand,

$\mu_{A\Gamma B}(a) = \vee_{a=a\alpha x\beta a}\left[\mu_A(a\alpha x) \wedge \mu_B(a)\right] \geq \left[\mu_A(a) \wedge \mu_B(a)\right] = \mu_{A\cap B}(a)$ and

$\nu_{A\Gamma B}(a) = \wedge_{a=a\alpha x\beta a}\left[\nu_A(a\alpha x) \vee \nu_B(a)\right] \leq \left[\nu_A(a) \vee \nu_B(a)\right] = \nu_{A\cap B}(a)$.

Thus, $A\cap B \subseteq A\Gamma B$. Hence $A\Gamma B = A\cap B$.

CHAPTER III

On intuitionistic Fuzzy bi- ideals in Γ- near- ring

Definition 3.1

An intuitionistic fuzzy ideal A= < μ_A; ν_A > of M is called an intuitionistic fuzzy bi- ideal of M if,

(i) $\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$

(ii) $\mu_A(y+x- y) \geq \mu_A(x)$

(iii) $\mu_A((x\alpha y\beta z) \wedge (x\alpha(y+z)- x\alpha z)) \geq \{\mu_A(x) \wedge \mu_A(z)\}$, for all $x,y,z \in M$ and $\alpha, \beta \in \Gamma$

(iv) $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\}$

(v) $\nu_A(y + x - y) \leq \nu_A(x)$

(vi) $\nu_A((x\alpha y\beta z) \vee (x\alpha(y+z)- x\alpha z)) \leq \{\nu_A(x) \vee \nu_A(z)\}$, for all $x,y,z \in M$ and $\alpha, \beta \in \Gamma$

Example 3.2

Let R be the set of all integers then R is a ring. Take M=Γ=R. Let $a,b \in M$, $\alpha \in \Gamma$ Suppose $a\alpha b$ is the product of $a, \alpha, b \in R$.

Then M is a Γ- near ring.

Define IFS A= < μ_A; ν_A > in R as follows.

$\mu_A(0) = 1$ and $\mu_A(\pm1) = \mu_A(\pm2) = \mu_A(\pm3) = \ldots = t$ and

$\nu_A(0) = 1$ and $\nu_A(\pm1) = \nu_A(\pm2) = \nu_A(\pm3) = \ldots = s$

where $t \in [0,1]$, $s \in [0,1]$ and $t + s \leq 1$.

By routine calculations,

Clearly, A is an intuitionistic fuzzy bi- ideal of a Γ- near ring M.

Lemma 3.3

If B is a bi-ideal of M then for any $0 < t, s < 1$, there exists an intuitionistic fuzzy k-bi-ideal $C = < \mu_c, \nu_c >$ of M such that $C_{<t,s>} = B$.

Proof

Let $C \to [0,1]$ be a function defined by

$$\mu_B(x) = \begin{cases} t & \text{if } x \in B, \\ s & \text{if } x \notin B. \end{cases} \qquad \nu_B(x) = \begin{cases} s & \text{if } y \in B, \\ 1 & \text{if } y \notin B. \end{cases}$$

for all $x \in M$ and the pair $s, t \in [0,1]$. Then $C_{<t,s>} = B$ is an intuitionistic fuzzy bi-ideal of M with $t + s \leq 1$.

Now suppose that B is a bi-ideal of M. For all $x, y \in B$, such that $x - y \in B$, we have

$\mu_c(x-y) \geq t = \{\mu_c(x) \wedge \mu_c(y)\}$,

$\nu_c(x-y) \leq s = \{\nu_c(x) \vee \nu_c(y)\}$,

$\mu_c(y+x-y) \geq t = \mu_c(x)$,

$\nu_c(y+x-y) \leq s = \nu_c(x)$,

Also, for all $x, y, z \in B$ and $\alpha, \beta \in \Gamma$ such that $x\alpha y\beta z \in B$, we have

$\mu_c((x\alpha y\beta z) \wedge (x\alpha(y+z)-x\alpha z))) \geq t = \{\mu_c(x) \wedge \mu_c(z)\}$,

$\nu_c((x\alpha y\beta z) \vee (x\alpha(y+z)-x\alpha z))) \leq s = \{\nu_c(x) \vee \nu_c(z)\}$

Thus $C_{<t,s>}$ is an intuitionistic fuzzy bi-ideal of M.

Lemma 3.4

Let B be a non-empty subset of M. Then B is a bi-ideal of M if and only if the IFS $\bar{B} = < \chi_B, \bar{\chi}_B >$ is an intuitionistic fuzzy ideal of M.

Proof

Let $x, y \in$ B. From the hypothesis $x - y \in B$.

(i) If $x, y \in$ B, then $\chi_B(x) = 1$, $\bar{\chi}_B(x) = 0$, $\chi_B(y) = 1$, and $\bar{\chi}_B(y) = 0$, thus

$\chi_B(x-y) = 1 \geq \{\chi_B(x) \wedge \chi_B(y)\}$

$\bar{\chi}_B(x-y) = 0 \leq \{\bar{\chi}_B(x) \vee \bar{\chi}_B(y)\}$.

(ii) If $x \in$ B, $y \notin B$, then $\chi_B(x) = 1$, $\bar{\chi}_B(x) = 0$, $\chi_B(y) = 0$, and $\bar{\chi}_B(y) = 1$, thus

$\chi_B(x-y) = 0 \geq \{\chi_B(x) \wedge \chi_B(y)\}$

$\bar{\chi}_B(x-y) = 1 \leq \{\bar{\chi}_B(x) \vee \bar{\chi}_B(y)\}$.

(iii) If $x \notin B$, $y \in$ B, then $\chi_B(x) = 0$, $\bar{\chi}_B(x) = 1$, $\chi_B(y) = 1$, and $\bar{\chi}_B(y) = 0$, thus

$\chi_B(x-y) = 0 \geq \{\chi_B(x) \wedge \chi_B(y)\}$

$\bar{\chi}_B(x-y) = 1 \leq \{\bar{\chi}_B(x) \vee \bar{\chi}_B(y)\}$.

(iv) If $x \notin B$, $y \notin B$, then $\chi_B(x) = 0$, $\bar{\chi}_B(x) = 1$, $\chi_B(y) = 0$, and $\bar{\chi}_B(y) = 1$, thus

$\chi_B(x-y) \geq 0 = \{\chi_B(x) \wedge \chi_B(y)\}$

$\bar{\chi}_B(x-y) \leq 1 = \{\bar{\chi}_B(x) \vee \bar{\chi}_B(y)\}$.

Thus, (i) of definition 3.1 holds good.

Let $x, y \in$ B, from the hypothesis $y + x - y \in$ B.

(i) If $x, y \in B$, then $\chi_B(x) = 1$, $\overline{\chi}_B(x) = 0$, $\chi_B(y) = 1$, and $\overline{\chi}_B(y) = 0$, thus

$\chi_B(y+x-y) = 1 \geq \chi_B(x)$

$\overline{\chi}_B(y+x-y) = 0 \leq \overline{\chi}_B(x)$

(ii) If $x \in B$, $y \notin B$, then $\chi_B(x) = 1$, $\overline{\chi}_B(x) = 0$, $\chi_B(y) = 0$, and $\overline{\chi}_B(y) = 1$, thus

$\chi_B(y+x-y) = 0 \geq \chi_B(x)$

$\overline{\chi}_B(y+x-y) = 1 \leq \overline{\chi}_B(x)$

(iii) If $x \notin B$, $y \in B$, then $\chi_B(x) = 0$, $\overline{\chi}_B(x) = 1$, $\chi_B(y) = 1$, and $\overline{\chi}_B(y) = 0$, thus

$\chi_B(y+x-y) = 0 \geq \chi_B(x)$

$\overline{\chi}_B(y+x-y) = 1 \leq \overline{\chi}_B(x)$

(iv) If $x \notin B$, $y \notin B$, then $\chi_B(x) = 0$, $\overline{\chi}_B(x) = 1$, $\chi_B(y) = 0$, and $\overline{\chi}_B(y) = 1$, thus

$\chi_B(y+x-y) \geq 0 = \chi_B(x)$

$\overline{\chi}_B(y+x-y) \leq 1 = \overline{\chi}_B(x)$

Thus (ii) of definition 3.1 holds good.

Let $x, y, z \in B$ and $\alpha, \beta \in \Gamma$, from the hypothesis $x\alpha y\beta z$, $x\alpha(y+z) - x\alpha z \in B$.

(i) If $x, z \in B$ then $\chi_B(x) = 1$, $\overline{\chi}_B(x) = 0$, $\chi_B(z) = 1$, and $\overline{\chi}_B(z) = 0$. Thus

$\chi_B(\mu((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z))) = 1 \geq \{\chi_B(x) \wedge \chi_B(z)\}$

$$\bar{\chi}_B(\mu((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z))) = 0 \le \left\{ \bar{\chi}_B(x) \vee \bar{\chi}_B(z) \right\}$$

(ii) If $x \in B$, $z \notin B$ then $\chi_B(x) = 1$, $\bar{\chi}_B(x) = 0$, $\chi_B(z) = 0$, and $\bar{\chi}_B(z) = 1$. Thus

$$\chi_B(\mu((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z))) = 0 \ge \{\chi_B(x) \wedge \chi_B(z)\}$$

$$\bar{\chi}_B(\mu((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z))) = 1 \le \left\{ \bar{\chi}_B(x) \vee \bar{\chi}_B(z) \right\}$$

(iii) If $x \notin B$, $z \in B$ then $\chi_B(x) = 0$, $\bar{\chi}_B(x) = 1$, $\chi_B(z) = 1$, and $\bar{\chi}_B(z) = 0$. Thus

$$\chi_B(\mu((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z))) = 0 \ge \{\chi_B(x) \wedge \chi_B(z)\}$$

$$\bar{\chi}_B(\mu((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z))) = 1 \le \left\{ \bar{\chi}_B(x) \vee \bar{\chi}_B(z) \right\}$$

(iv) If $x \notin B$, $z \notin B$ then $\chi_B(x) = 0$, $\bar{\chi}_B(x) = 1$, $\chi_B(z) = 0$, and $\bar{\chi}_B(z) = 1$. Thus

$$\chi_B(\mu((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z))) \ge 0 = \{\chi_B(x) \wedge \chi_B(z)\}$$

$$\bar{\chi}_B(\mu((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z))) \le 1 = \left\{ \bar{\chi}_B(x) \vee \bar{\chi}_B(z) \right\}$$

Thus (iii) of definition 3.1 holds good.

Conversely, Suppose that IFS $\bar{B} = <\chi_B, \bar{\chi}_B>$ is an intuitionistic fuzzy ideal of M. Then by lemma 3.3, χ_B is two-valued, Hence B is bi-ideal of M.

This complete the proof.

Theorem 3.5

If $\{A_i\}_{i \in A}$ is a family of intuitionistic fuzzy bi-ideals of M then $\cap A_i$ is an intuitionistic fuzzy bi-ideals of M, where $\cap A_i = \{\wedge \mu_{Ai}, \vee \nu_{Ai}\}$, $\wedge \mu_{Ai}(x) =$ inf $\{\mu_{Ai} | \in \wedge, x \in M\}$ and $\vee \nu_{Ai}(x) = \sup\{\nu_{Ai}(x) \ / \ i \in \vee, x \in M\}$

Proof

Let $x, y \in M$. Then we have,

$\wedge\mu_{Ai}(x-y) = \inf\{\{\mu_{Ai}(x) \wedge \mu_{Ai}(y)\} / i \in \wedge, x, y \in M\}$

$=\{\{\inf(\mu_{Ai}(x)) \wedge \inf(\mu_{Ai}(y))\} / i \in \wedge, x, y \in M\}$

$= \{\{\inf(\mu_{Ai}(x)) / i \in \wedge, x \in M)\} \wedge \{\inf(\mu_{Ai}(y) / i \in \wedge, y \in M)\}\}$

$= \{\wedge\mu_{Ai}(x) \wedge \wedge\mu_{Ai}(y)\}.$

Let $x, y \in M$. Then we have

$\wedge\mu_{Ai}(y+x-y) = \inf\{\mu_{Ai}(x) / i \in \wedge, x, y \in M\}$

$= \wedge\mu_{Ai}(x)$

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

$\wedge\mu_{Ai}((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z)) = \inf\{\{\mu_{Ai}(x)) \wedge \mu_{Ai}(z)\} / i \in \wedge, x, z \in M\}$

$= \{\{\inf(\mu_{Ai}(x)) \wedge \inf(\mu_{Ai}(z))\} / i \in \wedge, x, z \in M\}$

$= \{\{\inf(\mu_{Ai}(x)) / i \in \wedge, x \in M)\} \wedge \{\inf(\mu_{Ai}(z) / i \in \wedge, z \in M)\}\}$

$= \{\wedge\mu_{Ai}(x) \wedge \wedge\mu_{Ai}(z)\}.$

Let $x, y \in M$. Then we have

$\vee\nu_{Ai}(x-y) = \sup\{\{\nu_{Ai}(x) \vee \nu_{Ai}(y)\} / i \in \vee, x, y \in M\}$

$= \{\{\sup(\nu_{Ai}(x)) \vee \sup(\nu_{Ai}(y))\} / i \in \vee, x, y \in M\}$

$= \{\{\sup(\nu_{Ai}(x) / i \in \vee, x \in M)\} \vee \{\sup(\nu_{Ai}(y))\} / i \in \vee, y \in M)\}\}$

$= \{\vee\nu_{Ai}(x) \vee \vee\nu_{Ai}(y)\}.$

Let $x, y \in M$. Then we have

$\vee \nu_{A_i}(y+x-y) = \sup\{\nu_{A_i}(x) \,/\, i \in \vee, x,y \in M)\}$

$= \vee \nu_{A_i}(x).$

Let $x,y,z \in M$ and $\alpha,\beta \in \Gamma$

$\vee \nu_{A_i}((x\alpha y\beta z) \vee (x\alpha(y+z)-x\alpha z)) = \sup\{\{\nu_{A_i}(x) \vee \nu_{A_i}(z)\} \,/\, i \in \vee, x,z \in M\}$

$= \{\{\sup(\nu_{A_i}(x)) \vee \sup(\nu_{A_i}(z)\} \,/ i \in \vee, x,z \in M\}$

$= \{\{\sup(\nu_{A_i}(x) \,/ i \in \vee, x \in M)\} \vee \{\sup(\nu_{A_i}(z)\} \,/\, i \in \vee, z \in M)\}\}$

$= \{\vee \nu_{A_i}(x) \vee \vee \nu_{A_i}(z)\}.$

Hence $\cap A_i = \{\wedge \mu_{A_i}, \vee \nu_{A_i}\}$ is an intuitionistic fuzzy bi-ideal of M.

Theorem 3.6

If A is an intuitionistic fuzzy bi-ideal of M then A' is also an intuitionistic fuzzy bi-ideal of M.

Proof

Let $x,y \in M$. Then we have

$\mu'_A(x-y) = 1 - \mu_A(x-y)$

$= 1 - \{\mu_A(x) \wedge \mu_A(y)\},$

$\nu'_A(x-y) = 1 - \nu_A(x-y)$

$= 1 - \{\nu_A(x) \vee \nu_A(y)\}.$

Let $x,y \in M$. We have,

$\mu'_A(y+x-y) = 1 - \mu_A(y+x-y)$

$= 1 - \mu_A(x)$

$= \mu'_A(x).$

$\nu'_A(y+x-y) = 1 - \nu_A(y + x - y)$

$= 1 - \nu_A(x)$

$= \nu'_A(x).$

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We have

$\mu'_A((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z)) = 1 - \mu_A((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z))$

$= 1 - \{\mu_A(x) \wedge \mu_A(z)\}$

$= \{1 - \mu_A(x) \wedge 1 - \mu_A(z)\}$

$= \{\mu'_A(x) \wedge \mu'_A(z)\},$

$\nu'_A((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z)) = 1 - \nu_A((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z))$

$= 1 - \{\nu_A(x) \vee \nu_A(z)\}$

$= \{1 - \nu_A(x) \vee 1 - \nu_A(z)\}$

$= \{\nu'_A(x) \vee \nu'_A(z)\}.$

Therefore, A' is also an intuitionistic fuzzy bi-ideal of M.

Theorem 3.7

An IFSA of M is an intuitionistic fuzzy bi-ideal of M if and only if the level sets $U(\mu_A;t) = \{x \in M | \mu(x) \geq t\}$ and $L(\nu_A;t) = \{x \in M | \nu(x) \geq t\}$ are a bi-ideal of M when it is non-empty.

Proof

Let A be an intuitionistic fuzzy bi- ideal of M.

Then, $\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$.

$x, y \in U(\mu_A; t) \Rightarrow \mu_A(x) \geq t,\ \mu_A(y) \geq t$

$\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\} \geq t$

$\mu_A(x-y) \geq t \Rightarrow x - y \in U(\mu_A; t)$,

$\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$.

$x, y \in L(\nu_A; t) \Rightarrow \nu_A(x) \leq t,\ \nu_A(y) \leq t$

$\nu_A(x-y) \leq \{\nu_A(x) \vee \nu_A(y)\} \leq t$

$\nu_A(x-y) \leq t \Rightarrow x - y \in L(\nu_A; t)$.

$\mu_A(y+x-y) \geq \mu_A(x)$.

$x, y \in U(\mu_A; t) \Rightarrow \mu_A(x) \geq t,\ \mu_A(y) \geq t$

$\mu_A(y+x-y) \geq \mu_A(x) \geq t$

$\mu_A(y+x-y) \geq t \Rightarrow y + x - y \in U(\mu_A; t)$.

Let $\nu_A(y+x-y) \leq \nu_A(x)$

$x, y \in L(\nu_A; t) \Rightarrow \nu_A(x) \leq t,\ \nu_A(y) \leq t$

$\nu_A(y+x-y) \leq \nu_A(x) \leq t$

$\nu_A(y+x-y) \leq t \Rightarrow y + x - y \in L(\nu_A; t)$.

Also, Let $\mu_A((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z)) \geq \{\mu_A(x) \wedge \mu_A(z)\}$.

$x,y,z \in U(\mu_A;t)$, $\alpha,\beta \in \Gamma \Rightarrow \mu_A(x) \geq t$, $\mu_A(y) \geq t$, $\mu_A(z) \geq t$

$\mu_A((x\alpha y\beta z) \wedge (x\alpha(y+z)-x\alpha z)) \geq \{\mu_A(x) \wedge \mu_A(z)\} \geq t$

$(x\alpha y\beta z), (x\alpha(y+z)-x\alpha z) \in U(\mu_A;t)$

Thus $U(\mu_A;t)$ is a bi- ideal of M.

$\nu_A((x\alpha y\beta z) \vee (x\alpha(y+z)-x\alpha z)) \leq \{\nu_A(x) \vee \nu_A(z)\}$.

$x,y,z \in L(\nu_A;t)$, $\alpha,\beta \in \Gamma \Rightarrow \nu_A(x) \leq t$, $\nu_A(y) \leq t$, $\nu_A(z) \leq t$

$\nu_A((x\alpha y\beta z) \vee (x\alpha(y+z)-x\alpha z)) \leq \{\nu_A(x) \vee \nu_A(z)\} \leq t$

$\nu_A((x\alpha y\beta z) \vee (x\alpha(y+z)-x\alpha z)) \leq t$

$(x\alpha y\beta z), (x\alpha(y+z)-x\alpha z) \in L(\nu_A;t)$.

Thus, $L(\nu_A;t)$ is a bi- ideal of M.

Conversely, if $U(\mu_A;t)$ is a bi- ideal of M. let $t = \{\mu_A(x) \wedge \mu_A(y)\}$.

Then $x,y \in U(\mu_A;t) \Rightarrow x - y \in U(\mu_A;t)$

$\Rightarrow \mu_A(x-y) \geq t \Rightarrow \mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$.

Also, $x,y \in U(\mu_A;t) \Rightarrow y + x - y \in U(\mu_A;t)$

$\Rightarrow \mu_A(y+x-y) \geq \mu_A(x)$.

If $L(\nu_A;t)$ is a bi- ideal of M. Let $t = \{\nu_A(x) \vee \nu_A(y)\}$.

Then $x,y \in L(\nu_A;t) \Rightarrow x - y \in L(\nu_A;t)$

$\nu_A(x-y) \leq t \Rightarrow \nu_A(x-y) \leq \{\nu_A(x) \vee \nu_A(y)\}$.

Also, $x, y \in L(\nu_A; t) \Rightarrow y + x - y \in L(\nu_A; t)$

$\Rightarrow \nu_A(y+x-y) \leq \nu_A(x)$.

Next, define $t = \{\mu_A(x) \wedge \mu_A(z)\}$. Then $x, y, z \in U(\mu_A; t)$, $\alpha, \beta \in \Gamma$

$\Rightarrow (x\alpha y\beta z), (x\alpha(y+z) - x\alpha z) \in U(\mu_A; t)$

$\Rightarrow \mu_A((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z)) \geq t$

$\Rightarrow \mu_A((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z)) \geq \{\mu_A(x) \wedge \mu_A(z)\}$.

Next, define $t = \{\nu_A(x) \vee \nu_A(z)\}$. Then $x, y, z \in L(\nu_A; t)$, $\alpha, \beta \in \Gamma$

$\Rightarrow (x\alpha y\beta z), (x\alpha(y+z) - x\alpha z) \in L(\nu_A; t)$

$\Rightarrow \nu_A((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z)) \leq t$

$\Rightarrow \nu_A((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z)) \leq \{\nu_A(x) \vee \nu_A(z)\}$.

Consequently, A is an intuitionistic fuzzy bi-ideal of M.

Theorem 3.8

Let A be an intuitionistic fuzzy bi-ideal of M. If M is completely regular, then $\mu_A(a) = \mu_A(a\alpha a)$, $\nu_A(a) = \nu_A(a\alpha a)$ for all $a \in M$ and $\alpha \in \Gamma$.

Proof

Straight forward.

Definition 3.9

Let f be mappings from a set X to Y, and A be IFS on Y. Then the preimage of μ under f, denoted by $f^{-1}(A)$, is defined by:

$f^{-1}(\mu_A(x)) = \mu_A(f(x))$, $f^{-1}(\nu_A(x)) = \nu_A(f(x))$, for all $x \in X$.

Theorem 3.10

Let the pair of mappings $f: M \to N$ be a homomorphism of Γ- near-rings. If μ is an intuitionistic fuzzy bi-ideal of N, then the preimage $f^{-1}(A)$ of A under f is an intuitionistic fuzzy bi-ideal of M.

Proof

Let $x, y \in M$. Then we have

$$f^{-1}(\mu_A)(x-y) = \mu_A(f(x-y))$$

$$= \mu_A(f(x)-f(y))$$

$$\geq \{\mu_A(f(x)) \wedge \mu_A(f(y))\}$$

$$= \{f^{-1}(\mu_A(x)) \wedge f^{-1}(\mu_A(y))\}.$$

$$f^{-1}(\nu_A)(x-y) = \nu_A(f(x-y))$$

$$= \nu_A(f(x)-f(y))$$

$$\leq \{\nu_A(f(x)) \vee \nu_A(f(y))\}$$

$$= \{f^{-1}(\nu_A(x)) \vee f^{-1}(\nu_A(y))\}.$$

Let $x, y \in M$. Then we have

$$f^{-1}(\mu_A)(y+x-y) = \mu_A(f(y+x-y))$$

$$\geq \mu_A(f(x))$$

$$= f^{-1}(\mu_A)(x).$$

$$f^{-1}(\nu_A)(y+x-y) = \nu_A(f(x-y))$$

$$\leq \nu_A(f(x))$$

$= f^{-1}(\nu_A)(x).$

Let $x, y, z \in M$, $\alpha, \beta \in \Gamma$. Then

$$f^{-1}(\mu_A)((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z)) = \mu_A(f((x\alpha y\beta z) \wedge (x\alpha(y+z) - x\alpha z)))$$

$$= \mu_A((f(x\alpha y\beta z)) \wedge (f(x\alpha(y+z) - x\alpha z))$$

$$\geq \mu_A(f(x)) \wedge \mu_A(f(z))$$

$$= \{ f^{-1}(\mu_A(x)) \wedge f^{-1}(\mu_A(z)) \}.$$

Therefore, $f^{-1}(\mu_A)$ is an intuitionistic fuzzy bi- ideal of M.

$$f^{-1}(\nu_A)((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z)) = \nu_A(f((x\alpha y\beta z) \vee (x\alpha(y+z) - x\alpha z)))$$

$$= \nu_A((f(x\alpha y\beta z)) \vee (f(x\alpha(y+z) - x\alpha z))$$

$$\leq \nu_A(f(x)) \vee \nu_A(f(z))$$

$$= \{ f^{-1}(\nu_A(x)) \vee f^{-1}(\nu_A(z)) \}.$$

Therefore, $f^{-1}(\nu_A)$ is an intuitionistic fuzzy bi- ideal of M.

Therefore, $f^{-1}(A)$ is an intuitionistic fuzzy bi- ideal of M.

CHAPTER IV

Intuitionistic Fuzzy k-bi-ideals in Γ-near-rings

Definition 4.1

An IFS A= $(\mu_A;\gamma_A)$ in M called an intuitionistic fuzzy subgroup of M if

(i) $\mu_A(x-y) \geq \min\{\mu_A(x),\mu_A(y)\}$

(ii) $\gamma_A(x-y) \leq \max\{\gamma_A(x),\gamma_A(y)\}$ for all $x,y \in M$.

Definition 4.2

An intuitionistic fuzzy subgroup A= $(\mu_A;\gamma_A)$ of M is called an intuitionistic fuzzy bi-ideal of M if

(i) $\mu_A(x\alpha y\beta z) \geq \min\{\mu_A(x),\mu_A(z)\}$

(ii) $\gamma_A(x\alpha y\beta z) \leq \max\{\gamma_A(x),\gamma_A(z)\}$ for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$

Definition 4.3

An intuitionistic fuzzy bi-ideal A= $(\mu_A;\gamma_A)$ of M is called an intuitionistic fuzzy k-bi-ideal of M if for all $x,y \in M$.

(i) $\mu_A(x) \geq \min\{\max\{\mu_A(x+y),\mu_A(y+x)\},\mu_A(y)\}$.

(ii) $\gamma_A(x) \geq \max\{\min\{\gamma_A(x+y),\gamma_A(y+x)\},\gamma_A(y)\}$.

Theorem 4.4

An IFS A= $(\mu_A;\gamma_A)$ in M is an intuitionistic fuzzy k-bi-ideal of M. Then the fuzzy sets μ_A and γ'_A are fuzzy k- bi- ideal of M.

Proof

Let $A = (\mu_A; \gamma_A)$ be an intuitionistic fuzzy k- bi- ideal of M. Then clearly μ_A is a fuzzy k- bi- ideal of M. We claim that γ'_A is a fuzzy k- bi- ideal of M. Let $x, y \in M$.

Then, $\gamma'_A(x-y) = 1 - \gamma_A(x-y)$

$$\geq 1 - \max\{\gamma_A(x), \gamma_A(y)\}$$

$$= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\}$$

$$= \min\{\gamma'_A(x), \gamma'_A(y)\}.$$

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\gamma'_A(x\alpha y\beta z) = 1 - \gamma_A(x\alpha y\beta z)$$

$$\geq 1 - \max\{1 - \gamma_A(x), 1 - \gamma_A(z)\}$$

$$= \min\{1 - \gamma_A(x), 1 - \gamma_A(z)\}$$

$$= \min\{\gamma'_A(x), \gamma'_A(y)\}.$$

Now, $\gamma'_A(x) = 1 - \gamma_A(x)$

$$\geq 1 - \max\{\min\{\gamma_A(x+y), \gamma_A(y+x)\}, \gamma_A(y)\}$$

$$= \min\{1 - \min\{\gamma_A(x+y), \gamma_A(y+x)\}, 1 - \gamma_A(y)\}$$

$$= \min\{\max\{1 - \{\gamma_A(x+y), 1 - \gamma_A(y+x)\}, 1 - \gamma_A(y)\}$$

$$= \min\{\max\{\gamma'_A(x+y), \gamma'_A(y+x)\}, \gamma'_A(y)\}.$$

Hence γ'_A is a fuzzy k- bi- ideal of M.

Theorem 4.5

Let $f: M \rightarrow M'$ be a homomorphism of Γ-near-rings. If $B = (\mu_B; \gamma_B)$ is an intuitionistic fuzzy k-bi-ideal of M' then the preimage $f^{-1}(B) = (f^{-1}(\mu_B); f^{-1}(\gamma_B))$ of B under f is an intuitionistic fuzzy k-bi-ideal of M.

Proof

Assume that $B = (\mu_B; \gamma_B)$ is an intuitionistic fuzzy k-bi-ideal of M'. Let $x, y \in M$.

Then,
$$\begin{aligned} f^{-1}(\mu_B)(x-y) &= \mu_B(f(x-y)) \\ &= \mu_B(f(x)-f(y)) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\ &= \min\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\} \end{aligned}$$

$$\begin{aligned} f^{-1}(\gamma_B)(x-y) &= \gamma_B(f(x-y)) \\ &= \gamma_B(f(x)-f(y)) \\ &\leq \max\{\gamma_B(f(x)), \gamma_B(f(y))\} \\ &= \max\{f^{-1}(\gamma_B)(x), f^{-1}(\gamma_B)(y)\} \end{aligned}$$

Which implies that $f^{-1}(B) = (f^{-1}(\mu_B); f^{-1}(\gamma_B))$ is an intuitionistic fuzzy subgroup of M.

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} f^{-1}(\mu_B)(x\alpha y\beta z) &= \mu_B(f(x\alpha y\beta z)) \\ &= \mu_B(f(x)\alpha f(y)\beta f(z)) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(z))\} \\ &= \min\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(z)\} \end{aligned}$$

$$f^{-1}(\gamma_B)(x\alpha y\beta z) = \gamma_B(f(x\alpha y\beta z))$$

$$= \gamma_B(f(x)\alpha f(y)\beta f(z))$$

$$\leq \max\{\gamma_B(f(x)), \gamma_B(f(z))\}$$

$$= \max\{f^{-1}(\gamma_B)(x), f^{-1}(\gamma_B)(z)\}$$

Therefore $f^{-1}(B)$ is intuitionistic fuzzy bi- ideal of M. Now for $x, y \in M$, we have $f^{-1}(\mu_B)(x) = \mu_B(f(x))$

$$\geq \min\{\max\{\mu_B(f(x)+f(y)), \mu_B(f(y)+f(x))\}, \mu_B(f(y))\}$$

$$= \min\{\max\{\mu_B(f(x+y)), \mu_B(f(y+x))\}, \mu_B(f(y))\}$$

$$= \min\{\max\{f^{-1}(\mu_B)(x+y), f^{-1}(\mu_B)(y+x)\},$$

$f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$

$$\leq \max\{\min\{\gamma_B(f(x)+f(y)), \gamma_B(f(y)+f(x))\}, \gamma_B(f(y))\}$$

$$= \max\{\min \gamma_B\{(f(x+y)), \gamma_B(f(y+x))\}, \gamma_B(f(y))\}$$

$$= \max\{\min\{f^{-1}(\gamma_B)(x+y), f^{-1}(\gamma_B)(y+x)\}, f^{-1}(\gamma_B)(y)\}$$

Hence $f^{-1}(B) = (f^{-1}(\mu_B); f^{-1}(\gamma_B))$ of B under f is an intuitionistic fuzzy k- bi- ideal of M.

Definition 4.6

Let $A = (\mu_A; \gamma_A)$ be an intuitionistic fuzzy set in M. Then the intuitionistic level subset of A is an object having the form $(\mu_A^t; \gamma_A^{t'})$ where $\mu_A^t = \{x \in M / \mu_A(x) \geq t\}$ and $\gamma_A^{t'} = \{x \in M / \gamma_A(x) \geq t'\}$.

Theorem 4.7

An IFS $A = (\mu_A; \gamma_A)$ in M is an intuitionistic fuzzy k- bi- ideal of M if and only if the components $\mu_A^t \neq 0$ for any $t \in Im\mu_A$ and $\gamma_A^{t'} \neq 0$ for any $t' \in Im\gamma_A$ of the intuitionistic level subset are k- bi- ideals of M.

Proof

Assume that $A = (\mu_A; \gamma_A)$ in M is an intuitionistic fuzzy k- bi- ideal of M. Fix $t \in Im\mu_A$ such that $\mu_A^t \neq 0$ and $t' \in Im\gamma_A$ such that $\gamma_A^{t'} \neq 0$.

Suppose that, $x, y \in \mu_A^t$. Then $\mu_A(x) \geq t$; $\mu_A(y) \geq t$ which impl $\min\{\mu_A(x), \mu_A(y)\} \geq t$. We get $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t$ which implies that $x - y \in \mu_A^t$.

Let $x, y \in \mu_A^t$, $y \in M$ and $\alpha, \beta \in \Gamma$. Then $\mu_A(x) \geq t$ and $\mu_A(z) \geq t$. Thus $\mu_A(x\alpha y\beta z) \geq \min\{\mu_A(x), \mu_A(z)\} \geq t$ implies $x\alpha y\beta z \in \mu_A^t$.

Let $y \in \mu_A^t$. Also let $x + y \in \mu_A^t$ (or) $y + x \in \mu_A^t$. Then $\mu_A(y) \geq t$ and $\mu_A(x + y) \geq t$ (or) $\mu_A(y + x) \geq t$. Now $\mu_A(x) \geq \min\{\max\{\mu_A(x+y), \mu_A(y+x)\}, \mu_A(y)\} \geq t$ which implies that $x \in \mu_A^t$. Therefore μ_A^t is a k- bi- ideals of M.

Similarly let $x, y \in \gamma_A^{t'}$. Then $\gamma_A(x) \leq t'$; $\gamma_A(y) \leq t'$ implies $\max\{\gamma_A(x), \gamma_A(y)\} \leq t'$. Thus we have $\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\} \leq t'$ implies that $x - y \in \gamma_A^{t'}$.

Let $x, z \in \gamma_A^{t'}$; $y \in M$ and $\alpha, \beta \in \Gamma$. Then $\gamma_A(x) \leq t'$; $\gamma_A(z) \leq t'$. We have $\gamma_A(x\alpha y\beta z) \leq \max\{\gamma_A(x), \gamma_A(z)\} \leq t'$ implies that $x\alpha y\beta z \in \gamma_A^{t'}$.

Let $y \in \gamma_A^{t'}$. Also let $x + y \in \gamma_A^{t'}$ (or) $y + x \in \gamma_A^{t'}$. Then $\gamma_A(y) \leq t'$ and $\gamma_A(x + y) \leq t'$ (or) $\gamma_A(y + x) \leq t'$. Now $\gamma_A(x) \geq \max\{\min\{\gamma_A(x+y), \gamma_A(y+x)\}, \gamma_A(y)\} \leq t'$ which implies that $x \in \gamma_A^{t'}$. Hence $\gamma_A^{t'}$ is also a k- bi- ideals of M.

Conversely, Assume that the components μ_A^t and $\gamma_A^{t'}$ of the intuitionistic level subset are k- bi- ideals of M.

Fix any $x, y \in M$ and let $\mu_A(x) = t_1$, $\mu_A(y) = t_2$. Let $t = \min\{t_1, t_2\}$. Then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$ implies $x, y \in \mu_A^t$ which implies that $x - y \in \mu_A^t$. Therefore $\mu_A(x - y) \geq t = \min\{t_1, t_2\} = \min\{\mu_A(x), \mu_A(y)\}$.

Fix any $x, y \in M$ and let $\gamma_A(x) = t_3$ and $\gamma_A(y) = t_4$. Let $t' = \max\{t_3, t_4\}$. Then $\gamma_A(x) \leq t'$; $\gamma_A(y) \leq t'$ implies $x, y \in \gamma_A^{t'}$ which implies $- y \in \gamma_A^{t'}$. Therefore $\gamma_A(x - y) \leq t' = \max\{t_3, t_4\} = \max\{\gamma_A(x), \gamma_A(y)\}$.

Let $\mu_A(x) = t_1$, $\mu_A(z) = t_2$. Let $t = \min\{t_1, t_2\}$. Then $\mu_A(x) \geq t$ and $\mu_A(z) \geq t$ implies $x, z \in \mu_A^t$ which implies that $x\alpha y\beta z \in \mu_A^t$. Therefore $\mu_A(x\alpha y\beta z) \geq t = \min\{t_1, t_2\} = \min\{\mu_A(x), \mu_A(z)\}$.

Next, let $\gamma_A(x) = t_3$ and $\gamma_A(y) = t_4$. Let $t' = \max\{t_3, t_4\}$. Then $\gamma_A(x) \leq t'$; $\gamma_A(y) \leq t'$ implies $x, y \in \gamma_A^{t'}$ which implies $x\alpha y\beta z \in \gamma_A^{t'}$. Therefore $\gamma_A(x\alpha y\beta z) \leq t' = \max\{t_3, t_4\} = \max\{\gamma_A(x), \gamma_A(z)\}$.

Thus $A = (\mu_A; \gamma_A)$ in M is an intuitionistic fuzzy bi-ideal of M.

For any $x, y \in M$, let $\mu_A(y) = t_1$; $\mu_A(x+y) = t_2$; $\mu_A(y+x) = t_3$ ($t_i \in Im\gamma_A$). If we let $t = \min\{\max\{t_2, t_3\}, t_1\}$. Then $\mu_A(y) \geq t$ and $\mu_A(x + y) \geq t$ (or) $\mu_A(y + x) \geq t$ implies $y \in \mu_A^t$ and $x + y \in \mu_A^t$ (or) $y + x \in \mu_A^t$. Since μ_A^t is a k-bi-ideal of M, we get $x \in \mu_A^t$. (i.e)., $\mu_A(x) \geq t = \min\{\max\{t_2, t_3\}, t_1\} = \min\{\max\{\mu_A(x+y), \mu_A(y+x)\}, \mu_A(y)\}$.

For any $x, y \in M$, let $\gamma_A(y) = t_4$; $\gamma_A(x+y) = t_5$; $\gamma_A(y+x) = t_6$ ($t_i \in Im\gamma_A$). If we let $t' = \max\{\min\{t_5, t_6\}, t_4\}$. Then $\gamma_A(y) \leq t'$ and $\gamma_A(x + y) \leq t'$ (or) $\gamma_A(y + x) \leq t'$ which implies that $y \in \gamma_A^{t'}$ and $x + y \in \gamma_A^{t'}$ (or) $y + x \in \gamma_A^{t'}$ implies that $x \in \gamma_A^{t'}$ [since $\gamma_A^{t'}$ is a k-bi-ideal of M]. Therefore $\gamma_A(x) \leq t' = \max\{\min\{t_5, t_6\}, t_4\} = \max\{\min\{\gamma_A(x+y), \gamma_A(y+x)\}, \gamma_A(y)\}$.

Hence $A = (\mu_A; \gamma_A)$ in M is an intuitionistic fuzzy k-bi-ideal of M.

Definition 4.8

A k- bi- ideal B of M is said to be an characteristic if $f(B) = B$ for all $f \in Aut(M)$, where Aut (M) is the set of all automorphism of M. An intuitionistic fuzzy k- bi - ideal $A = (\mu_A; \gamma_A)$ of M is said to be an intuitionistic fuzzy characteristic if $\mu_A(f(x)) = \mu_A(x)$ and $\gamma_A(f(x)) = \gamma_A(x)$ for all $x \in M$ and $f \in Aut(M)$.

Theorem 4.9

Let $A = (\mu_A; \gamma_A)$ be an intuitionistic fuzzy k- bi- ideal of M. If A is intuitionistic fuzzy characteristic then the components μ_A^t($t \in Im\mu_A$) and $\gamma_A^{t'}$($t' \in Im\gamma_A$) of the intuitionistic level subset are characteristic.

Proof

Let $A = (\mu_A; \gamma_A)$ be an intuitionistic fuzzy characteristic and Let $f \in Aut(M)$. For any $t \in [0,1]$, if $y \in f(\mu_A^t)$ then $\mu_A(y) = \mu_A(f(x))$ for some $x \in \mu_A^t$ which implies that $\mu_A(y) = \mu_A(x) \geq t$. It follows that $y \in \mu_A^t$. Conversely if $y \in \mu_A^t$ then $\mu_A(y) \geq t$ implies $t \leq \mu_A(y) = \mu_A(f(x)) = \mu_A(x)$, for some $x \in M$ with $y = f(x)$. It follows that $y \in f(\mu_A^t)$. Thus $(\mu_A^t) = \mu_A^t$. In a similar manner, we can prove that $f(\gamma_A^{t'}) = \gamma_A^{t'}$. Hence μ_A^t and $\gamma_A^{t'}$ are characteristic.

CHAPTER V

Intuitionistic Fuzzy k- bi- ideal of Boolean like semi rings:

Definition 5.1

An IFS $A = (\mu_A; \gamma_A)$ in R called an intuitionistic fuzzy subgroup of R if

(i) $\mu_A(x - y) \geq \wedge \{\mu_A(x), \mu_A(y)\}$

(ii) $\gamma_A(x - y) \leq \vee \{\gamma_A(x), \gamma_A(y)\}$, for all $x, y \in R$

Definition 5.2

An intuitionistic fuzzy subgroup $A = (\mu_A; \gamma_A)$ of R is called an intuitionistic fuzzy bi- ideal of R if

(i) $\mu_A(xyz) \geq \wedge \{\mu_A(x), \mu_A(z)\}$

(ii) $\gamma_A(xyz) \leq \vee \{\gamma_A(x), \gamma_A(z)\}$, for all $x, y, z \in R$

Definition 5.3

An intuitionistic fuzzy bi- ideal $A = (\mu_A; \gamma_A)$ of R is called an intuitionistic fuzzy k- bi- ideal of R if for all $x, y \in R$.

(i) $\mu_A(x) \geq \wedge \{ \vee \{\mu_A(x+y), \mu_A(y+x), \mu_A(y)\}$

(ii) $\gamma_A(x) \geq \vee \{ \wedge \{\gamma_A(x+y), \gamma_A(y+x), \gamma_A(y)\}$.

Theorem 5.4

An IFS $A = (\mu_A; \gamma_A)$ in R is an intuitionistic fuzzy k- bi- ideal of R. Then the fuzzy sets μ_A and γ_A' are fuzzy k- bi- ideal of R.

Proof

Let $A = (\mu_A; \gamma_A)$ be an intuitionistic fuzzy k- bi- ideal of R. Then clearly μ_A is a fuzzy k- bi- ideal of R. We claim that γ_A' is a fuzzy k- bi- ideal of M. Let $x, y \in R$.

Then, $\gamma_A'(x-y) = 1 - \gamma_A(x-y)$

$\geq 1 - \vee \{\gamma_A(x), \gamma_A(y)\}$

$= \wedge \{1 - \gamma_A(x), 1 - \gamma_A(y)\}$

$= \wedge \{\gamma_A'(x), \gamma_A'(y)\}$.

Let $x, y, z \in R$. Then

$\gamma_A'(xyz) = 1 - \gamma_A(xyz)$

$\geq 1- \vee\{1 - \gamma_A(x), 1 - \gamma_A(z)\}$

$= \wedge \{1 - \gamma_A(x), 1 - \gamma_A(z)\}$

$= \wedge \{\gamma_A'(x), \gamma_A'(y)\}$.

Now, $\gamma_A'(x) = 1 - \gamma_A(x)$

$\geq 1 - \vee \{ \wedge \{\gamma_A(x+y), \gamma_A(y+x)\}, \gamma_A(y)\}$

$= \wedge\{1- \wedge\{\gamma_A(x+y), \gamma_A(y+x)\}, 1- \gamma_A(y)\}$

$=\wedge \{ \vee \{1 - \{\gamma_A(x+y), 1- \gamma_A(y+x)\}, 1- \gamma_A(y)\}$

$=\wedge \{ \vee \{\gamma_A'(x+y), \gamma_A'(y+x)\}, \gamma_A'(y)\}$.

Hence γ_A' is a fuzzy k- bi- ideal of R.

Theorem 5.5

Let $f: R \rightarrow R'$ be a homomorphism of Boolean like semi rings. If $B= (\mu_B; \gamma_B)$ is an intuitionistic fuzzy k- bi- ideal of R' then the preimage $f^{-1}(B) = (f^{-1}(\mu_B); f^{-1}(\gamma_B))$ of B under f is an intuitionistic fuzzy k- bi- ideal of R.

Proof

Assume that $B= (\mu_B; \gamma_B)$ is an intuitionistic fuzzy k- bi- ideal of R'. Let $x, y \in R$.

Then, $f^{-1}(\mu_B)(x- y) = \mu_B(f(x- y))$

$= \mu_B(f(x)- f(y))$

$\geq \wedge\{\mu_B(f(x)), \mu_B(f(y))\}$

$= \wedge \{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\}$

$$f^{-1}(\gamma_B)(x-y) = \gamma_B(f(x-y))$$

$$= \gamma_B(f(x)-f(y))$$

$$\leq \vee \{\gamma_B(f(x)), \gamma_B(f(y))\}$$

$$= \vee \{f^{-1}(\gamma_B)(x), f^{-1}(\gamma_B)(y)\}$$

Which implies that $f^{-1}(B) = (f^{-1}(\mu_B); f^{-1}(\gamma_B))$ is an intuitionistic fuzzy subgroup of R.

Let $x,y,z \in R$. Then,

$$f^{-1}(\mu_B)(xyz) = \mu_B(f(xyz))$$

$$= \mu_B(f(x)f(y)f(z))$$

$$\geq \wedge \{\mu_B(f(x)), \mu_B(f(z))\}$$

$$= \wedge \{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(z)\}$$

$$f^{-1}(\gamma_B)(xyz) = \gamma_B(f(xyz))$$

$$= \gamma_B(f(x)f(y)f(z))$$

$$\leq \vee \{\gamma_B(f(x)), \gamma_B(f(z))\}$$

$$= \vee \{f^{-1}(\gamma_B)(x), f^{-1}(\gamma_B)(z)\}$$

Therefore $f^{-1}(B)$ is intuitionistic fuzzy bi-ideal of R. Now for $x,y \in R$, we have

$$f^{-1}(\mu_B)(x) = \mu_B(f(x))$$

$$\geq \wedge \{\vee \{\mu_B(f(x)+f(y)), \mu_B(f(y)+f(x))\}, \mu_B(f(y))\}$$

$$= \wedge \{\vee \{\mu_B(f(x+y)), \mu_B(f(y+x))\}, \mu_B(f(y))\}$$

$$= \wedge \{\vee \{f^{-1}(\mu_B)(x+y), f^{-1}(\mu_B)(y+x)\}, f^{-1}(\mu_B)(y)\}$$

$$f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$$

$$\leq \vee \{ \wedge \{ \gamma_B(f(x)+f(y)), \gamma_B(f(y)+f(x))\}, \gamma_B(f(y))\}$$

$$= \vee \{ \wedge \{ \gamma_B(f(x+y)), \gamma_B(f(y+x))\}, \gamma_B(f(y))\}$$

$$= \vee \{ \wedge \{f^{-1}(\gamma_B)(x+y), f^{-1}(\gamma_B)(y+x)\}, f^{-1}(\gamma_B)(y)\}$$

Hence $f^{-1}(B) = (f^{-1}(\mu_B);f^{-1}(\gamma_B))$ of B under f is an intuitionistic fuzzy k- bi- ideal of M.

Definition 5.6

Let $A= (\mu_A;\gamma_A)$ be an intuitionistic fuzzy set in R. Then the intuitionistic level subset of A is an object having the form $(\mu_A^t;\gamma_A^{t'})$ where $\mu_A^t = \{x \in R/\mu_A(x) \geq t\}$ and $\gamma_A^{t'} = \{x \in R/\gamma_A(x) \geq t'\}$.

Theorem 5.7

An IFS $A= (\mu_A;\gamma_A)$ in R is an intuitionistic fuzzy k- bi- ideal of R if and only if the components $\mu_A^t \neq 0$ for any $t \in Im\mu_A$ and $\gamma_A^{t'} \neq 0$ for any $t' \in Im\gamma_A$ of the intuitionistic level subset are k- bi- ideals of R.

Proof

Assume that $A= (\mu_A;\gamma_A)$ in R is an intuitionistic fuzzy k- bi- idealof R. Fix $t \in Im\mu_A$ such that $\mu_A^t \neq 0$and $t' \in Im\gamma_A$ such that $\gamma_A^{t'} \neq 0$.

Suppose that, $x,y \in \mu_A^t$. Then $\mu_A(x) \geq t$; $\mu_A(y) \geq t$ which implies $\wedge\{\mu_A(x),\mu_A(y)\} \geq t$. We get $\mu_A(x - y) \geq \wedge\{\mu_A(x),\mu_A(y)\} \geq t$ which implies that $x - y \in \mu_A^t$.

Let $x,z \in \mu_A^t$; $y \in R$. Then, $\mu_A(x) \geq t$ and $\mu_A(z) \geq t$. Thus $\mu_A(x\alpha y\beta z) \geq \wedge\{\mu_A(x),\mu_A(z)\} \geq t$ implies $xyz \in \mu_A^t$.

Let $y \in \mu_A^t$. Also let $x + y \in \mu_A^t$ (or) $y + x \in \mu_A^t$. Then $\mu_A(y) \geq t$ and $\mu_A(x + y) \geq t$ (or) $\mu_A(y + x) \geq t$. Now $\mu_A(x) \geq \wedge \{ \vee \{\mu_A(x+y),\mu_A(y+x)\},\mu_A(y)\} \geq t$ which implies that $x \in \mu_A^t$. Therefore μ_A^t is a k- bi- ideals of R.

Similarly let $x,y \in \gamma_A^{t'}$. Then $\gamma_A(x) \le t'$; $\gamma_A(y) \le t' \Rightarrow \vee \{\gamma_A(x),\gamma_A(y)\} \le t'$. Thus we have $\gamma_A(x-y) \le \vee \{\gamma_A(x),\gamma_A(y)\} \le t'$ implies that $x-y \in \gamma_A^{t'}$.

Let $x,z \in \gamma_A^{t'}; y \in R$. Then $\gamma_A(x) \le t'$; $\gamma_A(z) \le t'$. We have $\gamma_A(xyz) \le \vee\{\gamma_A(x),\gamma_A(z)\} \le t'$ implies that $xyz \in \gamma_A^{t'}$.

Let $y \in \gamma_A^{t'}$. Also let $x + y \in \gamma_A^{t'}$ (or) $y + x \in \gamma_A^{t'}$. Then $\gamma_A(y) \le t'$ and $\gamma_A(x + y) \le t'$ (or) $\gamma_A(y + x) \le t'$. Now $\gamma_A(x) \ge \vee \{ \wedge \{\gamma_A(x+y),\gamma_A(y+x),\gamma_A(y)\} \le t'$ which implies that $x \in \gamma_A^{t'}$. Hence $\gamma_A^{t'}$ is also a k- bi- ideals of R.

Conversely , Assume that the components μ_A^t and $\gamma_A^{t'}$ of the intuitionistic level subset are k- bi- ideals of R.

Fix any $x,y \in R$ and let $\mu_A(x) = t_1$, $\mu_A(y) = t_2$. Let $t = \wedge\{t_1,t_2\}$. Then $\mu_A(x) \ge t$ and $\mu_A(y) \ge t$ implies $x,y \in \mu_A^t$ which implies that $x-y \in \mu_A^t$. Therefore $\mu_A(x-y) \ge t = \wedge\{t_1,t_2\} = \wedge\{\mu_A(x),\mu_A(y)\}$.

Fix any $x,y \in R$ and let $\gamma_A(x) = t_3$, $\gamma_A(y) = t_4$. Let $t' = \vee\{t_3,t_4\}$. Then $\gamma_A(x) \le t'$; $\gamma_A(y) \le t'$ implies $x,y \in \gamma_A^{t'}$ which implies $x-y \in \gamma_A^{t'}$. Therefore $\gamma_A(x-y) \le t' = \vee\{t_3,t_4\} = \vee\{\gamma_A(x),\gamma_A(y)\}$.

Let $\mu_A(x) = t_1$, $\mu_A(z) = t_2$. Let $t = \wedge\{t_1,t_2\}$. Then $\mu_A(x) \ge t$ and $\mu_A(z) \ge t$ implies $x,z \in \mu_A^t$ which implies that $xyz \in \mu_A^t$. Therefore $\mu_A(xyz) \ge t = \wedge\{t_1,t_2\} = \wedge\{\mu_A(x),\mu_A(z)\}$.

Next let $\gamma_A(x) = t_3$ and $\gamma_A(y) = t_4$. Let $t' = \vee\{t_3,t_4\}$. Then $\gamma_A(x) \le t'$; $\gamma_A(y) \le t'$ implies $x,y \in \gamma_A^{t'}$ which implies $xyz \in \gamma_A^{t'}$. Therefore $\gamma_A(xyz) \le t' = \vee\{t_3,t_4\} = \vee \{\gamma_A(x),\gamma_A(z)\}$.

Thus $A= (\mu_A;\gamma_A)$ in R is an intuitionistic fuzzy bi- ideal of R.

For any $x, y \in R$, let $\mu_A(y) = t_1$; $\mu_A(x+y) = t_2$; $\mu_A(y+x) = t_3$($t_i \in Im\gamma_A$). If we let $t = \wedge\{\vee\{t_2, t_3\}, t_1\}$. Then $\mu_A(y) \geq t$ and $\mu_A(x+y) \geq t$ (or) $\mu_A(y+x) \geq t$ implies $y \in \mu_A^t$ and $x+y \in \mu_A^t$ (or) $y+x \in \mu_A^t$. Since μ_A^t is a k- bi- ideal of R, we get $x \in \mu_A^t$. (i.e)., $\mu_A(x) \geq t = \wedge\{\vee\{t_2, t_3\}, t_1\} = \wedge\{\vee\{\mu_A(x+y), \mu_A(y+x)\}, \mu_A(y)\}$.

For any $x, y \in R$, let $\gamma_A(y) = t_4$; $\gamma_A(x+y) = t_5$; $\gamma_A(y+x) = t_6$ ($t_i \in Im\gamma_A$). If we let $t' = \vee\{\wedge\{t_5, t_6\}, t_4\}$. Then $\gamma_A(y) \leq t'$ and $\gamma_A(x+y) \leq t'$ (or) $\gamma_A(y+x) \leq t'$ which implies that $y \in \gamma_A^{t'}$ and $x+y \in \gamma_A^{t'}$ (or) $y+x \in \gamma_A^{t'}$ implies that $x \in \gamma_A^{t'}$ [since $\gamma_A^{t'}$ is a k- bi- ideal of R]. Therefore $\gamma_A(x) \leq t' = \vee\{\wedge\{t_5, t_6\}, t_4\} = \vee\{\wedge\{\gamma_A(x+y), \gamma_A(y+x)\}, \gamma_A(y)\}$.

Hence A= $(\mu_A; \gamma_A)$ in R is an intuitionistic fuzzy k- bi- ideal of R.

A STUDY ON BOOLEAN LIKE GAMMA NEAR-RINGS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

T. KAVITHA

Reg.No: 19SPMT14

Under the guidance of

Ms. M. KANAGA M.Sc., B.Ed., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON BOOLEAN LIKE GAMMA NEAR-RINGS** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by

T. KAVITHA (Reg. No: 19SPMT14)

M. Kanagi

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous,
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON BOOLEAN LIKE GAMMA NEAR-RINGS"** submitted for the degree of Master of Science is my original work carried out under the guidance of **Ms. M. Kanaga M.Sc., B.Ed., SET.,** Assistant Professor, Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi.**

T. Kavitha.

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

1. PRELIMINARIES

Definition: 1.1

Let A and B be non-empty sets, then any subset R of the Cartesian product $A \times B$ is called a binary relation from A to B, **binary relation** is simply called a relation.

We shall write $a\ R\ b$ for $(a, b) \in R$.

Definition: 1.2

A binary relation R on a non-empty set P is called a **partial order** if for all $a, b, c \in P$, R is reflexive ($a\ R\ a$), anti-symmetric ($a\ R\ b, b\ R\ a \Rightarrow a\ = b$) and transitive ($a\ R\ b, b\ R\ c \Rightarrow a\ R\ c$).

Definition: 1.3

A non-empty set P together with a partial order R on it is called a **partially ordered set** or a **poset**. R is usually denoted by "$\leq$ " , we denote the poset by (P,$\leq$).

Definition: 1.4

Let (P, $\leq$) be a partially ordered set. If for every a, b $\in P$, we have $a \leq b$ or $b \leq a$ then "$\leq$ " is called a simple ordering on P, and the poset (P,$\leq$) is called a **totally ordered set** or a **chain.**

.Definition: 1.5

Let $(P,\leq)$ be a partially ordered set. An element $a \in P$ is called a **maximal element** if for $x \in P$, $a \leq x \Rightarrow a = x$.

Definition: 1.6

Let $(P,\leq)$ be a partially ordered set. An element $b \in P$ is called a **minimal element** if for $x \in P, x \leq b \Rightarrow b = x$.

Proposition: 1.7

Let P be a finite non-empty poset with partial order "$\leq$". Then P has at least one maximal element. Similarly P has the least one minimal element.

Definition: 1.8

Let $(P,\leq)$ be a partially ordered set and let $Q \subseteq P$. An element $m \in P$ is called an **upper bound** for Q if for all $x \in Q, x \leq m$. An element $\ell \in P$ is called a **lower bound** for Q if for all $x \in Q$, $\ell \leq x$.

Definition: 1.9

Let $(P,\leq)$ be a partially ordered set and let $Q \subseteq P$. An element $m \in P$ is called a **least upper bound** for Q if 'm' is an upper bound of Q and $m \leq m'$, whenever m' is an upper bound of Q. In this case we write $l.u.b\ Q = m = \sup Q = \vee Q$. In particular , for $m, n \in P$, we denote by $l.u.b\ \{m, n\}$ by $m \vee n$. An element $\ell \in P$is called a **greatest lower bound** for Q if 'ℓ' is a lower bound of Q and $\ell' \leq \ell$ whenever ℓ' is a lower bound of Q.

We write it as $g.l.b\ Q = \ell = \inf\ Q = \wedge Q$. For m,n ∈ P, g.l.b {m, n} is denoted by $m \wedge n$.

Definition: 1.10

Let $(S, \leq)$ be a partially ordered set. S is said to be a **join semi-lattice** if every non-empty finite subset of S has a least upper bound in S.

Definition: 1.11

Let $(S, \leq)$ be a partially ordered set. S is said to be a **meet semi-lattice** if every non-empty finite subset of S has a greatest lower bound in S.

Note: 1.12

If S is a join semi-lattice then the binary operation $\vee$ is commutative, associative and idempotent, similarly if S is a meet semi-lattice then the binary operation $\wedge$ is commutative, associative and idempotent. Further $m \leq n\ iff\ m \vee n = n\ iff\ m \wedge n = m$.

Definition: 1.13

Let $(L, \leq)$ be a partially ordered set. L is said to be a **lattice** if L is both join and meet semi-lattice. A lattice is a partiallay ordered set $(L, \leq)$ in which every pair of elements has a least upper bound and a greatest lower bound.

Definition: 1.14

A lattice $(L, \leq)$ is called a **distributive lattice** if it satisfies any one of the following equivalent conditions:

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $for\ all\ a, b, c \in L$.

(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $for\ all\ a, b, c \in L$.

Definition: 1.15

Complemented distributive lattices are called **Boolean algebras**.

Definition: 1.16

Relatively complemented distributive lattices bounded below are called **Generalized Boolean Algebras**.

Definition: 1.17

A non-empty set N is said to be a **near-ring**, if in N there are defined two operations, denoted by $+$ and $\cdot$ respectively such that for all n, n_1, n_2, n_3 in N:

(1) $n_1 + n_2$ is in N

(2) $(n_1 + n_2) + n_3 = n_1 + (n_2 + n_3)$

(3) There is an element 0 in N such that $n + 0 = 0 + n = n$

(4) There exists an element $-n$ in N such that $n + (-n) = 0 = (-n) + n$

(5) $n_1 . n_2$ is in N.

(6) $(n_1 \cdot n_2) \cdot n_3 = n_1 \cdot (n_2 \cdot n_3)$

(7) $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$ (right distributive law)

Definition:1.18

An element x of $[a, b]$ is called a relative complement of u in $[a, b]$ if $u \wedge x = a$ and $u \vee x = b$. A lattice L is said to be **relatively complemented** if for any triplet of its elements a, b, u $(a \leq u \leq b)$, there can be found at least one complemented of u in $[a, b]$; in other words if every interval of L is a complemented sub lattice.

2. BOOLEAN LIKE RINGS

Definition: 2.1

Let R be a commutative ring with unit element. Let $\otimes$ be the binary operation defined by $a \otimes b = a + b - ab$ for all $a, b \in R$, this binary operation $\otimes$ is called the **dual ring product** of R. For any, $a, b \in R, a \otimes b$ is also denoted by $a \, \Delta \, b$. The ring product is also called as the **logical product** in R. The dual ring product is also called as the **logical sum** in R.

Definition: 2.2

Let R be a commutative ring with unit element. An unary operation $*$ defined on R by $a^* = 1 - a$ for all $a \in R$ is called the **ring complement** of R.

Remark: 2.3

In a commutative ring R with unit element, the following notations are

followed:

a) For any $a, b \in R, a + b$ is called as the **ring sum** of a and b.
b) For any $a, b \in R, a \oplus b = a + b - 1$ is called the **dual ring sum** of a and b.
c) For any $a, b \in R$, $a - b$ is called the **ring subtraction** of a and b.
d) For any $a, b \in R$, $a \, \Theta b = a + b - 1$ is called the **dual ring subtraction** of a and b.

Theorem: 2.4

If $P = P\,(0, 1, \times, \Delta, *, +, \oplus, -, \Theta)$ is any true preposition of the ring R, then its dual preposition $P' = P(1, 0, \Delta, \times, *, \oplus, +, \Theta, -)$ obtained by replacing each operation in P by its dual operation is also a true preposition.

Example: 2.5

If P is the preposition in a ring R given by $a\,(b + c) = ab + ac$, then its dual preposition is given by $a \,\Delta\, (b \oplus c) = a \,\Delta\, b \oplus a \,\Delta\, c$.

Example: 2.6

In a commutative ring R with unit element, the dual of the preposition $a \,\Delta\, b = a + b - ab$ is given by $a \times b = (a \oplus b)\, \Theta\, (a \,\Delta\, b)$ for any $a, b \in R$.

Definition: 2.7

Let $(R, +, \times)$ be a commutative ring with unit element. Then the operations $\times, \Delta, *, 0$ and 1 are called the **basic logical concepts** of R.

Definition: 2.8

Let $\times, \Delta, *, 0$ and 1 be the basic logical concepts of a ring R. Then the system

(R, $\times$, Δ, $*$, 0, 1) is called the **logical algebra** of the ring R.

Remark: 2.9

Let $(R, \times, \Delta, *, 0, 1)$ be the logical algebra of the ring $(R, +, \times)$ then the following are true.

a) $(R, \times)$ is a closed, commutative, associative system in which the null element 0 and the universe element 1 of the ring satisfy, $0 \cdot a = a \cdot 0$; $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

b) (R, Δ) is a closed commutative, associative system in which the null element 0 and the universe element 1 of the ring satisfy, $0 \,\Delta\, a = a \,\Delta\, 0$; $1 \,\Delta\, a = a \,\Delta\, 1 = 1$ for all $a \in R$

c) $(a^*)^* = a$, $1^* = 0$ and $a^* = b^*$ implies $a = b$ for all $a, b \in R$.

d) $(ab)^* = a^* \,\Delta\, b^*$ for all $a, b \in R$.

e) $(a \,\Delta\, b)^* = a^* b^*$ for all $a, b \in R$.

f) $a(b \,\Delta\, c) = ab \,\Delta\, ac$ if and only if $aa^* \, bc = 0$ for all $a, b \in R$.

g) $a \,\Delta\, bc = (a \,\Delta\, b)(a \,\Delta\, c)$ if and only if $a \,\Delta\, a^* \,\Delta\, b \,\Delta\, c = 1$ if and only if $aa^* b^* c^* = 0$ for all $a, b, c \in R$.

Theorem: 2.10

In any commutative ring R with unit element, for any $a, b \in R$ $ab(a \,\Delta\, b) = ab$ if and only if $aa^* bb^* = 0$.

Proof:

Let R be a commutative ring with unit element. Let $a, b \in R$.

Assume that $ab\,(a \,\Delta\, b) = ab$

$\Rightarrow \quad ab\left(a + b - ab\right) = ab$

$\Rightarrow \quad aba + abb + abab = ab$

$\Rightarrow \; aba + abb - abab - ab = 0$

Now we have $aa^{*}bb^{*} = a\left(1 - a\right)b\,(1 - b)$

$= \left(a - aa\right)(b - bb)$

$= ab - abb - aab + aabb$

$= 0.$

Conversely, suppose that $aa^{*}bb^{*} = 0$

$\Rightarrow \quad a\left(1 - a\right)b\left(1 - b\right) = 0$

$\Rightarrow \quad ab - abb - aab + aabb = 0$

$\Rightarrow \quad abb + aab - aabb = ab$

Now we have that

$ab\,(\,a\,\Delta\,b\,) = ab\,(\,a\;+b - ab\,)$

$= aba\;+abb - aab$

$= ab.$

Hence the theorem.

Definition: 2.11

Let $(R, +, *, -, 0, 1)$ be a Boolean ring with unit element 1. Let $(R, \times, \Delta, *, 0, 1)$ be the logical algebra of the Boolean ring. Now the complete ring $(R, +, \oplus, \times, \Delta, *)$ can be defined in terms of **the logical operations** as follows:

$a\;+b\;= ab^*\,\Delta\,a^*b$ for all $a, b\;\in R$.

$a\;\oplus b\;=\;(\,a\,\Delta\,b^*\,)(\,a^*\,\Delta\,b\,)$ for all $a, b\;\in R$.

Definition: 2.12:

A commutative ring with unit element is called **Boolean like ring** if the complete ring is logically definable in which the addition + is defined by $a\;+b\;= ab^*\,\Delta\,a^*b$ for all $a, b\;\in R$.

Remark: 2.13

We know that a commutative ring with unit element in which every element is idempotent, is $a^2 = a$ for all $a \in R$ is Boolean ring and vice-versa. But in the case of Boolean like ring it is not so.

Definition: 2.14

A commutative ring with unit element is said to be a **Boolean like ring** if for all elements $a, b \in R$, $aa^*bb^* = 0$ and $a + a = 0$ i.e., $a = -a$.

Theorem: 2.15

If R and R' are both Boolean like rings then the direct product $R \times R'$ is also a Boolean like ring.

Proof:

Let $(R, +, \times)$ and $(R', +, \times)$ be two Boolean like rings. Since R and R' are commutative rings with identity, it follows that $R \times R'$ is also a commutative ring with identity with usual addition and multiplication.

Now we show that $R \times R'$ is a Boolean ring.

Let $(a, a'), (b, b') \in R \times R'$.

Since $a, b \in R$ we have $a + b = ab^* \Delta a^*b$

Since $a', b' \in R'$ we have $a' + b' = a' (b')^* \Delta (a')^* b'$

Now $(a, a') (b, b')^* \Delta (a, a')^* (b, b')$

$$(a,a')\,(b,b')^*\,\Delta\,(a,a')^*\,(b,b') = (a,a')[(1,1)-(b,b')]\,\Delta[(1,1)-(a,a')](b,b')$$

$$= (a,a')(1-b,1-b')\,\Delta\,(1-a,1-a)'(b,b')$$

$$= (a(1-b),a'(1-b'))\,\Delta\,((1-a)b,(1-a')b'$$

$$= (ab^*,a'(b')^*)\,\Delta\,(a^*,(a')^*\,(b,b')$$

$$= (ab^*,a'(b)^*) + (a^*b,(a')^*b' - (ab^*,a'(b')^*(a^*b,(a')^*b'))$$

$$= (ab^*\Delta\, a^*b, a'(b')\Delta(a')^*b')$$

$$= (a+b,a'+b')$$

$$= (a,a') + (b,b').$$

Therefore it follows that $R \times R'$ is a Boolean ring.

Definition: 2.16

An element 'a' of a Boolean like ring is said to be an **idempotent** if $a^2 = a$.

Definition: 2.17

An element 'a' of a Boolean like ring is said to be a **nilpotent element** if $a^n = 0$ for some $n \geq 1$.

Definition: 2.18

An element 'a' of a Boolean like ring is said to be an **unipotent element** if $a^{[n]} = a\Delta a\Delta a\Delta \ldots . \Delta a(n - factors) = 1$ for some n ≥ 1.

Remark: 2.19:

For any element $a \in R, a^n = 0$ for some n iff $a^{[n]} = 1$.

Theorem: 2.20

An element 'a' of a Boolean like ring R is

1) nilpotent only if $a^2 = 0$.
2) Unipotent only if $a^2 = 1$.
3) Idempotent only if $a^2 = a$.

Proof:

(1) Suppose 'a' is nilpotent.

Let n be the least positive integer such that $a^n = 0$.

Since $a^{n+4} a^{n+2}$, it follows that this n must be either 1 or 2 or 3.

If $a = 0$ then clearly $a^n = 0$

Suppose $a \neq 0$

If $n = 2$ clearly $a^2 = 0$

If $n = 3 \Rightarrow a^3 = 0$

$\Rightarrow a^4 = a^3 a = 0a = 0$

$\Rightarrow a^2 = a^4 = 0$

Thus in any case $a^2 = 0$.

(2) Suppose 'a' is unipotent

$\Rightarrow a^*$ is nilpotent.

$\Rightarrow (a^*)^2 = 0$

$\Rightarrow a^* a^* = 0$

$\Rightarrow (a \, \Delta \, a)^* = 0$

$\Rightarrow ((a \, \Delta \, a)^*) = 0^* = 1$

$\Rightarrow a \, \Delta \, a = 1$

$\Rightarrow a^2 = 1$

(3) If 'a' is idempotent then clearly $a^2 = a$

Theorem: 2.21

In any Boolean like ring R, for any nilpotent elements a, a', we have that $aa' = 0$ if and only if $a \,\Delta\, a' = a + a'$.

Proof:

Let a, a' be any nilpotent elements of R.

Suppose that $aa' = 0$

Now $a \,\Delta\, a' = a + a' + aa' = a + a'$

Conversely suppose that $a \,\Delta\, a' = a + a'$

Now $a + a' = a(a')^* \,\Delta\, a^*a'$ (since $a + b = ab^* \,\Delta\, a^*b$ for all $a, b \in R$)

$= a\,(1 - a') \,\Delta\, (1 - a)a'$ (since $a^* = 1 - a$)

$= a\,(1 - a') + (1 - a)a' - aa'\,(1 - a)(1 - a')$ ($\because a \,\Delta\, b = a + b - ab$)

$= a - aa' + a' - aa' - aa'\,(1 - a - a' + aa')$

$= a + a' - aa'\,(1 - a - a' + aa')$

$= a + a' - aa'$

$\Rightarrow aa' = 0$

Thus we have $aa' = 0$ if and only if $a \,\Delta\, a' = a + a'$

ODERING ON BOOLEAN LIKE-RINGS

Definition: 2.21

Let R be a Boolean like-ring. Define a relation " $\leq$ " on R by $a \leq b$ if $a = ab$.

Proposition: 2.22

Let R be a Boolean like-ring. If $a \, \Delta \, a = a$ for every $a \in R$, then $(R, \leq)$ is a partially ordered set.

Proof:

For any $a \in R$ we have $a \, \Delta \, a = a$

$\Rightarrow \; a + a - aa = a$

$\Rightarrow \quad a - a^2 = 0$

$\Rightarrow \quad a = a^2$

$\Rightarrow \quad a = aa$

$\Rightarrow a \leq a$

Thus " $\leq$ " is reflexive.

Suppose $a \leq b$ and $b \leq a$ then $a = ab$ and $b = ba$

Now $a = ab$

$= ba$

$= b.$

Therefore $a = b$.

Thus " $\leq$" is anti-symmetric.

Suppose $a \leq b$ and $b \leq c$ then $a = ab$ and $b = bc$.

Now $a = ab$

$$a = abc$$

$$= ac$$

$$\Rightarrow a \leq c.$$

Thus " $\leq$ " is transitive.

Therefore $(R, \leq)$ is a partially ordered set.

Theorem: 2.23

Let R be a Boolean like-ring and $a \Delta a = a$ for every $a \in R$. Then the poset R is a lattice in which $a \wedge b = ab$ and $a \vee b = a \Delta b + 2ab$.

Proof:

Now we show that R is a lattice.

Claim: (i) $g.l.b \{a, b\} = a \wedge b = ab$.

Let $c = ab$

Consider $ac = a(ab)$

$$= aab$$

$$= ab$$

$$= c$$

$$\Rightarrow \quad c \leq a$$

Similarly, Consider $bc = b(ab)$

$$= abb$$

$$= ab$$

$$= c$$

$$\Rightarrow \quad c \leq b$$

Suppose $d \leq a$ and $d \leq b$

$d \leq a \Rightarrow d = da$

$d \leq b \Rightarrow d = db$

Now $dd = dadb$

$\Rightarrow \quad d^2 = addb$

$\Rightarrow \quad d = adb$

$\Rightarrow \quad d = dab$

$\Rightarrow \quad d = dc$

$\Rightarrow \quad d \leq c.$

Thus $g.l.b\ \{a, b\} = c.$

Claim: (ii) $l.u.b\{a, b\} = a \vee b = a \, \Delta \, b + 2ab$

Let $c = a \, \Delta \, b + 2ab$

Now $ac = a(a \, \Delta \, b + 2ab)$

$= a\big(a + b - ab\big) + a2ab$

$= aa + ab - aab + 2aab$

$= a + ab + aba$

$= a + ab + ab$

$= a.$

$\Rightarrow \quad a \leq c.$

Similarly, Consider $bc = b(a \, \Delta \, b + 2ab)$

$= b(a \, \Delta \, b) + b2ab$

$= b\big(a + b - ab\big) + 2bab$

$$bc = ba + bb - bab + 2a$$

$$= ba + bb - abb + 2abb$$

$$= ba + b - ab + 2ab$$

$$= ba + b + ab$$

$$= b.$$

$\Rightarrow b \leq c.$

Suppose $a \leq d$ and $b \leq d$

$a \leq d \Rightarrow a = ad$

$b \leq d \Rightarrow b = bd$

Now $ab = adbd$

Consider $cd = (a \Delta b + 2ab)d$

$$= (a \Delta b)d + (2ab)d$$

$$= (a + b - ab)d + 2abd$$

$$= ad + bd - abd + 2abd$$

$$= ad + bd + abd$$

$$= a + b + ab$$

$$= a \Delta b + 2ab$$

$$= c.$$

$\Rightarrow \quad c \leq d.$

Thus $l.u.b\{a, b\} = c.$

Hence R is a lattice in which $a \wedge b = ab$ and $a \vee b = a \Delta b + 2ab$.

Theorem: 2.24

Let R be a Boolean like-ring. Then the lattice R is distributive.

Proof:

Now we show that the lattice R is a distributive lattice.

Claim: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Consider $a \wedge (b \vee c) = a\,(b\,\Delta\,c + 2bc)$

$= a\,(b\,\Delta\,c) + 2abc$

$= a\,(b + c - bc) + 2abc$

$= ab + ac - abc + 2abc$

$= ab + ac + abc$

$(a \wedge b) \vee (a \wedge c) = (a \wedge b)\Delta\,(a \wedge c) + 2\,(a \wedge b)$

$= (ab\,\Delta\,ac) + 2(ab)(ac)$

$= ab + ac + abac$

$= ab + ac + abc.$

Therefore $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Thus R is a distributive lattice.

Theorem: 2.25

Let R be a Boolean like-ring with identity 1 and $a\,\Delta\,a = a$ for every $a \in R$. Then R is a Distributive lattice in which $a \wedge b = ab$ and $a \vee b = a\,\Delta\,b + 2ab$. This distributive lattice R is ring complemented, where the ring complement of $a \in R$ is the element $a^* = 1 - a$. In otherwords R is a Boolean algebra.

Proof:

R is a distributive lattice in which $a \wedge b = ab$ and $a \vee b = a \Delta b + 2ab$. In order to show that this distributive lattice R is ring complemented, it is enough to check that the ring complement of every element a in R is the element $a^* = 1 - a$.

Claim: (i) $a \wedge a^* = 0$

Consider $a \wedge a^* = a(1-a)$

$= a - aa$

$= a - a$

$= 0.$

Therefore $a \wedge a^* = 0$.

Claim: (ii) $a \vee a^* = 1$

Consider $a \vee a^* = a\Delta a^* + 2aa^*$

$= a + a^* - aa^* + 2aa^*$

$= a + a^* + aa^*$

$= a + (1-a) + a(1-a)$

$= a + 1 - a + a - aa$ (since $a \cdot a = a$)

$= 1.$

$a \vee a^* = 1.$

Therefore a^* is a ring complement of a.

Here R is a complemented distributive lattice. i.e.R is a Boolean algebra.

Theorem: 2.26

Let R be any Boolean like-ring (with or without identity) and define in R a meet and a join operations by the formulae $a \wedge b = ab$ and $a \vee b = a + b + ab$, then a generalized Boolean algebra is obtained.

Proof:

Let R be any Boolean likr-ring and $a \Delta a = a$ for every $a \in R$. By theorems 2.23,2.24 and 2.25, if $a \leq b \Rightarrow a = ba$ then $(R, \leq)$ is a distributive lattice in which $a \wedge b = ab$ and $a \vee b = a \Delta b + 2ab$.

In order to show that this distributive lattice $(R, \leq)$ is a generalized Boolean algebra, it is enough to check that for any triplet of its elements a, b, u $(a \leq u \leq b)$, $u \vee x = b$ for some element x in

$[a, b]$.

If there were an element $x \in [a, b]$ such that $u \wedge x = a$ and $u \vee x = b$, then $u \wedge x = a$

$$\Rightarrow \quad ux = a \qquad (1)$$

Simiarly $u \vee x = b$

$$\Rightarrow \quad (u \Delta x + 2ux) = b$$

$$\Rightarrow \quad u \Delta x + 2a = b \qquad \text{from (1)}$$

$$\Rightarrow \quad u + x - ux + 2a = b$$

$$\Rightarrow \quad u + x - a + 2a = b$$

$$\Rightarrow \; x = u + a + b.$$

Claim(i) $u \wedge x = a$, where $x = u + a + b$.

Now $u \wedge x = ux$

$$= u\,(u + a + b)$$

$$= u^2 + ua + ub$$

$u \wedge x = u + a + u$ (since $a \leq u \leq b$)

$= a.$

Therefore $u \wedge x = a.$

Claim:(ii) $u \vee x = b$, where $x = u + a + b$.

Now $u \vee x = (u \Delta x) + 2ux$

$= \big(u \Delta (u + a + b)\big) + 2u(u + a + b)$

$= u + u + a + b - uu - ua - ub + 2uu + 2ua + 2ub$

$= a + b + u(u + a + b)$

$= a + b + ux$ (since $x = u + a + b$)

$= a + b + a$

$= b.$

Therefore $u \vee x = b$

Therefore R is a relatively complemented distributive lattice. Clearly R is a relatively bounded below (since 0 is the least element). Hence R is a relatively complemented distributive lattice bounded below.

Therefore R is a generalized Boolean algebra.

Theorem: 2.27

Every complemented distributive lattice is relatively complemented.

Proof:

Let L be a complemented distributive lattice and u^* be the complemented of u in L i.e. $u \wedge u^* = 0$ and $u \vee u^* = 1.$

Claim: For any triplet of its elements $a, b, u(a \leq u \leq b), u \wedge u' = a$ and $u \vee u' = b$ for some element $u^{'}$ in $[a, b]$.

Put $u^{'} = (a \vee u^{*}) \wedge b = a \vee (u^{*} \wedge b)$ (by modular law)

Consider $u \wedge u^{'}$

$$u \wedge u^{'} = u \wedge ((a \vee u^{*}) \wedge b)$$

$$= u \wedge (a \vee u^{*}) \qquad \text{(since } u \leq b\text{)}$$

$$= (u \wedge a) \vee (u \wedge u^{*})$$

$$= (u \wedge a) = a. \qquad \text{(since } a \leq u\text{)}$$

Also consider $u \vee u^{'}$

$$u \vee u^{'} = u \wedge (a \vee (u^{*} \wedge b))$$

$$= u \vee (u^{*} \wedge b)$$

$$= (u \vee u^{*}) \wedge (u \vee b)$$

$$= (1 \wedge b)$$

$$= b.$$

Hence L is relatively complemented.

3. BOOLEAN LIKE NEAR- RINGS

Definition: 3.1

A non-empty set N together with two binary operations " $+$ " (called addition) and " $\cdot$ " (called multiplication) is said to be a **Near-ring** if

(i) $(N, +)$ is a group (not necessarily abelian)

(ii) $(N, \cdot)$ is a semi group

(iii) $(a + b) \cdot c = a \cdot c + b \cdot c \ for \ all \ a, b, c \in N.$ (right distributive law)

The Near-ring is denoted by simply N.

Note: 3.2

(i) The additive identity 0 in N is called the zero element of N.

(ii) For $a, b \in N$, the product $a \cdot b$ is denoted by ab.

(iii) Clearly $0n = 0$ for all $n \in N$.

(iv) The additive inverse of $a \in N$ is denoted by $-a$.

(v) If there exists an element $1 \in N$ such that $a \cdot 1 = 1 \cdot a$ for all $a \in N$.

then we say that N is a near-ring with unit element.

Definition: 3.3

An element $d \in N$ is called a **distributive element** if for $n, n' \in N$, $d\,(n + n') = dn + dn'$.

Definition: 3.4

A Near-ring N is said to be a **Zero-Symmetric** if $n0 = 0$ for all $n \in N$.

Definition: 3.5

A Near-ring N is said to be a **Weak Commutative** if for $all\ x, y, z \in N, xyz = xzy$.

Definition: 3.6

A Near-ring N with unit element is called a Boolean like near-ring provided

for all elements $ab \in N$, $aa * bb * = 0$ and $a \, \Delta \, a = a$.

Lemma: 3.7

If N is a Boolean like near-ring then $ab = aba$. For $a, b \in N$.

Proof:

Let N be a Boolean like near-ring and . $a, b \in N$

Now $(ab - aba)\, a = aba - aba^2$

$= aba - aba$

$= 0.$

Therefore $(ab - aba)a = 0.$ (1)

Also $a\big(ab - aba\big) = (a\,(ab - aba))^2$

$= a\,\big(ab - aba\big)\, a\,(ab - aba)$

$= a0.$ [from (1)]

and $(ab - aba)\, aba = ababa - (aba)^2$

$= (ab)^2\, a - (aba)^2$

$= aba - aba$

$= 0.$

As above, it follows that $aba\,\big(ab - aba\big) = aba0$ (2)

Similarly $ab\left(ab-aba\right) = ab0$

Now $(ab-aba)^2 = \left(ab-aba\right)(ab-aba)$

$$= ab\left(ab-aba\right) - aba\,(ab-aba)$$

$$= ab0 - aba0$$

$$= \left(ab-aba\right)0$$

$$= \left(ab-aba\right)a$$

$$= 0. \qquad \text{[from (1)]}$$

Therefore $ab-aba = 0.$

Hence $ab = aba$ for all $a, b \in N.$

Theorem: 3.8

If N is a Boolean like near-ring then $abc = acb$ for $a, b, c \in N$

i.e. N is weak commutative.

Proof:

Let N be a Boolean like near-ring and $a, b, c \in N.$

Consider $abc-acb = abc - acbc$

$$= \left(a-ac\right)bc$$

$$= \left(a-ac\right)b\left(a-ac\right)c$$

$$= \left(a-ac\right)b0$$

$$= ab0 - acb0 \qquad (1)$$

Replacing b by bc in (1) and by lemma 3.7

$$abc-acb = abc0 - acb0 \qquad (2)$$

From (1) and (2), $ab0 = abc0$ for all $a, b, c \in N$. (3)

Substituting $b = a$ in (3), we have

$aa0 = aac0 \Rightarrow a0 = ac0$ for all $a, c \in N$ (4)

By (1) and (4) we get

$$abc - acb = ab0 - acb0$$

$$= a0 - a0$$

$$= 0.$$

Therefore $abc = acb$.

Hence N is Weak Commutative.

ORDERING ON BOOLEAN LIKE NEAR-RINGS

Definition: 3.9

Let N be a Boolean like near-ring. Define a **relation** " $\leq$ " on N by $a \leq b$ *if* $a = ba$.

Proposition: 3.10

If N is a Boolean like near-ring then $(N, \leq)$ is a partially ordered set.

Proposition: 3.11

Let N be a Boolean like near-ring and $, b \in N$. If $a \leq b$ then $a = ab = ba$.

Proof:

Let N be a Boolean like near-ring and $a, b \in N$ We know by 3.10

$a \leq b \Rightarrow a = ba$

Now $ab = bab$ (since $a = ba$)

$= bba$ (since $abc = acb$)

$ab = ba.$

Therefore $ab = ba$.

Theorem: 3.12

Let N be a Boolean like near-ring. If N has the least element, then the Boolean like near-ring N is zero-symmetric.

Proof:

We claim that every Boolean like ner-ring N with least element is Zero-symmetric.

Suppose N has the least element ℓthen $\ell \leq$ a for all $a \in N \Rightarrow \ell = a\ell$

In particular $\ell \leq 0$, where 0 is the zero element of N

Therefore $\ell = 0\ell$

$\Rightarrow \ell = 0$

Thus 0 is the least element of N.

i.e. $0 \leq a$ for all $a \in N$..

Therefore $a0 = 0a = 0$. (since $a = ab = ba$)

Therefore N is a zero-symmetric Boolean like near-ring.

Theorem: 3.13

Let N be a Boolean like near-ring. If N is a meet semi lattice then we have the following.

1. N is zero-symmetric.
2. $ab = a \wedge b$ iff $ba = a \wedge b$ iff $ab = ba$ for all $a, b \in N$..
3. $a \wedge b = ab \wedge ba$.
4. $(a \wedge b) \wedge (ab - ba) = 0$.

Proof:

Suppose N is a meet semi lattice

1. Clearly $0a = 0$ for all $a \in N$

For any $a \in N, g.l.b \{0, a\}$ exists and let it be 'e'

therefore $e \leq 0, e \leq a$

$\Rightarrow e = 0e$ and $e = ae$

$\Rightarrow e = 0$ and $0 = a0$

$\Rightarrow a0 = 0.$

Therefore N is zero-symmetric.

2. $ab = a \wedge b$ iff $ba = a \wedge b$ iff $ab = ba$ for all $a, b \in N$..

Suppose $ab = a \wedge b$

Then $ab \leq a$ and $ab \leq b$

$\Rightarrow ab = aab$ and $ab = bab$

$\Rightarrow ab = bab = bba = ba.$

Conversely, suppose that $ab = ba$

Now $aab = ab$ and hence $ab \leq a$

and $bab = ba = ab$ and hence $ab \leq b$

Therefore ab is a lower bound of $\{a, b\}$.

Suppose $c \leq a$ and $c \leq b$ then $c = ac, c = bc$

now $c = ac = abc$

$\Rightarrow c \leq ab$

Therefore $g.l.b \{a, b\} = a \wedge b = ab.$

Similarly we can prove that $ba = a \wedge b$ iff $ab = ba$

Therefore $g.l.b\,\{a, b\} = a \wedge b = ba.$

3. $a \wedge b = ab \wedge ba$

Let $c = a \wedge b$ and $d = ab \wedge ba$

then $c \leq a$ and $c \leq b$

$\Rightarrow c = ac$ and $c = bc$

$\Rightarrow c = ac = abc$ and $c = bc = bac$

$\Rightarrow c \leq ab$ and $c \leq ba$

Since $d = ab \wedge ba$ we have $c \leq d$

Also $d \leq ab$ and $d \leq ba$

since $ab \leq a$ and $ba \leq b$

we have $d \leq ab \leq a$ and $d \leq ba \leq b$

$\Rightarrow d \leq a$ and $d \leq b$

Since $c = a \wedge b$ we have $d \leq c$

Therefore $c = d.$

4. $(a \wedge b) \wedge \left(ab - ba\right) = 0$

Let $x = (a \wedge b) \wedge (ab - ba)$

then $x \leq a, x \leq b, x \leq (ab - ba)$

$\Rightarrow x = ax,\ x = bx,\ x = \left(ab - ba\right)x$

So, $x = \left(ab - ba\right)x$

$= abx - bax$

$= ax - bx$

$x = x - x$

$= 0.$

Therefore $(a \wedge b) \wedge (ab - ba) = 0.$

Definition: 3.14

Suppose $(P, \leq)$ is partially ordered set. For any subset A of P.

Let $L(A) = \{ x \in P / x \leq a, \forall a \in A\}$ and $U(A) = \{ x \in P / a \leq x, \forall a \in A\}$.

For convenience we write $L(x)$ for $L(\{x\})$ and $U(x)$ for $U(\{x\})$

Write $L(P) = \{(L(A) / A$ is a non-empty finite subset of $P\}$ and

$U(P) = \{U(P) / A$ is a non-empty finite subset of $P\}$.

Proposition: 3.15

Let P be a poset. Then $L(P)$ is a meet semi-lattice under set inclusion.

Proof:

Let $L(A), L(B) \in L(P)$

Then $L(A \cup B) \subseteq L(P)$

Clearly $L(A \cup B) \subseteq L(A)$ and $L (A \cup B) \subseteq L(B)$

Let $L(C)$ be a lower bound of $L(A)$ and $L(B)$

i.e $L(C) \subseteq L(A)$ and $L(C) \subseteq L(A)$

Let $x \in L(C)$ that implies $x \in L(A)$ and $x \in L(B)$

$\Rightarrow x \in L(A \cup B)$

Therefore $L(C) \subseteq L(A \cup B) = \inf \{L(A), L(B)\}$.

Hence $L(P)$ is a meet semi-lattice.

Proposition: 3.16

Let P be a poset. Then $U(P)$ is a meet semi-lattice under set inclusion.

Proof:

Let $U(A), U(B) \in U(P)$

Then $U(A \cup B) \subseteq U(A)$ and $U(A \cup B) \subseteq U(B)$

Let $U(C)$ be a lower bound of $U(A)$ and $U(B)$

i.e $U(C) \subseteq U(A)$ and $U(C) \subseteq U(B)$

Let $x \in U(C)$ that implies $x \in U(A)$ and $x \in U(B)$

$\Rightarrow x \in U(A \cup B)$

Therefore $U(A \cup B) = \inf \{U(A), U(B)\}$.

Hence $U(P)$ is a meet semi-lattice.

Definition: 3.17

A poset P is called **distributive** if both the meet semi-lattices $L(P)$ and $U(P)$ are distributive.

Theorem: 3.18

Let N be a Boolean like near-ring. If N is a distributive poset then the Boolean like near-ring N is Boolean like ring.

Proof:

Let N be a distributive poset.

Claim: N is a Boolean ring.

Since N is a distributive poset, $L(N)$and $U(N)$are meet distributive semi-lattices.

For any $L(A), L(B), L(C) \in L(N)$ such that

$L(A) \wedge L(B) \subseteq L(C)$ there exists $L(X)$, $L(Y)$ in $L(N)$ such that $L(A) \subseteq L(X)$, $L(B) \subseteq L(Y)$ and $L(X) \wedge L(Y) = L(C)$.

Similarly for any $U(A), U(B), U(C) \in U(N)$ such that $U(A) \subseteq U(X)$, $U(B) \subseteq U(Y)$ and $U(X) \wedge U(Y) = U(C)$.

Let $a, b \in N$, Clearly $L(a) \wedge L(b) \subseteq L(b)$. Therefore there exists $L(a) \subseteq L(X)$, $L(b) \subseteq L(Y)$ and $L(X) \wedge L(Y) = L(b)$.

therefore $b \in L(X)$ and $b \in L(Y)$

Choose $x \in X$. Then $b \leq x$ and $a \leq x$

$\Rightarrow$ $a = xa$ and $b = xb$

Now $ab = xab$

$= xba$

$= ba.$ (since $b \leq x$)

$\Rightarrow$ $ab = ba.$

Therefore the Boolean like near-ring N is commutative.

Hence N is a Boolean like ring.

4.BOOLEAN LIKE GAMMA NEAR-RINGS

Definition: 4.1

A **Γ-near-ring** Γ_N is a system consisting of

(i) a group $(\Gamma_N, +)$ (not necessarily Abelian)

(ii) a non-empty set Γ

(iii) a mapping $(a, \alpha, b) \rightarrow a\alpha b$ of $\Gamma_N \times \Gamma \times \Gamma_N \rightarrow \Gamma_N$ satisfying the following conditions:

(a) $(a + b)\alpha c = a\alpha c + b\alpha c \ \forall\ a, b, c \in \Gamma_N$ and $\alpha \in \Gamma$.

(b) $(a\alpha b)\beta c = a\alpha(b\beta c) \ \forall\ a, b, c \in \Gamma_N$ and $\alpha, \beta \in \Gamma$.

Note: 4.2

(i) The identify 0 in $(\Gamma_N, +)$ is called the zero element of Γ_N.

(ii) Clearly $0\gamma b = 0 \ \forall\ \gamma \in \Gamma, b \in \Gamma_N$.

(iii) The inverse of a $a \in \Gamma_N$ is denoted by $-a$.

(iv) $(-a)\gamma b = -a\gamma b$.

Definition:4.3

An element $d \in \Gamma_N$ is called a **distributive element** if for all, $n, n' \in \Gamma_N,\ \gamma \in \Gamma$,

$d\gamma(n + n') = d\gamma n + d\gamma n'$.

Definition: 4.4

A Γ- Near ring Γ_N is said to be a **Zero-Symmetric** if $n\gamma 0 = 0$ for all $n \in \Gamma_N,\ \gamma \in \Gamma$.

Definition: 4.5

A Γ-Near ring Γ_N is said to be **Weak Commutative** if for all $x, y, z \in \Gamma_N, \gamma \in \Gamma$, $x\gamma y\gamma z = x\gamma z\gamma y$.

Definition: 4.6

A Γ Near-ring Γ_N with unit element is called a **Boolean like *Γ*-near-ring** provided $a\gamma a^*\gamma b\gamma b^* = 0, a\gamma a = a$ and $a \,\Delta\, a = a$ for all $a, b \in \Gamma_N, \gamma \in \Gamma$.

Lemma: 4.7

If Γ_N is Boolean like Γ-near-ring then $a\gamma b = a\gamma b\gamma a$ for all $a, b \in \Gamma_N, \gamma \in \Gamma$.

Proof:

Let Γ_N is a Boolean like Γ-near-ring and $a, b \in \Gamma_N, \gamma \in \Gamma$.

Now $(a\gamma b - a\gamma b\gamma a)\,\gamma a = a\gamma b\gamma a - a\gamma b\gamma a\gamma a$

$= a\gamma b\gamma a - a\gamma b\gamma a$

$= 0.$

Therefore $(a\gamma b - a\gamma b\gamma a)\,\gamma a = 0.$ (1)

and $a\gamma\big(a\gamma b - a\gamma b\gamma a\big) = \big(a\gamma(a\gamma b - a\gamma b\gamma a)\big)\,\gamma(a\gamma b - a\gamma b\gamma a))$

$= a\gamma\Big(\big((a\gamma b - a\gamma b\gamma a)\gamma a\big)\gamma(a\gamma b - a\gamma b\gamma a)\Big)$

$= a\gamma 0\gamma(a\gamma b - a\gamma b\gamma a)$ from (1)

$= a\gamma 0.$ (2)

Also $(a\gamma b - a\gamma b\gamma a)\,\gamma a\gamma b\gamma a = a\gamma b\gamma a\gamma b\gamma a - a\gamma b\gamma a\gamma a\gamma b$

$= a\gamma b\gamma a - a\gamma b\gamma a$

$= 0.$

As above it follows that

$a\gamma b\gamma a\gamma\left(a\gamma b - a\gamma b\gamma a\right) = a\gamma b\gamma a\gamma 0$ (3)

Similarly $a\gamma b\gamma\left(a\gamma b - a\gamma b\gamma a\right) = a\gamma b\gamma 0$

Now $(a\gamma b - a\gamma b\gamma a) = (a\gamma b - a\gamma b\gamma a)\gamma(a\gamma b - a\gamma b\gamma a)$

$= a\gamma b\gamma\left(a\gamma b - a\gamma b\gamma a\right) - a\gamma b\gamma a\gamma(a\gamma b - a\gamma b\gamma a)$

$= a\gamma b\gamma 0 - a\gamma b\gamma a\gamma 0$ from(2)

$= (a\gamma b - a\gamma b\gamma a)\gamma 0$

$= \left(a\gamma b - a\gamma b\gamma a\right)\gamma\,(a\gamma b - a\gamma b\gamma a)\gamma a$ from(1)

$= \left(a\gamma b - a\gamma b\gamma a\right)\gamma a$

$= 0.$ from(1)

Therefore $\left(a\gamma b - a\gamma b\gamma a\right) = 0.$

Hence $a\gamma b - a\gamma b\gamma a$ for all $a, b \in \Gamma_N, \gamma \in \Gamma$.

Theorem: 4.8

If Γ_N is a Boolean like Γ-near-ring then $a\gamma b\gamma c = a\gamma c\gamma b$ for $a, b, c \in \Gamma_N, \gamma \in \Gamma$ i.e. Γ_N is weak commutative.

Proof:

Let Γ_N be a Boolean like Γ-near-ring and $a, b, c \in \Gamma_N, \gamma \in \Gamma$

Consider $a\gamma b\gamma c - a\gamma c\gamma b$

$a\gamma b\gamma c - a\gamma c\gamma b = a\gamma b\gamma c - a\gamma c\gamma b\gamma c$ (since $= a\gamma b\gamma a$)

$= \left(a - a\gamma c\right)\gamma b\gamma c$

$= \left(a - a\gamma c\right)\gamma b\gamma\left(a - a\gamma c\right)\gamma c$

$$a\gamma b\gamma c - a\gamma c\gamma b = \left(a - a\gamma c\right)\gamma b\gamma 0$$

$$= a\gamma b\gamma 0 - a\gamma c\gamma b\gamma 0. \qquad (1)$$

Replacing b by $b\gamma c$ in (1) and theorem 4.8

$$a\gamma b\gamma c - a\gamma c\gamma b = a\gamma b\gamma c\gamma 0 - a\gamma c\gamma b\gamma 0. \qquad (2)$$

From (1) and (2),

$a\gamma b\gamma 0 = a\gamma b\gamma c\gamma 0$ for all $a, b, c \in \Gamma_N, \gamma \in \Gamma$. (3)

Substituting $b = a$ in (3) we have

$$a\gamma a\gamma 0 = a\gamma a\gamma c\gamma 0$$

$\Rightarrow a\gamma 0 = a\gamma c\gamma 0$ for $a, c \in \Gamma_N, \gamma \in \Gamma$ (4)

By (1) and (4) we get

$$a\gamma b\gamma c - a\gamma c\gamma b = a\gamma b 0 - a\gamma c\gamma b\gamma 0$$

$$= a\gamma 0 - a\gamma 0$$

$$= 0.$$

Therefore $a\gamma b\gamma c = a\gamma c\gamma b$.

Hence Γ_N is Weak Commutative.

Lemma: 4.9

Let Γ_N be a Boolean like Γ-near-ring. If d is a distributive element in Γ_N, then $d + d = 0$ and hence $d = -d$.

Proof:

Let Γ_N be a Boolean like like Γ-near-ring and d is a distributive element.

Consider $d + d$

$$d + d = (d + d)\,\gamma\,(d + d)$$

$$d + d = d\gamma\,(d + d) + d\gamma\,(d + d)$$

$$= d\gamma d + d\gamma d + d\gamma d + d\gamma d$$

$$= d + d + d + d. \qquad \text{(Since } a\gamma a = a\text{)}$$

Therefore $d + d = 0$

$\Rightarrow d = -d.$

Definition: 4.10

A Γ-near-ring Γ_N is called a **distributive Γ-near-ring** every element of Γ_N is a distributive element.

Definition: 4.11

An element $e \in \Gamma_N$ is called a **left identity** if $e\gamma a = a, \forall\, a \in \Gamma_N$.

ORDERING ON BOOLEAN LIKE Γ-NEAR-RINGS

Definition: 4.12

Let Γ_N be a Boolean like Γ-near-ring. Define a **relation** "≤" on Γ_N by a $a \leq b$ if $a = b\gamma a$ for all $\gamma \in \Gamma$.

Proposition: 4.13

Let Γ_N a Boolean like Γ-near-ring and $a, b \in \Gamma_N$. If $a \leq b$ then $a = b\gamma a = a\gamma b$.

Proof:

Let Γ_N a Boolean like Γ-near-ring and $a, b \in \Gamma_N$.

We know by 4.12 $a \leq b \Rightarrow a = b\gamma a$

Now $a\gamma b = b\gamma a\gamma b$ (since $a = b\gamma a$)

$= b\gamma b\gamma a$ (since $a\gamma b\gamma c = a\gamma c\gamma b$)

$= b\gamma a.$ (since $a\gamma a = a$)

Therefore $a\gamma b = b\gamma a$.

Lemma: 4.14

Let Γ_N Boolean like Γ-near-ring. If Γ_N is u-directed then Γ_N is a commutative Boolean like Γ-near-ring.

Proof:

Let Γ_N is u-directed.

Let $a, b \in \Gamma_N$

Since Γ_N is u-directed, $\exists\, c \in \Gamma_N$ such that $c = a \vee b$

Now $a \leq c$ and $b \leq c$

$\Rightarrow$ $a = c\gamma a$ and $b = c\gamma b$ $\forall\, \gamma \in \Gamma$.

Now $a\gamma b = c\gamma a\gamma b$

$= c\gamma b\gamma a$ (since $a\gamma a = a$)

$= b\gamma a$

Therefore $a\gamma b = b\gamma a$ for all $a, b \in \Gamma_N$.

Thus Γ_N is a commutative Boolean like Γ-near-ring.

Theorem: 4.15

Let Γ_N be a Boolean like Γ-near-ring. If Γ_N is a Γ-meet semi lattice then we have the following.

1. Γ_N is a zero-symmetric.
2. Let $a, b \in \Gamma_N$, $a\gamma b = a \wedge b \ \forall\, \gamma \in \Gamma$ iff $a\gamma b = b\gamma a \ \forall\, \gamma \in \Gamma$.
3. $a \wedge b = a\gamma b \wedge b\gamma a$ for all $a, b \in \Gamma_N$ and $\forall\, \gamma \in \Gamma$.
4. $(a \wedge b) \wedge (a\gamma b - b\gamma a) = 0$ for all $a, b \in \Gamma_N$ and $\forall\, \gamma \in \Gamma$.

Proof:

Suppose Γ_N is a Γ-meet semi lattice

For any $a \in \Gamma_N$, g.l.b $\{0, a\}$ exist and let it be 'e

Therefore $e \leq 0,\ e \leq a$

$$\Rightarrow \quad e = 0\gamma e \quad \text{and} \quad e = a\gamma e \quad \forall\, \gamma \in \Gamma$$

$$\Rightarrow \quad e = 0 \quad \text{and} \quad 0 = a\gamma 0$$

$$\Rightarrow \quad a\gamma 0 = 0.$$

Therefore Γ_N is zero-symmetric.

2. $a\gamma b = a \wedge b$ iff $b\gamma a = a \wedge b$ iff $a\gamma b = b\gamma a$ for all $a, b \in \Gamma_N$.

Suppose $a\gamma b = a \wedge b$

then $a\gamma b \leq a$ and $a\gamma b \leq b$

$\Rightarrow a\gamma b = a\gamma a\gamma b$ and $a\gamma b = b\gamma a\gamma b$

$\Rightarrow a\gamma b = b\gamma a\gamma b = b\gamma b\gamma a = b\gamma a$

Conversely, suppose that $a\gamma b = b\gamma a$.

Now $a\gamma a\gamma b = a\gamma b,$ hence $a\gamma b \leq a$

and $b\gamma a\gamma b = b\gamma a = a\gamma b,$ hence $a\gamma b \leq a$

Therefore $a\gamma b$ is a lower bound of $\{a, b\}$.

Suppose $c \leq a$ and $c \leq b$ then $c = a\gamma c, c = b\gamma c$.

Now $c = a\gamma c = a\gamma b$

$\Rightarrow c \leq a\gamma b$

Therefore $g.l.b\ \{a, b\} = a \wedge b = a\gamma b$.

Similarly we can prove that $b\gamma a = a \wedge b$ iff $a\gamma\gamma b = b\gamma a$.

Therefore $g.l.b\ \{a, b\ \} = a \wedge b = b\gamma a$.

3. For all $\gamma \in \Gamma$

$a \wedge b = a\gamma b \wedge b\gamma a$

Let $c = a \wedge b$ & $d = a\gamma b \wedge b\gamma a$

$\Rightarrow \; c \leq a$ and $c \leq b$

$\Rightarrow \; c = a\gamma c$ and $c = b\gamma c$

$\Rightarrow \; c = a\gamma c = a\gamma b\gamma c$ and $c = b\gamma c = b\gamma a\gamma c$

$\Rightarrow \; c \leq a\gamma b$ and $c = b\gamma a$.

Since $d = a\gamma b \wedge b\gamma a$ we have $c \leq d$.

Also $d \leq a\gamma b$ and $d \leq b\gamma a$.

Now $a\gamma b \leq a$ and $b\gamma a \leq b$

$\Rightarrow \; d \leq a\gamma b \leq a$ and $d \leq b\gamma a \leq b$ Since $c = a \wedge b,$ we have $d \leq c$.

Therefore $c = d$.

4. $(a \wedge b) \wedge (a\gamma b - b\gamma a) = 0$

Let $x = (a \wedge b) \wedge (a\gamma b - b\gamma a)$

then $x \leq (a \wedge b)$ and $x \leq (a\gamma b - b\gamma a)$

$\Rightarrow \; x \leq a, \; x \leq b,$ and $x \leq (a\gamma b - b\gamma a)$

$\Rightarrow \; x = a\gamma x, x = b\gamma x,$ and $x = (a\gamma b - b\gamma a)\gamma x$

$\Rightarrow \; a\gamma x = b\gamma x,$ and $x = a\gamma b\gamma x - b\gamma a\gamma x$

$$
\begin{aligned}
x &= a\gamma a\gamma x - b\gamma b\gamma x \\
&= a\gamma x - b\gamma x \\
&= x - x \\
&= 0.
\end{aligned}
$$

Therefore $(a \wedge b) \wedge (a\gamma b - b\gamma a) = 0$ for all $\gamma \in \Gamma$.

A STUDY ON BOOLEAN LIKE GAMMA NEAR-RINGS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

T. KAVITHA

Reg.No: 19SPMT14

Under the guidance of

Ms. M. KANAGA M.Sc., B.Ed., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON BOOLEAN LIKE GAMMA NEAR-RINGS** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by

T. KAVITHA (Reg. No: 19SPMT14)

Signature of the Guide **Signature of the Coordinator**

Signature of the Director **Signature of the Principal**

Signature of the Examiner

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Rev. Dr. Sr. A.S.J. Lucia Rose M.Sc., PGDCA., M.Phil., Ph.D., and to our Director **Rev. Dr. Sr. F. Joyce Baby M.A., M.Phil., Ph.D.,** is kindly permitting me to do this project.

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I also thank my staff members for their encouragement in all my efforts. Finally, I thank all those who extended their help regarding this project.

Place: Thoothukudi

Date:

CONTENT

INTRODUCTION

G. Boole in 1824 introduced an important class of lattices, which were named as Boolean algebras. In 1944, Alfred L Foster and Bernstein introduced the concept of Boolean ring, which is a generalization of the Boolean algebra in which the abstract algebraic structure of a ring and logical properties are preserved. Later in 1945, A L Foster introduced the concept of Boolean like ring, which is generalization of the concept of Boolean ring. From the collection of Gunter Pliz the theory of Near-rings is a generalization of the theory or rings. Now in this project "Study on Boolean Like Gamma Near-Rings" a generalization of the theories of Boolean like rings, near-rings and gamma near rings, I have introduced BOOLEAN LIKE GAMMA NEAR-RINGS.

The project divided into 4 chapters.

In chapter 1 "Preliminaries", I have collected literature from "Introduction to Lattice Theory " by Gabor Szasz [8], "Near-rings" by Gunter Pliz [13], "Boolean Algebra" by Roman Sikorski [14], which are used in later chapters.

In chapter 2 "Boolean Like Rings", it is proved that the direct product of Boolean like rings is also a Boolean like ring. Further an ordering on a Boolean like ring is defined and some interesting result are proved.

In chapter 3 "Boolean Like Near-Rings", some standard definitions relating to Boolean like near-rings and some important results like, the weak commutativity and every Boolean like near-ring with least element is zero-symmetric and is the Boolean like near-ring is a distributive poset then the Boolean like near-ring is Boolean like ring; are proved.

In chapter 4 "Boolean Like Gamma Near-Ring", it is proved that gamma near ring be a Boolean like gamma near-ring. If gamma near-ring is u-directed then gamma near ring is a commutative Boolean like gamma near-ring and some standard definitions relating to Boolean like gamma near-rings and some important results also are proved.

CONCLUSION

It was a wonderful learning experience for me while working on this project. This project took me through the various phases of Boolean algebras. Here I have discussed the “Study on Boolean Like Gamma Near-Rings” a generalization of the theories of Boolean like rings, near-rings and gamma near-rings.

REFERENCES

[1] Alfred L-Foster "The Idempotent element of a commutative ring form a Boolean Algebra" Duke math Jr.Vol 12 (1945).

[2] Alfred L-Foster "The theory of Boolean like rings", Trans. Amar. Math. Soc, 59(1946) PP (166-187).

[3] Birkhoff. G "Lattice theory", Amar. Math. Soc, 1973.

[4] Blackett D. V "Simple and Semi simple near-rings", Proc. Amar. Math. Soc.4 (1953) 772-785.

[5] Booth. G.L. "A note on Γ-near-rings", Stud. Sci. Math. Hung.23 (1988) 471-475.

[6] Chales Gilbert Fain "Some Structure theorem for near-rings", Proc. Oklahama 1968.

[7] Frohlich A "Distributively generated near-rings", Proc. London Math. Soc(3) 76-94.

[8] Gabor Szaz "Introduction to lattice Theory", Academic Press New York 1963.

[9] Gopala Rao A. P. C. U "Boolean Like Rings and their generalization", M. Phil Disseration, 2008.

[10] Ligh. Steve "On Boolean near-rings", Bull. Australian Math. Soc 1(1969) 375-379.

[11] Ligh. Steve "On distributive generated near-rings", Proc. Edinburgh Math. Soc (1969) 239-243.

[12] Mc Coy N.H. "A representation of generalized Boolean Rings", Duke Math J.3 (1937) 455-459.

[13] Pliz. G "Near-rings Theory and its applications", North Holland 1983.

[14] Sikorski R "Boolean algebra", Springer New York 1969.

[15] Stone M. H "The Theory of representation for Boolean Algebra", Trans. Amar. Math Soc 40, 37-111.

[16] Stone M "The Theory of representation for Boolean Algebras", Trans. Amar. Math. Soc 1936.

A STUDY ON Q – FUZZY NORMAL SUBGROUPS

A project submitted to

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Submitted by

R. KRENCY

Reg. No: 19SPMT15

Under the guidance of

DR. R. MARIA IRUDHAYA ASPIN CHITRA, M.Sc., M.Phil., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON *Q* – FUZZY NORMAL SUBGROUPS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **R. KRENCY (Reg. No: 19SPMT15)**

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled "A STUDY ON Q-FUZZY NORMAL SUBGROUPS" submitted for the degree of Master of Science is my work carried out under the guidance of **Dr. R. Maria Irudhaya Aspin Chitra, M.Sc., M.Phil., Ph.D.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's college (Autonomous), Thoothukudi.**

Station: Thoothukudi **Signature of the Student**

Date : 10.04.2021

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I express my sincere gratitude and heartfelt thanks to our principal **Rev. Dr. Sr. A.S.J. Lucia Rose M.Sc., PGDCA., M.Phil., Ph.D.,** and to our Director **Rev. Dr. Sr. F. Mary Joyce Baby M.A., M.Phil., Ph.D.,** for kindly permitting me to do this project.

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I also thank my staff members for their encouragement in all my efforts. Finally, I thank all those who extended their help regarding this project.

Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER - 1

PRELIMINARIES

Definition: 1.1

Let X be any non-empty set. A **fuzzy subset** μ of X is a function $\mu{:}X \to [0,1]$.

Definition: 1.2

Let G be a group. A fuzzy subset μ of G is called a **fuzzy subgroup** if for all $x, y \in G$,

(i) $(xy) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x^{-1}) = \mu(x)$

Definition: 1.3

Let G be a group. A fuzzy subgroup μ of G is said to be **normal** if for all $x, y \in G$, $\mu(xyx^{-1}) = \mu(y)$ or $\mu(xy) = \mu(yx)$.

Definition: 1.4

Let μ be a fuzzy subgroup of a group G. For any $t \in [0,1]$, we define the **level subset** of μ is the set, $\mu^t = \{x \in G / \mu(x) \geq t\}$.

Definition: 1.5

A mapping $\mu{:}M \times Q \to [0,1]$, where M and Q are arbitrary non empty sets is called **$\boldsymbol{Q}$-fuzzy set** of M and is denoted by $A_Q = \{[(x, q), \mu(x, q)]/x \in M, q \in Q\}$

Definition: 1.6

A Q-fuzzy set μ is called a **$\boldsymbol{Q}$-fuzzy subgroup** of a group G, if for $x, y \in G$, $q \in Q$,

(i) $\mu\,(xy, q) \geq \min\{\mu\,(x, q), \mu\,(y, q)\}$

(ii) $\mu\,(x^{-1}, q) = \mu\,(x, q)$

Definition: 1.7

Let X be a non-empty set. A Fuzzy Multiset A drawn from X is characterized by a function **'Count Membership'** of A denoted by CM_A such that CM_A:$X \rightarrow Q$ where Q is the set of all crisp finite set drawn from the unit interval [0,1]. Then for any $x \in X$,

the value of $CM_A(x)$ is a crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $M_A(x)$.

It is denoted by, ($\mu_{A_1}(x)$, $\mu_{A_2}(x)$, ……, $\mu_{A_k}(x)$).

Example: 1.8

Let $X = \{x, y, z, w\}$ be a universal non empty set. For each $x \in X$, we can write a **Fuzzy Multiset** as follows.

$A = \{[x, (0.8, 0.7, 0.7, 0.6)], [y, (0.8, 0.5, 0.2)], [z, (1, 0.5, 0.5)]\}$ where

$M_A(x) = (0.8, 0.7, 0.7, 0.6)$ with $0.8 \geq 0.7 \geq 0.7 \geq 0.6$.

Definition: 1.9

Let X be a non-empty set. An **intuitionistic fuzzy set**A on X is an object having the form $A = \{[x, \mu_A(x), \gamma_A(x)]/x \in X\}$ where μ_A:$X \rightarrow [0,1]$ and γ_A:$X \rightarrow [0,1]$ are the degree of membership and non-membership functions respectively with

$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$.

Definition: 1.10

Let X and Q are arbitrary non empty sets. An **intuitionistic $\boldsymbol{Q}$-fuzzy set**A is an object having the form $A = \{[(x, q), \mu_A(x, q), \gamma_A(x, q)]: x \in X, q \in Q\}$ where the functions μ_A:$X \times Q \rightarrow [0,1]$ and γ_A:$X \times Q \rightarrow [0,1]$ denote the degree of membership and non-membership of each element $(x, q) \in X \times Q$ to the set A respectively and for all $x \in X$ $and\ q \in Q$, $0 \leq \mu_A(x, q) + \gamma_A(x, q) \leq 1$.

Definition: 1.11

Let μ be an anti-fuzzy subgroup of a group G. For any $t\in$ [0,1], we define the **level subset** of μ is the set $\mu_t = \{x\in G/\mu(x) \leq t\}$.

Definition: 1.12

A fuzzy set μ of a group G is called an **anti-fuzzy subgroup** of G, if for all

$x, y\in G$,

(i) $(xy \leq$ max $\{\mu(x), \mu(y)\}$

(ii) $\mu(x^{-1}) = \mu(x)$

Definition: 1.13

An anti-fuzzy subgroup μ of a group G is called an **anti-fuzzy normal subgroup** of G if for all $x, y\in G$, $(xyx^{-1})) = (y)$ or $(xy) = (yx)$.

Definition: 1.14

Let X be a field and F be a fuzzy set in X with membership function μ_F. Suppose the following

(i) $\mu_F((x + y), q \quad \geq min\ \{\mu_F(x, q), \mu_F(y, q)\}$

(ii) $\mu_F(-x, q) \qquad \geq \mu_F(x, q)$

(iii) $\mu_F((xy), q) \quad \geq min\{\mu_F(x, q), \mu_F(y, q)\}$

(iv) $\mu_F\ (x^{-1}, q) \quad \geq \mu_F(x, q), x \neq 0\ inX$

F is a **fuzzy field** in X and denote it by (F, X). Also (F, X) is called a fuzzy field of X.

Definition: 1.15

Let $(G, \cdot)$ be a group and Q be a non-empty set. A Q-fuzzy subgroup A of G is said to be a ***Q*-fuzzy characteristic subgroup** (QFCSG) of G if $A\ (x, q) = A\ (f\ (x, q))$, for all x in G and f in $Q - AutG$ and q in Q.

Definition: 1.16

A Q-fuzzy subset A of a set X is said to be **normalized** if there exist x in X such that $(x, q) = 1$.

Definition: 1.17

Let A be a Q-fuzzy subgroup of a group (G , ·). For any a in G, aA defined by $(aA)(x, q) = (a^{-1}x, q)$ for every x in G and q in Q, is called a **Q-fuzzy coset of G.**

Definition: 1.18

Let A be a Q-fuzzy subgroup of a group (G, ·) and

$H = \{ x \in G \,/\, A(x, q) = A(e, q) \}$,

then $O(A)$, order of A is defined as $\boldsymbol{O(A) = O(H)}$.

CHAPTER - 2

$\boldsymbol{Q}$- Fuzzy Normal Subgroups

Definition: 2.1

Let G be a group. A Q- fuzzy subgroup μ of G is said to be **normal** if for all x, $y \in G$ and $q \in Q$,

$$\mu\ (xyx^{-1}, q) = \mu\ (y, q)\ or\ \mu\ (xy, q) \geq \mu\ (yx, q).$$

Definition: 2.2

Let μ be a Q-fuzzy subgroup of a group G. For any $t \in$ [0,1], we define the **level subset** of μ is the set,

$\mu^t = \{x \in G,\ q \in Q\ /\ \mu\ (x,q) \geq t\}$.

Theorem: 2.3

Let G be a group and μ be a Q-fuzzy subset of$G$. Then μ is a Q – fuzzy subgroup of G iff the level subsets,μ^t,∈ [0,1], are subgroup of G.

Proof:

Let μ be a Q-fuzzy subgroup of G and the level subset

$\mu^t = \{x \in G,\ q \in Q\ /\ \mu\ (x,q) \geq t,\ t \in [0,1]\}$.

Let $x, \in \mu^t$. Then $\mu\ (x, q) \geq t$ & $\mu\ (y, q) \geq t$.

Now $\mu\ (xy^{-1}, q)$ $\geq \{\mu\ (x, q), (y^{-1},)\}$

$= min\ \{\ \mu\ (x\ ,\ q)\ ,\ \mu\ (y\ ,\ q)\ \}$

$\geq min\ \{\ t\ ,\ t\ \}$

Therefore, $\mu\ (xy^{-1}, q)\ \geq t$

This implies $x\ y^{-1} \in \mu^t$.

Thus ,μ^t is a subgroup of G.

Conversely, let us assume that μ^t be a subgroup of G.

Let x,$\in\mu^t$. Then μ $(x, q) \geq$ nd μ $(y, q) \geq t$.

Also, μ $(x\, y^{-1}, q)$ $\geq t$, since $x\, y^{-1} \in \mu^t$

$= min\ \{t\ ,\ t\}$

$= min\{\mu\ (x,q), \mu\ (y,q)\}$

(i.e.), $(x\, y^{-1}, q)$ $\geq min\ \{\mu\ (x,), (y,)\}$.

Hence, μ is a Q-fuzzy subgroup of G.

Definition: 2.4

Let G be a group and μ be a Q-fuzzy subgroup of G.

Let (μ) = $\{a\in G\ /\ (axa^{-1}, q) = \mu\ (x, q)$, for all $x\in G$, $q\in Q\}$. Then (μ) is called the

$\boldsymbol{Q}$-fuzzy Normalizer of $\boldsymbol{\mu}$.

Theorem: 2.5

Let G be a group and μ be a Q-fuzzy subset of G. Then μ is a $Q-$ fuzzy normal subgroup of G iff the level subsets μ^t, $t\in$ [0,1], are normal subgroup of G.

Proof:

Let μ be a Q- fuzzy normal subgroup of G and the level subsets μ^t,

$t\in$ [0,1], is a subgroup of G. Let $x\in G$ and $a\in\mu^t$, $then$ μ $(a, q) \geq t$.

Now, μ $(xax^{-1}, q) = (a, q) \geq t$,

since μ is a Q-fuzzy normal subgroup of G.

That is, μ $(xax^{-1}, q) \geq t$.

Therefore, $xax^{-1}\in\mu^t$

Hence, μ^t is a normal subgroup of G.

Theorem: 2.6

Let μ be a Q-fuzzy subgroup of a group G.

Then (i) (μ) is a subgroup of G.

(ii) μ is a Q-fuzzy nor mal $\Leftrightarrow(\mu) = G$.

(iii) μ is a Q-fuzzy normal subgroup of the group (μ).

Proof:

(i) Let $a,\in N(\mu)$ then $\mu(axa^{-1}, q) = \mu\,(x, q)$, for all $x\in G$ and

$\mu\,(bxb^{-1}, q) = \mu\,(x, q)$, for all $x\in G$.

Now $\mu\,(abx(ab)^{-1}, q)$ $= (abx\,b^{-1}a^{-1}, q)$

$= \mu\,(bxb^{-1},q)$

$= \mu\,(x, q)$

Thus we get, $(abx(ab)^{-1},)$ $= \mu\,(x,)$.

$\Rightarrow ab\in N(\mu)$

Therefore, (μ) is a subgroup of G.

(ii) Clearly N $(\mu) \subseteq$, is a Q-fuzzy normal subgroup of G.

Let $a\in G$, then $\mu\,(axa^{-1}, q) = \mu\,(x, q)$.

Then $a\in(\mu) \Longrightarrow G\subseteq N(\mu)$.

Hence $(\mu) = G$.

Conversely, let$(\mu) = G$.

Clearly, $\mu\,(axa^{-1}, q) = \mu\,(x,)$, for all $x\in G \; and \; a\in G$.

Hence, μ is a Q – fuzzy normal subgroup of G.

(iii) From (ii), μ is a Q −fuzzy normal subgroup of a group (μ).

Definition: 2.7

Let μ be a Q-fuzzy subset of G.

Let ${}_xf$: $G{\times}Q \to G{\times}Q$ [f_x:$G{\times}Q \to G{\times}Q$] be a function defined by

${}_xf(a , q) = (xa , q)$[$f_x(a , q) = (ax , q)$]. A ***Q*-fuzzy left (right) coset**

${}_x\mu$ (μ_x) is defined to be (μ) ($f_x(\mu)$). It is easily seen that

$({}_x\mu)\,(y, q) = \mu\,(x^{-1}y, q)$ *and* $(\mu_x)\,(y, q) = \mu\,(yx^{-1}, q)$, for every (y, q) in $G \times Q$.

Theorem: 2.8

Let μ be a Q-fuzzy subset of G. Then the following conditions are equivalent for each x, y in G.

(i) $\mu\,(xyx^{-1}, q) \geq \mu\,(y, q)$

(ii) $\mu\,(xyx^{-1}, q) = \mu\,(y, q)$

(iii) $\mu\,(xy , q) = \mu\,(yx , q)$

(iv) ${}_x\mu = \mu_x$

(v) ${}_x\mu_{x^{-1}} = \mu$

Proof:

Straight forward

Theorem: 2.9

If μ is a Q-fuzzy subgroup of G, then $g\mu g^{-1}$ is also a Q-fuzzy subgroup of G for all $g \in G$ and $q \in Q$.

Proof:

Let μ be a Q-fuzzy subgroup of G. Then

(i) $(g\mu g^{-1})\,(xy, q) = \mu\,(g^{-1}(xy), q)$

$= \mu(g^{-1}(xgg^{-1}y)g, q)$

$= \mu\,((g^{-1}xg)(g^{-1}yg), q)$

$\geq min\{\mu\,(g^{-1}xg, q), \mu\,(g^{-1}yg, q)\}$

$\geq min\{g\mu g^{-1}\,(x, q), g\mu g^{-1}\,(y, q)\},$

$for\ all\ x\ , y\ in\ G\ and\ q \in Q.$

(*ii*) $g\mu g^{-1}(x, q) = \mu(g^{-1}xg, q)$

$= \mu\,((g^{-1}xg)^{-1}, q)$

$= \mu\,(g^{-1}x^{-1}g, q)$

$= g\mu g^{-1}(x^{-1}, q), for\ all\ x,y\ in\ G\ and\ q \in Q.$

Hence, $g\mu g^{-1}$ is a Q-fuzzy subgroup of G.

Theorem: 2.10

If μ is a Q-fuzzy normal subgroup of G, then $g\mu g^{-1}$ is also a Q-fuzzy normal subgroup of G, for all $g \in G\ and\ q \in Q$.

Proof:

Let μ be a Q-fuzzy normal subgroup of G. then $g\mu g^{-1}$ is a subgroup of G.

Now, $g\mu g^{-1}\,(xyx^{-1}, q) = (g^{-1}(xyx^{-1}), q)$

$= \mu(xyx^{-1}, q)$

$= \mu(\ y, q)$

$= \mu(gyg^{-1}, q)$

$= g\mu g^{-1}\,(y, q).$

Theorem: 2.11

Let μ and λ be two Q-fuzzy subgroups of G. Then $\lambda \cap \mu$ is a Q-fuzzy subgroup of G.

Proof:

Let λ and μ be two Q-fuzzy subgroups of G

(i) $(\lambda \cap \mu)(xy^{-1}, q) = min(\lambda(xy^{-1}, q), \mu(xy^{-1}, q))$

$\geq min\{ min\{\lambda(x, q), \lambda(y, q)\}, min\{\mu(x, q), \mu(y, q)\}\}$

$\geq min\{ min\{\lambda(x, q), \mu(x, q)\}, min\{\lambda(y, q), \mu(y, q)\}\}$

$= \min\{(\lambda \cap \mu)(x, q), (\lambda \cap \mu)(y, q)\}$

Thus, $(\lambda \cap \mu)(xy^{-1}, q) \geq \{(\lambda \cap \mu)(x, q), (\lambda \cap \mu)(y, q)\}$

(ii) $(\lambda \cap \mu)(x,) = \{\lambda(x, q), \mu(x, q)\}$

$= \{\lambda(x^{-1}, q), \mu(x^{-1}, q)\}$

$= \{(\lambda \cap \mu)(x^{-1}, q)\}$.

Hence, $\lambda \cap \mu$ is a Q-fuzzy subgroup of G.

Remark: 2.12

If μi, $i \in \Delta$ is a Q-fuzzy subgroup of G, then $\cap_{i\in\Delta} \mu_i$ is a Q-fuzzy subgroup of G.

Theorem: 2.13

The intersection of any two Q-fuzzy normal subgroups of G is also a Q-fuzzy normal subgroup of G.

Proof:

Let λ and μ be two Q-fuzzy normal subgroups of G.

According to theorem 2.11, $\lambda \cap \mu$ is a Q-fuzzy subgroup of G.Now for all x, y in G, we have,

$(\lambda \cap \mu)(xyx^{-1}, q) = min(\lambda(xyx^{-1}x, q), \mu(xyx^{-1}, q))$

$= min(\lambda(y, q), \mu(y, q))$

$= (\lambda \cap \mu)(\mathrm{y}, q)$

Hence, $(\lambda \cap \mu)$ is a Q-fuzzy normal subgroup of G.

Remark: 2.14

If μi, $i\in \Delta$ is a Q-fuzzy normal subgroup of G, then$\cap_{i\in\Delta} \mu_i$ is a Q-fuzzy normal subgroup of G.

Definition: 2.15

The mapping$f$: $G\times Q \rightarrow H\times Q$ is said to be a group ***Q*-homomorphism** if

(i) f: $G \rightarrow H$ is a group homomorphism

(ii) f $(xy,)$ = $((x)(y),q)$, for all $x,y\in G$ and $q\in Q$.

Theorem: 2.16

Let f: $G\times Q \rightarrow H\times Q$ is a group Q- homomorphism.

(i) If μ is a Q −fuzzy normal subgroup of , then $f^{-1}(\mu)$ is a Q-fuzzy normal subgroup of G.

(ii) If f is an epimorphism and μ is a Q-fuzzy normal subgroup of G, then (μ) is a Q-fuzzy normal subgroup of H.

Proof:

(i) Let f: $G \times Q \rightarrow H \times Q$ is a group Q-homomorphism and let μ be a Q-fuzzy Normal subgroup of H.

Now for all x, $y \in G$, we have

$$
\begin{aligned}
f^{-1}(\mu)\,(xyx^{-1}, q) &= \mu\,(f\,(xyx^{-1},q)) \\
&= \mu\,(f\,(x)\,f\,(y)f\,(x)^{-1},q) \\
&= \mu\,(\,f(y)\,,\,q) \\
&= f^{-1}(\mu)\,(y,q)
\end{aligned}
$$

Hence, $f^{-1}(\mu)$ is a Q-fuzzy normal subgroup of G.

(ii) Let μ be a Q-fuzzy normal subgroup of G.

Then (μ) is a Q- fuzzy subgroup of H.

Now, for all u, v in H, we have

f (μ) (uvu^{-1}, q) = sup μ $(y,)$

$= sup\ \mu\ (xyx^{-1}, q)$

$f(y) \quad = \ uvu^{-1}$

$f(x) = u; \quad f(y) = v$

= sup μ $(y,)$

$= f(\mu)(v\ , q)$

f $(y) = v$ (since f is an epimorphism)

Hence, f (μ) is a Q-fuzzy normal subgroup of H.

Definition: 2.17

Let λ and μ be two Q-fuzzy subsets of G. The product of λ and μ is defined to be the **Q-fuzzy subset** $\lambda\,\mu$ of G is given by

$$\lambda\mu(x\ , q) = \sup_{yz = x} \min\ (\ \lambda(y\ , q)\ , \mu(z\ , q)\ )\ , x \in G.$$

Theorem: 2.18

If λ & μ are Q-fuzzy normal subgroups of G, then $\lambda\ \mu$ is a Q-fuzzy normal subgroup of .

Proof:

Let λ & μ be two Q-fuzzy normal subgroups of G.

(i) $\lambda\mu\ (xy, q)$ = sup min $\{\lambda\ (x_1y_1,), \mu\ (x_2y_2,q)\}$

By substituting, $x_1y_1\ = x,\ \ x_2y_2 = y$

$\geq$ sup min $\{$min $\{\lambda\ (x_1, q), \lambda\ (y_1, q)\}$,min $\{\mu\ (x_2, q), \mu\ (y_2, q)\}\}$

By substituting, $x_1y_1 = x$, $x_2y_2 = y$

$\geq$ min {sup min {λ (x_1, q), λ (y_1,q)}, sup min {μ (x_2,q),μ (y_2,q)}}

By substituting, x_1y_1= x, x_2y_2= y

$\lambda\mu(xy\ ,\ q)$ $\geq min$ { $\lambda\mu$ (x , q) , $\lambda\mu$ (y , q) }

(ii) $\lambda\mu$ (x^{-1}, q) = sup min{μ(z^{-1}, q), λ(y^{-1}, q)}

$(yz)^{-1}$ = x^{-1}

= sup min {(z, q), λ(y, q)}, x= yz

= sup min {(y, q), μ(z, q)}, x= yz

= $\lambda\mu$(x , q).

Hence, $\lambda\mu$ is a normal Q-fuzzy subgroup of G.

CHAPTER - 3

Intuitionistic $\boldsymbol{Q}$-Fuzzy Normal Subgroups

Definition: 3.1

Let $A = \{[(x, q), \mu_A(x, q), \gamma_A(x, q)]: x \in X, q \in Q\}$ be an Intuitionistic Q-fuzzy set on X. Then for $\alpha, \beta \in [0,1]$, the set $A^{[\alpha\beta]} = \{x \in X, q \in Q / \mu_A(x, q) \geq \alpha \text{ } and \text{ } \mu_A(x, q) \leq \beta\}$ is called the **(α, β)- level subsets of $\boldsymbol{A}$.**

Theorem: 3.2

Let G be a group and A be an Intuitionistic Q- Fuzzy subset of G. Then A is a Intuitionistic Q- Fuzzy subgroup of a group G if the Intuitionistic (α, β)- level subsets $A^{[\alpha\beta]}\alpha, \beta \in [0,1]$ are subgroups of G.

Proof:

Let A be an Intuitionistic Q- Fuzzy subgroup of G and the (α, β)- level subset

$A^{[\alpha\beta]} = \{x \in X, q \in Q / \mu_A(x, q) \geq \alpha \text{ } and \text{ } \gamma_A(x, q) \leq \beta\}$.

Let $x, y \in A^{[\alpha\beta]}$. Then $\mu_A(x, q) \geq \alpha \text{ } and \text{ } \gamma_A(x, q) \leq \beta$

Now $\mu_A(xy^{-1}, q) \geq \min\{\mu_A(x, q), \mu_A(y^{-1}, q)\}$

$= \min\{\mu_A(x, q), \mu_A(y, q)\}$

$= \min\{\alpha, \alpha\} = \alpha$

Hence, $\mu A\,(xy^{-1}, q) \geq \alpha$.

Similarly, $(xy^{-1}, q) \leq \max\{(x, q), \gamma_A(y^{-1}, q)\}$

$= \max\{\gamma A\,(x, q), \gamma_A(y, q)\}$

$= max\,\{\beta, \beta\} = \beta$

Hence, $\gamma A\,(x\gamma_A, q) \leq \beta$.

This implies$xy^{-1} \in A^{[\alpha\beta]}$.

Thus, $A^{[\alpha\beta]}$ is a subgroup of G.

Conversely, let us assume that $A^{[\alpha\beta]}$ is a subgroup of G.

Let x, $y \in A^{[\alpha\beta]}$. Then, $\mu_A(x, q) \geq \alpha$ and $\gamma_A(x, q) \leq \beta$

Also, $\mu_A(xy^{-1}, q) \geq \alpha = min\ \{\alpha, \alpha\}$

$= \min \{((x, q), \mu_A(y, q)\}$

$\mu_A(xy^{-1}, q) \quad \geq \min \{\mu_A(x, q), \mu_A(y, q)\}$

Also, $(xy^{-1}q) \leq \beta \quad = max\ \{\beta, \beta\}$

$= \max \{\gamma_A(x, q), \gamma_A\ (y, q)\}$

$(xy^{-1}, q) \quad \leq \max \{\gamma_A(x, q), \gamma_A\ (y, q)\}$

Hence, A is an Intuitionistic Q-Fuzzy subgroup of G.

Definition: 3.3

Let G be a group and A be a Intuitionistic Q-Fuzzy subgroup of a group G. Let $N(A) = \{a \in G\ / \mu_A(axa^{-1}, q) = \mu_A(x, q), \gamma_A(axa^{-1}, q) = \gamma_A(x, q)\ \forall x \in G, q \in Q\}$.

Then, (A) is called the **Intuitionistic Q-fuzzy normalizer of A.**

Theorem: 3.4

Let G be a group and A be an Intuitionistic Q-fuzzy subset of G. Then A is an Intuitionistic Q-fuzzy normal subgroup of Gif the level subsets $A^{[\alpha\beta]}\alpha$, $\beta \in [0,1]$ is a subgroup of G.

Proof:

Let A be an Intuitionistic Q-fuzzy normal subgroup of G and the level subsets $A^{[\alpha\beta]}$, α, $\beta \in [0,1]$ is a subgroup of G.

Let $x \in G\ and\ a \in A^{[\alpha\beta]}\ then\ \mu_A(a, q) \geq \alpha$.

Now $\mu_A(xax^{-1}, q) = \mu_A(a, q) \geq \alpha$.

Since A is an Intuitionistic Q-fuzzy normal subgroup of G,

That is, $\mu_A\ (xax^{-1}, q) \geq \alpha$

$\Rightarrow xax^{-1} \in\ A^{[\alpha\beta]}$

Similarly, $\gamma_A(xax^{-1}, q) = \gamma_A(a, q) \leq \beta$

Since, A is an Intuitionistic Q-fuzzy normal subgroup of G,

That is, $\gamma_A(xax^{-1}, q) \leq \beta$

$\Rightarrow xax^{-1} \in\ A^{[\alpha\beta]}$

Hence, $A^{[\alpha\beta]}$ is a normal subgroup of G.

Theorem: 3.5

Let A be a Intuitionistic Q-fuzzy normal subgroup of a group G. Then

(i) (A) is a subgroup of a group G.

(ii) A is an Intuitionistic Q-fuzzy normal subgroup of a group G iff $(A) = G$.

(iii) A is an Intuitionistic Q-fuzzy normal subgroup of a group (A).

Proof:

(i) Let $a, b \in \mathrm{N}(A) \Rightarrow \mu_A\ (xax^{-1}, q) \geq \alpha,\ \ \mu_A\ (xbx^{-1}, q) \geq \alpha\ \ for\ all\ x \in G, q \in Q$.

Now, $\mu_A\ (abx(ab)^{-1}, q) \quad = \mu_A(abxb^{-1}a^{-1}, q)$

$= \mu_A(bxb^{-1}, q) = \mu_A(x, q)$.

Thus, $\mu_A(abx(ab)^{-1}, q) \quad = \mu_A(x, q)$.

Let $a, b \in \mathrm{N}(A) \Rightarrow \gamma_A(xax^{-1}, q) \leq \beta, \gamma_A(xbx^{-1}, q) \leq \beta\ for\ all\ x \in G, q \in Q$.

Now, $\gamma_A(abx(ab)^{-1}, q) \quad = \gamma_A(abxb^{-1}a^{-1}, q)$

$= \gamma_A(bxb^{-1}, q)$

$= \gamma_A(x, q)$.

Thus, $\gamma_A\ (abx(ab)^{-1}, q) \quad = \gamma_A(x, q)$

This implies $ab \in \mathrm{N}(\lambda)$.

That is, N (A) is a subgroup of a group G.

(ii) From (i), N(A) $\subseteq G$, A is an Intuitionistic Q-fuzzy normal subgroup of a group G.

Let $a \in G \Rightarrow \mu_A(axa^{-1}, q) = \mu_A(x, q) \Rightarrow a \in N(A)$.

$$\gamma_A(axa^{-1}, q) = \gamma_A(x, q) \Rightarrow a \in N(A).$$

Hence, $(A) = G$.

Conversely, let $(A) = G$,

Clearly, $\mu_A(axa^{-1}, q) = \mu_A(x, q)$ and $\gamma_A(axa^{-1}, q) = \gamma_A(x, q)\ \forall\ a, x \in G, q \in Q$.

Hence, A is an Intuitionistic Q-fuzzy normal subgroup of a group G.

(iii) From (ii), A is an Intuitionistic Q-fuzzy normal subgroup of a group (A).

Theorem: 3.6

If A is an Intuitionistic Q-fuzzy subgroup of a group G. Then $g\mu_A g^{-1}$&$g\,\gamma_A g^{-1}$ are also an Intuitionistic Q-fuzzy subgroups of a group G for all $g \in G\ and\ q \in Q$.

Proof:

Let A is an Intuitionistic Q-fuzzy subgroup of a group G.

Then **(i)** $(g\mu_A g^{-1})(xy, q) = \mu_A(g^{-1}(xy)g, q)$

$$= \mu_A(g^{-1}(xgg^{-1}y)g, q)$$

$$= \mu_A(g^{-1}xg)(g^{-1}yg), q)$$

$$\geq min\ \{\mu_A(g^{-1}xg, q), \mu_A(g^{-1}yg, q)\}$$

for all $x, y \in G\ and\ \ q \in Q$

Similarly, $(g\ \ \gamma_A g^{-1})(xy, q) = \gamma_A(g^{-1}(xy)g, q)$

$$= \gamma_A(g^{-1}(xgg^{-1}y)g, q)$$

$$= \gamma_A(g^{-1}xg)(g^{-1}yg), q)$$

$$\leq \max\ \{\ \gamma_A(g^{-1}xg, q),\ \ \gamma_A(g^{-1}yg, q)\}$$

for all x, $y \in G$ and $q \in Q$.

(ii) $(g\mu_A g^{-1})(x, q)$ $= \mu_A(g^{-1}xg, q)$

$= \mu_A((g^{-1}xg)^{-1}, q)$

$= \mu_A(g^{-1}x^{-1}g, q)$

$= g\mu_A g^{-1}(x^{-1}, q)$

Similarly, $(g\ \gamma_A g^{-1})(x, q)$ $= \gamma_A\ (g^{-1}xg, q)$

$= \gamma_A((g^{-1}xg)^{-1}, q)$

$= \gamma_A(g^{-1}x^{-1}g, q)$

$= \gamma_A g^{-1}(x^{-1}, q)$ for all x, $y \in G$ and $q \in Q$.

Hence, $g\mu_A g^{-1}$&$g\ \gamma_A g^{-1}$ are also an Intuitionistic Q-fuzzy subgroups of a group G.

Theorem: 3.7

Let A and B be two Intuitionistic Q-fuzzy subgroups of a group G. Then $A \cap B$ is also an Intuitionistic Q-fuzzy subgroup of a group G.

Proof:

Let A and B be two Intuitionistic Q-fuzzy subgroups of a group G.

(i) $(A \cap B)(xy^{-1}, q)$ $= \min\{\mu_A(xy^{-1}, q), \mu_B(xy^{-1}, q)\}$

$\geq \min\ \{\min\ \{\mu_A(x, q), \mu_A(y, q)\}, \min\{\mu_B(x, q), \mu_B(y, q)\}\}$

$\geq \min\ \{min\ \{\{\mu_A(x, q), \mu_B(x, q)\}, \min\{\mu_A(y, q), \mu_B(y, q)\}\}$

$= \min\ \{(A \cap B)(x, q), (A \cap B)(y, q)\}$

Thus, $(A \cap B)(xy^{-1}, q) \geq \min\{(A \cap B)(x, q), (A \cap B)(y, q)\}$

Similarly,$(A \cap B)\ (xy^{-1}, q) = \max\ \{\ \gamma_A(xy^{-1}, q), \gamma B(xy^{-1}, q)\}$

$\leq \max\ \{\max\ \{\ \gamma_A(x, q),\ \ \gamma_A(y, q), \max\{\gamma_B(x, q), \gamma_B(y, q)\}\}$

$\leq \max\ \{\max\{\ \gamma_A(x, q), \gamma_B(x, q)\}, \max\{\ \gamma_A(y, q), \gamma_B(y, q)\}\}$

$= \max\{(A \cap B)(x, q), (A \cap B)(y, q)\}$

Thus $(A \cap B)(xy^{-1}, q) \leq max\{(A \cap B)(x, q), (A \cap B)(y, q)\}$

(ii) $(A \cap B)(x, q) = \min\{\mu_A(x, q), \mu_B(x, q)\}$

$= \min\{\mu_A(x^{-1}, q), \mu_B(x^{-1}, q)$

$= \min\{(A \cap B)(x^{-1}, q)\}$

$(A \cap B)(x, q) = \max\{\gamma_A(x, q), \gamma_B(x, q)\}$

$= \max\{\gamma_A(x^{-1}, q), \gamma_B(x^{-1}, q)$

$= \max\{(A \cap B)(x^{-1}, q)\}$

Hence, $A \cap B$ is also an Intuitionistic Q-fuzzy subgroup of a group G.

Theorem: 3.8

The intersection of any two Intuitionistic Q-fuzzy normal subgroup of a group G is also an Intuitionistic Q-fuzzy normal subgroup of a group G.

Proof:

Let A and B be two Intuitionistic Q-fuzzy normal subgroup of a group G.

According to previous theorem, $A \cap B$ is also an Intuitionistic Q-fuzzy subgroup of a Group G.

Now we have to prove that $A \cap B$ is a normal subgroup.

Now $\forall\, x, y \in G$ and $q \in Q$. we have,

$(A \cap B)(xyx^{-1}, q) = \min\{\mu_A(xyx^{-1}, q), \mu_B(xyx^{-1}, q)\}$

$= \min\{\mu_A(y, q), \mu_B(y, q)\}$

$= (A \cap B)(y, q)$

Similarly, $\forall x, y \in G$ and $q \in Q$. we have,

$(A \cap B)(xyx^{-1}, q) = \max\{\gamma_A(xyx^{-1}, q), \gamma_B(xyx^{-1}, q)\}$

$= \max\{\gamma_A(y, q), \gamma_B(y, q)\}$

$= (A \cap B)(y, q)$

Hence, $A \cap B$ is a Intuitionistic Q- fuzzy normal subgroup of a group G.

Definition: 3.9

The mapping f: $G\times Q \rightarrow H\times Q$ is said to be a **group Intuitionistic $\boldsymbol{Q}$-fuzzy homomorphism** if

(i) f:$G \rightarrow H$ is a group homomorphism

(ii) $f(xy, q) = (f(x)f(y), q)$ $\forall$ x,y∈G, q∈Q.

Theorem: 3.10

Let f: $G\times Q \rightarrow H\times Q$ be intuitionistic Q-fuzzy group homomorphism

(i) If A is a intuitionistic Q-fuzzy normal subgroup of a group H, then $f'(A)$ is a intuitionistic$Q$-fuzzy normal subgroup of a group G.

Proof:

Let f: $G\times Q \rightarrow H\times Q$ be a group intuitionistic Q-fuzzy homomorphism and let A be an intuitionistic Q-fuzzy normal subgroup of a group H.

Now for all x, y∈G, q∈Q

we have, $f'(\mu_A)(xyx^{-1}, q)$ $= \mu_A\ (f(xyx^{-1}, q))$

$= \mu_A\ (f\ (x)\ f\ (y)(f(x))^{-1}, q)$

$= \mu_A(f(y), q)$

$= f'(\mu_A)(y, q)$

Similarly, $f'(\gamma_A)(xyx^{-1}, q)$ $= \gamma A(f(xyx^{-1}, q))$

$= \gamma_A(f\ (x)\ f\ (y)(f(x))^{-1}, q)$

$= \gamma_A(f(y), q)$

$= f'(\gamma_A)(y, q)$

Hence, $f'(A)$ is a intuitionistic Q-fuzzy normal subgroup of a group G.

CHAPTER - 4

Properties of Anti- Q-Fuzzy Normal Subgroups

Definition: 4.1

A Q-fuzzy set μ of a group G is called an **Anti- Q-fuzzy subgroup** of G,

if for all x, $y \in G$, $q \in Q$,

(i) $\mu(xy, q) \leq \max\{\mu(x, q), \mu(y, q)\}$

(ii) $\mu(x^{-1}, q) = \mu(x, q)$

Definition: 4.2

An anti- Q- fuzzy subgroup μ of a group G is called an **anti-Q- fuzzy normal subgroup** of G if for all x, $y \in G$ and $q \in Q$,

$(xyx^{-1}, q) = \mu(y, q)$ or $\mu(xy,q) = \mu(yx,q)$.

Definition: 4.3

Let μ be an anti-Q-fuzzy subgroup of a group G. For any $t \in [0, 1]$,

we define the **level subset of μ** as, $\mu_t = \{x \in G, q \in Q \,/\, \mu(x, q) \leq t\}$.

Theorem: 4.4

Let μ be a Q-fuzzy subset of a group G. Then μ is an anti-Q-fuzzy subgroup of G iff the level subsets μ_t, $t \in [0, 1]$ are subgroups of G.

Proof:

Let μ be an anti-Q-fuzzy subgroup of G and the level subset.

$\mu_t = \{x \in G \,/\, \mu(x, q) \leq t, t \in [0, 1]\}$

Let $x, \in \mu_t$.Then $\mu(x, q) \leq$ & $\mu(y, q) \leq t$

Now , $\mu(xy^{-1}, q) \leq max\{\mu(x, q), \mu(y^{-1}, q)\}$

$= \max\{\mu(x, q), \mu(y, q)\}$

$\leq max \{t, t\}$

Therefore, μ $(xy^{-1}, q) \leq t$, hence $xy^{-1} \in \mu_t$.

Thus, μ_t is a subgroup of G.

Conversely, let us assume that μ_t be a subgroup of G.

Let $x, \in \mu_t$. Then μ $(x, q) \leq$ and μ $(y, q) \leq t$.

Also, $(xy^{-1}, q) \leq t$, since $xy^{-1} \in \mu_t$

$= max \{t , t\}$

$= max \{ \mu (x , q) , \mu (y , q) \}$

That is, $(xy^{-1}, q) \leq max \{\mu (x, q), \mu (y, q)\}$.

Hence, μ is an anti-Q-fuzzy subgroup of G.

Definition: 4.5

Let μ be an anti- Q-fuzzy subgroup of a group G. Then

N(μ) = $\{a \in G$ / μ $(axa^{-1}, q) = \mu(x , q)$, for all $x \in G$, $q \in Q$ }, is called an **anti-$\boldsymbol{Q}$ −fuzzy Normaliser of $\boldsymbol{\mu}$.**

Theorem: 4.6

Let μ be a Q −fuzzy subset of G. Then μ is an anti- Q- fuzzy normal subgroup of G if the level subsets μ_t, $t \in [0,1]$ are normal subgroups of G.

Proof:

Let μ be an anti-Q- fuzzy normal subgroup of G and the level subsetsμ_t , $t \in [0,1]$, is a subgroup of G.

Let $x \in G$ and $a \in \mu_t$, then μ $(a, q) \leq t$.

Now, $(xax^{-1}, q) = \mu$ $(a, q) \leq t$,

Since μ is an anti-Q-fuzzy normal subgroup of G, μ $(xax^{-1}, q) \leq t$.

Therefore, $xax^{-1} \in \mu_t$.

Hence, μ_t is a normal subgroup of G.

Theorem: 4.7

Let μ be an anti- Q-fuzzy subgroup of a group G. Then

(i) (μ) is a subgroup of G.

(ii) μ is an anti- Q-fuzzy normal $\Leftrightarrow(\mu) = G$.

(iii) μ is an anti Q-fuzzy normal subgroup of the group (μ).

Proof:

(i) Let $a, \in$ N(μ) then,

$\mu\ (axa^{-1}, q) = \mu\ (x, q), for\ all\ x\in G.$

$\mu\ (bxb^{-1}, q)\ \ = \mu\ (x, q), for\ all\ x\in G.$

Now, $(abx(ab)^{-1}, q)= \mu\ (abxba^{-1},q)$

$=\mu\ (bxb^{-1}, q)$

$= \mu\ (x, q)$

Thus, we get, $\mu\ ((ab)^{-1}, q) = \mu\ (x,)\Longrightarrow ab \in N(\mu)$

Therefore,N(μ) is a subgroup of G.

(ii) Clearly N$(\mu) \subseteq G$, is an anti- Q-fuzzy normal subgroup of G.

Let $a\in G$, then $\mu\ (axa^{-1}, q)\ = \mu\ (x, q)$.

Then $a\in(\mu) \Longrightarrow G\subseteq N(\mu)$.

Hence, $(\mu) = G$.

Conversely, let $(\mu) =G$.

Clearly, $\mu\ (axa^{-1}, q) =\mu\ (x, q)$, for all $x\in G\ and\ a\in G$.

Hence μ is an anti- Q – fuzzy normal subgroup of G.

(iii) From (ii), μ is an anti- Q-fuzzy normal subgroup of a group (μ).

Definition: 4.8

Let μ be a Q-fuzzy subset of G.

Let $: G \times Q \to G \times Q$ $[f_x: G \times Q \to G \times Q]$ be a function defined by,

${}_xf\,(a\,,\,q) = (xa\,,\,q)[f_x(a\,,\,q) = (ax\,,\,q)]$.

A ***Q*-fuzzy left (right) coset** ${}_x\mu$ (μ_x) is defined to be (μ) $(f_x(\mu))$.

It is easily seen that$({}_x\mu)(y\,,\,q) = \mu(x^{-1}y\,,\,q)$ *and*

$(\mu_x)(y, q) = \mu\,(yx^{-1}, q)$, for every $(y\,,\,q)$ in $G{\times}Q$.

Theorem: 4.9

Let μ be a Q-fuzzy subset of G. Then the following conditions are equivalent for each x, in G.

(i) $(xyx^{-1}, q) \geq \mu\,(y, q)$

$(ii)\mu\,(xyx^{-1}, q) = \mu\,(y, q)$

(iii) $\mu\,(xy\,,\,q) = \mu\,(yx\,,\,q)$

(iv) ${}_x\mu = \mu_x$

(v) ${}_x\mu_x{}^{-1} = \mu$

Proof:

Straight forward

Theorem: 4.10

If μ is an anti- Q-fuzzy subgroup of G, then $g\mu g^{-1}$ is also an anti- Q-fuzzy subgroup of G for all $g{\in}G$ and $q{\in}Q$.

Proof:

Let μ be an anti- Q-fuzzy subgroup of G.Then

(i) $(g\mu g^{-1})\,(xy, q) = \mu\,(g^{-1}(xy)\,g,q)$

$= \mu\ (g^{-1}(xgg^{-1}y)\ g, q)$

$= \mu\ ((g^{-1}xg)\ (g^{-1}yg), q)$

$\leq max\ \{\mu\ (g^{-1}xg, q), \mu\ (g^{-1}yg, q)\}$

$\leq max\ \{g\mu g^{-1}\ (x, q), g\mu g^{-1}\ (y, q)\}$,

$forall\ x\ , y\ in\ G\ and\ q \in Q.$

(*ii*) $g\mu g^{-1}\ (x, q) = \mu\ (g^{-1}xg, q)$

$= \mu\ ((g^{-1}\, xg)^{-1}\ , q)$

$= \mu\ (g^{-1}x^{-1}g, q)$

$= g\mu g^{-1}(x^{-1}, q), forall\ x, in\ G\ and\ q \in Q.$

Hence, $g\mu g^{-1}$ is an anti- Q-fuzzy subgroup of G.

Theorem: 4.11

If μ is an anti- Q-fuzzy normal subgroup of G, then $g\mu g^{-1}$ is also an anti- Q-fuzzy normal subgroup of G, for all $g \in G$ and $q \in Q$.

Proof:

Let μ be an anti- Q-fuzzy normal subgroup of G. then $g\mu g^{-1}$ is a subgroup of G.

Now $g\mu g^{-1}\ (xyx^{-1}, q) = \mu\ (g^{-1}(xyx^{-1})\ g, q)$

$= \mu\ (xyx^{-1}, q)$

$= \mu(\ y, q)$

$= \mu\ (gyg^{-1}, q)$

$= g\mu g^{-1}\ (y, q)$.

Thus, $g\mu g^{-1}$ is also an anti- Q-fuzzy normal subgroup of G.

Theorem: 4.12

The intersection of any two anti –Q-fuzzy subgroups of G is also an anti –Q-fuzzy subgroup of G.

Proof:

Let λ and μ be two anti- Q-fuzzy subgroups of G.

$(\lambda \cap \mu)$ $(xy^{-1},)$ $=min$ $(\lambda$ (xy^{-1}, q), $(xy^{-1}, q))$

$\leq min$ { max {$\lambda(x , q)$, $\lambda(y , q)$}, max {$\mu(x , q)$, μ (y , q)}}

$\leq max$ { min{$\lambda(x , q)$, $\mu(x , q)$}, min{$\lambda(y, q)$, $\mu(y , q)$}}

= max {$(\lambda \cap \mu)$ $(x,)$, $(\lambda \cap \mu)$ $(y,)$}

Thus,$(\lambda \cap \mu)$ (xy^{-1},q) $\leq max$ {$(\lambda \cap \mu)$ (x, q), $(\lambda \cap \mu)$ (y, q)}

Therefore, $(\lambda \cap \mu)$ is an anti Q −fuzzy subgroup of G.

Remark: 4.13

If μ_i, $i\in \Delta$ is an anti- Q-fuzzy subgroup of G, then $\cap_{i\in\Delta} \mu_i$ is an anti- Q-fuzzy subgroup of G.

Theorem: 4.14

The intersection of any two anti- Q-fuzzy normal subgroups of G is also an anti- Q- fuzzy normal subgroup of G.

Proof:

Let λ and μ be two anti- Q-fuzzy normal subgroups of G.

According to theorem 4.12, $\lambda \cap \mu$ is an anti- Q-fuzzy subgroup of G.

Now for all x, y in G, we have

$(\lambda \cap \mu)$ (xyx^{-1}, q) $= max$ $((xyx^{-1}, q)$, $(xyx^{-1}, q))$

$= max$ $(\lambda$ (y , q) , $\mu(y , q)$)

= $(\lambda \cap \mu)$(y, q)

Hence, $(\lambda \cap \mu)$ is an anti- Q-fuzzy normal subgroup of G.

Remark: 4.15

If μ_i, $i \in \Delta$ are anti- Q-fuzzy normal subgroup of G, then $\cap_{i\in\Delta}\ \mu_i$ is an anti- Q-fuzzy normal subgroup of G.

Definition: 4.16

The mapping f: $G\times Q \rightarrow H\times Q$ is said to be a **group *Q*-homomorphism** if

(i)f: $G \rightarrow H$ is a group homomorphism

(ii) f $(xy, q) = (f$ (x) f $(y), q)$, for all x, $y\in G$ and $q\in Q$.

Definition: 4.17

The mapping f: $G\times Q \rightarrow H\times Q$ is said to be a **group anti- *Q*-homomorphism**

if, (i) f: $G \rightarrow H$ is a group homomorphism

(ii) f $(xy, q) = (f$ $(y)(x), q)$, for all x, $y \in G$ and $q\in Q$.

Theorem: 4.18

Let f: $G \times Q \rightarrow H \times Q$ be a group anti- Q- homomorphism.

(i) If μ is an anti- Q $-$fuzzy normal subgroup of H, then $f^{-1}(\mu)$ is an anti- Q-fuzzy normal subgroup of G.

(ii) If f is an epimorphism and μ is an anti Q-fuzzy normal subgroup of G, then (μ) is an anti-Q-fuzzy normal subgroup of H.

Proof:

(i) Let f: $G \times Q \rightarrow H \times Q$ is a group anti- Q-homomorphism and let μ be an anti- Q-fuzzy Normal subgroup of H.

Now, for all x, $y \in G$, we have

$f^{-1}(\mu)(xyx^{-1}, q)$ $= \mu$ $(f$ (xyx^{-1}, q)

$= \mu$ $((x)^{-1}(y)f(x), q)$

$= \mu (f(y) , q)$

$= f^{-1}(\mu) (y, q)$

Hence, $f^{-1}(\mu)$ is an anti- Q-fuzzy normal subgroup of G.

(ii) Let μ be an anti- Q-fuzzy normal subgroup of G.

Then $f (\mu)$ is an anti- Q- fuzzy subgroup of H.

Now, for all u, $in\ H$, we have $(\mu)(uvu^{-1}, q) = \inf \mu (y, q)$

$= \inf \mu (xyx^{-1}, q)$

$f (y) = uvu^{-1}$

$f(x) = u ; f(y) = v$

$= \inf \mu (y, q)$

$= f(\mu)(v , q)$

$f (y) = v$ (since f is an epimorphism)

Hence, (μ) is an anti- Q-fuzzy normal subgroup of H.

Definition: 4.19

Let λ and μ be two Q-fuzzy subsets of G. The product of λ and μ is defined to be the **Q-fuzzy subset $\lambda\mu$** of G is given by,

$$\lambda \mu(x , q) = \inf_{yz = x} \max (\lambda(y , q) , \mu(z , q)) , x \in G.$$

Theorem: 4.20

If $\lambda \& \mu$ are anti- Q-fuzzy normal subgroups of G, then $\lambda\mu$ is an anti- Q-fuzzy normal subgroup of G.

Proof:

Let λ & μ be two anti- Q-fuzzy normal subgroups of G.

(i) $\lambda\mu (xy, q) = inf\ max \{\lambda (x_1y_1, q), \mu (x_2y_2, q)\}$

By substituting, $x_1y_1= x$, $x_2y_2= y$

$\leq$ inf max {max {λ (x_1, q), λ (y_1, q)},max {μ (x_2, q), μ (y_2, q)}}

By substituting, $x_1y_1= x$, $x_2y_2= y$

$\leq max$ {$infmax$ {λ (x_1, q), λ (y_1,q)}, $infmax$ {μ (x_2,q),μ (y_2,q)}}

By substituting, $x_1y_1= x$, $x_2y_2= y$

$\lambda\, \mu\, (xy\, ,\, q) \quad \leq max\, \{\, \lambda\mu\, (x\, ,\, q)\, ,\, \lambda\mu\, (y\, ,\, q)\, \}$

(ii) $\lambda\mu$ (x^{-1}, q) = inf max {μ (z^{-1}, q), λ (y^{-1}, q)}

By substituting, $(yz)^{-1}= x^{-1}$

= inf max {μ (z,), λ (y, q)}

By substituting, $x= yz$

= inf max {λ (y,), μ (z,)}

By substituting, x =yz

= $\lambda\mu(x\, ,\, q)$.

Hence, $\lambda\, \mu$ is an anti- Q-fuzzy normal subgroup of G.

Cartesian Product of Anti -Q-Fuzzy Normal Subgroups

Theorem: 4.21

If μ & δ are two anti-Q-fuzzy subgroups of a group G, then $\mu \times \delta$ is also an anti-Q- fuzzy subgroup of the group $G \times G$.

Proof:

Let μ & δ be two anti-Q-fuzzy subgroups of a group G.

Let (x_1, y_1),(x_2, y_2)∈ $G \times G$ and q∈Q.

Then, $(\mu{\times}\delta)\{((x_1,\ y_1), (x_2, y_2)^{-1}, q)\}$ = $(\mu{\times}\delta)\{((x_1, y_1), (x_2^{-1}, y_2^{-1}), q)\}$

$= (\mu\times\delta)\{((x_1x_2^{-1},\ \ y_1y_2^{-1}), q)\}$

$= \max\{\mu(x_1x_2^{-1}, q), \delta(\ y_1y_2^{-1}, q)\}$

$= \max\{\mu(x_1, q), \mu(x_2^{-1}, q), \delta(\ y_1, q), \delta(y_2^{-1}, q)\}$

$= \max\ \{\ \ (x_1, q), \mu\ (x_2, q), \delta\ (\ y_1, q), \delta(\ y_2, q)\}$

$= \max\ \{(\mu \times \delta)((x_1,\ y_1), q), (\mu \times \delta)((x_2,y_2\ ),q)\}$

Therefore, $(\mu\times\delta)$ is an anti-Q-fuzzy subgroup of $G \times G$.

Theorem: 4.22

If μ & δ are two anti-Q-fuzzy normal subgroups of a group G, then $\mu \times \delta$ is also an anti-Q-fuzzy normal subgroup of the group $G \times G$.

Proof:

Straight forward.

CHAPTER - 5

A Review on Q-Fuzzy Subgroups in Algebra

Theorem: 5.1

If μ is a Q-fuzzy subgroup of a group G if and only if $(\mu^c)^c$ is a Q- fuzzy subgroup of a group G.

Proof:

Suppose μ is a Q-fuzzy subgroup of a group G then for all x, $y \in G$ and $q \in Q$,

$\mu\,(xy, q) \geq \min\{\mu\,(x, q), \mu\,(y, q)\}$

Now, $1 - \mu^c\,(xy, q) \geq \min\{1 - \mu^c(x, q), 1 - \mu^c(y, q)\}$

$\Leftrightarrow \mu^c(xy, q) \quad \leq 1 - \min\{\mu^c\,(x, q), \mu^c\,(y, q)\}$

$\mu^c(xy, q) \quad \leq \max\{\mu^c\,(x, q), \mu^c\,(y, q)\}$

$[\mu^c\,(xy, \mathrm{q})]^c \leq [\max\{\mu^c(x, \mathrm{q}), \mu^c(\mathrm{y}, \mathrm{q})\}]^c$

$1 - \mu^c\,(xy, q) \geq \min\{1 - \mu^c(x, q), 1 - \mu^c(y, q)\}$

$\mu\,(xy, q) \quad \geq \min\{\mu\,(x, q), \mu\,(y, q)\}$

We have, $\mu\,(x, q) = \mu\,(x^{-1}, q)$ for all $x \in G$ *and* $q \in Q$.

$\Leftrightarrow 1 - \mu^c(x, q) \quad = 1 - \mu^c\,(x^{-1}, q)$

$\mu^c(x, q) \quad = \mu^c\,(x^{-1}, q)$

$[\mu^c\,(x, \mathrm{q})]^c \quad = [\mu^c\,(x^{-1}, \mathrm{q})]^c$

$1 - \mu^c\,(x, q) \quad = 1 - \mu^c\,(x^{-1}, q)$

$\mu\,(x, q) \quad = \mu\,(x^{-1}, q)$

Hence, $(\mu^c)^c$ is a Q- fuzzy subgroup of a group G.

Theorem: 5.2

If A is a Q- fuzzy subgroup of a group$G$ if and only if

$A\,(xy^{-1}, q) \geq \min\,\{A\,(x, q), A\,(y, q)\}$ for all $x, y \in G$ and $q \in Q$.

Proof:

Let A be a Q- fuzzy subgroup of a group G.

Then for all $x, y \in G$ and $q \in Q$

$A\,(xy, q) \qquad \geq \min\,\{A\,(x, q), A\,(y, q)\}$ an $(x^{-1}, q) = A\,(x, q)$

Now, $A\,(xy^{-1}, q) \geq \min\,\{A\,(x, q), A\,(y^{-1}, q)\}$

$A\,(xy^{-1}, q) \qquad \geq \min\,\{A\,(x, q), A\,(y, q)\}$ by given condition.

Therefore, $\Leftrightarrow A\,(xy^{-1}, q) \geq \min\,\{A\,(x, q), A\,(y, q)\}$

Theorem: 5.3

If A is an anti-Q-fuzzy subgroup of group G then gAg^{-1} is also an anti-Q-fuzzy subgroup of group G for all $g \in G$ and $q \in Q$.

Proof:

Let A be an anti-Q-fuzzy subgroup of group G.

Then for all $g \in G$ and $q \in Q$

(i) $gAg^{-1}(xy, q) \quad = (g^{-1}(xy), q)$

$= A\,(g^{-1}(xgg^{-1}y)\,g, q)$

$= A\,((g^{-1}xg)(g^{-1}yg)\,, q)$

$\leq \max\{A((g^{-1}xg), q), A((g^{-1}yg), q)\}$ $for\ all\ x, y \in G$ and $q \in Q$.

(ii) $gAg^{-1}(x, q) \quad = A\,(g^{-1}xg, q)$

$= A\,((g^{-1}xg)^{-1}), q)$

$= A\,(g^{-1}x^{-1}g, q)$

$= gAg^{-1}(x^{-1}, q) \quad for\ all\ x, y \in G\ and\ q \in Q.$

Hence, gAg^{-1} is also an anti-Q-fuzzy subgroup of group G for all $g \in G$ and $q \in Q$.

Theorem: 5.4

Let G be a group. Let μ be a Q-fuzzy normal subgroup of a group G if and only if μ^c is an anti-Q-fuzzy normal subgroup of group G.

Proof:

Let G be a group. Let μ be a Q-fuzzy normal subgroup of a group G.

That is $\mu\ (xyx^{-1},) = \mu\ (y, q)$

Now we have to show that μ^c is an anti-Q-fuzzy subgroup of a group G.

$\mu\ (xy, q) \quad \geq \min\ \{\mu\ (x, q), \mu\ (y, q)\}$

$1 - \mu^c(xy, q) \ \geq \min\ \{1 - \mu^c\ (x, q), 1 - \mu^c(y, q)\}$

$\mu^c\ (xy, q) \quad \leq 1 - \min\ \{1 - \mu^c(x, q), 1 - \mu^c(y, q)\}$

$\mu^c(xy, q) \quad \leq \max\ \{\mu^c(x, q), \mu^c(y, q)\}$

Hence, μ^c is an anti-Q-fuzzy subgroup of a group G.

Given μ is a Q-fuzzy normal subgroup of a group G.

That is $\mu\ (xyx^{-1},) \ =\mu\ (y, q)$

$1 - \mu^c\ (xyx^{-1},) \quad = 1 - \mu^c(y, q)$

$\mu^c\ (xyx^{-1},) \quad = \mu^c(y, q)$

Therefore, μ^c is an anti-Q-fuzzy normal subgroup of a group G.

Theorem: 5.5

Let A be a -Q-fuzzy normal subgroup of a group G with identity e.

Then$A\ (xy, q) = A\ (yx, q)$ for all $x, y \in G\ and\ q \in Q$.

Proof:

Given A is a Q-fuzzy normal subgroup of a group G.

That is $A\ (xy^{-1}x,\ q) = A\ (y,\ q)$

Now $(xy,\ q) = A\ (xy\ (xx^{-1}),\ q)$

$= A\ ((xyx)\ x^{-1},\ q)$

$= A\ (x\ (yx)\ x^{-1},\ q)$

$= A(yx,\ q)$

Therefore $A\ (xy,\ q) = A\ (yx,\ q)$

Definition: 5.6

Let X be a field. Let $F\ and\ Q$ be any two fuzzy sets in X. A mapping $\mu_F\colon X \times Q \rightarrow [0,1]$ is called ***Q*-fuzzy set** in X.

Definition: 5.7

Let μ_F be a Q-fuzzy set in a field X is said to be Q-fuzzy field in X if for $x,y \in \mu_F\ and\ q \in Q$.

(i) $\mu_F((x + y),\ q) \geq \min\ \{\mu_F(x,\ q),\ \mu_F(y,\ q)\}$

(ii) $\mu_F\ (-x,\ q) \geq \mu_F\ (x,\ q)$

(iii) $\mu_F((xy),\ q) \geq \min\ \{\mu_F\ (x,\ q),\ \mu_F\ (y,\ q)\}$

(iv) $\mu_F\ (x^{-1},\ q \geq \mu_F\ (x,\ q),\ x \neq 0\ in\ X$.

Theorem: 5.8

If μ_F be a Q-fuzzy field in X and λ_F be a subset of μ_F. Then $\lambda_F$is a Q-fuzzy subfield of μ_F in X.

Proof:

Given μ_F is a Q-fuzzy field in X.

Let x, $y \in \lambda_F$ and $q \in Q$.

From the definition,

(i) $\mu_F\ ((x + y), q) \geq \min\ \{\mu_F\ (x, q), \mu_F\ (y, q)\}$

(ii) $\mu_F\ (-x, q) \quad \geq\ \mu_F\ (x, q)$

(iii) $\mu_F((xy), q)\ \geq \min\ \{\mu_F\ (x, q), \mu_F\ (y, q)\}$

(iv) $\mu_F\ (x^{-1}, q)\ \ \geq \mu_F\ (x, q), x \neq 0\ in\ X\ for\ all\ x, \in \lambda_F\ and\ q \in Q.$

Hence, λ_F is a fuzzy field in X.

Therefore, λ_F is a Q-fuzzy subfield of μ_F in X.

CHAPTER - 6

A study on Q-fuzzy normal subgroups and cosets

Definition: 6.1

Let A be a Q-fuzzy subgroup of a group $(G, \cdot)$. Then for any $a\ and\ b$ in G, a **$\boldsymbol{Q}$-fuzzy middle coset** aAb of G is defined by $(aAb)(x, q) = A(a^{-1}xb^{-1}q)$, for every x in G and q in Q.

Definition: 6.2

Let A be a Q-fuzzy subgroup of a group $(, \cdot)$ and a in G. Then the **pseudo $\boldsymbol{Q}$-fuzzy coset** (aA) is defined $by((aA)^{\ p})\ (x, \ \ )= p(a)\ A(x, q)$, for every x in G and for some p in P and q in Q.

Definition: 6.3

A Q-fuzzy subgroup A of a group G is called a **generalized characteristic $\boldsymbol{Q}$-fuzzy subgroup** (GCQFSG) if for all x and y in G, $(x) = (y)$ implies

$A\ (x, q) =(y, q)$, q in Q.

Some Properties of $\boldsymbol{Q}$-fuzzy normal subgroups

Theorem: 6.4

Let $(, \cdot)$ be a group and Q be a non-empty set. If A and B are two Q-fuzzy normal subgroups of G, then their intersection $A \cap B$ is a Q-fuzzy normal subgroup of G.

Proof:

Let $x\ and\ y$ in G and q in Q and A ={ [(x, q), A(x, q)] / x in G and q in Q }and B = { [(x, q), $B(x, q)$] / $x\ in\ G\ and\ q\ in\ Q$ } be a Q-fuzzy normal subgroups of G.

Let $C = A \cap B\ and\ C = \{[(, q), C(x, q)]\ /\ x\ in\ G\ and\ q\ in\ Q\}$,

Where $(x, q) =\ n\{A(x, q), B(x, q)\}$.Then, Clearly, C is a Q-fuzzy subgroup of G, since A and B are two Q-fuzzy subgroups of G.

And, $C(xy, q) = min\ \{A(xy, q), B(xy, q)\ \}, = min\{\ A(yx, q), B(yx, q)\ \}\ =\ C(yx, q)$.

Therefore, $(xy, q) =\ (yx, q)$, for all x and y in G and q in Q.

Hence, $A \cap B$ is a Q-fuzzy normal subgroup of a group G.

Theorem : 6.5

Let $(G, \cdot\ )$ be a group and Q be a non-empty set. The intersection of a family of Q-fuzzy normal subgroups of G is a Q-fuzzy normal subgroup of G.

Proof:

Let $\{\ A_i\}_{\in I}$ be a family of Q-fuzzy normal subgroups of G and $A = \bigcap_{i\in I} A_i$. Then for x and y in G and q in Q, clearly the intersection of a family of Q-fuzzy subgroups of a group G is a Q-fuzzy subgroup of a group G.

Now, $(\ , q\ )\ =\ \inf_{i\in I} A_i(xy, q)$

$$= \inf_{i\in I} A_i(yx, q)$$

$$= A(yx, q)$$

Therefore, $(\ xy, q\ )\ = A(yx, q)$ for all $x, y\ in\ G$ and $q\ in\ Q$.

Hence, the intersection of a family of Q-fuzzy normal subgroups of G is a Q-fuzzy normal subgroup of G.

Theorem :6.6

If A is a Q-fuzzy characteristic subgroup of a group G, then A is a Q-fuzzy normal subgroup of a group G.

Proof:

Let A be a Q-fuzzy characteristic subgroup of a group G, x *and* y in G and q in Q.

Consider the map $f{:}G \times Q \rightarrow G \times Q$ defined by $(x, q) = (y^{-1}, q)$.

Clearly, f in $Q - AutG$.

Now, $(xy, q) = ((xy, q))$

$= ((xy)^{-1}, q)$

$= A(yx, q)$.

Therefore, $(xy, q) = (yx, q)$, for all x *and* y *in* G *and* q *in* Q.

Hence, A is a Q-fuzzy normal subgroup of a group G.

Theorem: 6.7

A Q-fuzzy subgroup A of a group G is a Q-fuzzy normal subgroup of G if and only if A is constant on the conjugate classes of G.

Proof

Suppose that A is a Q −fuzzy normal subgroup of a group G.

Let x *and* y *in* G *and* q *in* Q.

Now, $A\,(y^{-1}xy, \ ) = (xyy^{-1}, q)$

$= A(x, q)$.

Therefore, $(y^{-1}xy, q) = (x, q)$, for all x *and* y *in* G *and* q *in* Q.

Hence, $(x) = \{y^{-1}xy / y \in G\}$.

Hence, is constant on the conjugate classes of G.

Conversely, suppose that A is constant on the conjugate classes of G.

Then, $A\,(xy, q) = (xyxx^{-1}, \ )$

$= ((yx)x^{-1}, q)$

$= A(yx, q)$.

Therefore, $(xy, q) = (yx, q)$, for all x *and* y *in* G *and* q *in* Q.

Hence, A is a Q-fuzzy normal subgroup of a group G.

Theorem: 6.8

Let A be a Q-fuzzy normal subgroup of a group G. Then for any y in G and q in Q, we have $(yxy^{-1}, q) = (y^{-1}xy, \ )$, for every x *in* G.

Proof:

Let A be a Q-fuzzy normal subgroup of a group G.

For any y *in* G *and* q *in* Q, we have,

$A\,(\,y^{-1}, q) \qquad = A(x, q)$

$= (xyy^{-1}, \ )$

$= (y^{-1}\,, q\,).$

Therefore, $A\,(\,y^{-1}, q) = A(y^{-1}xy, q\,)$, for all x and y in G and q in Q.

Theorem: 6.9

A Q-fuzzy subgroup A of a group G is normalized if and only if $(e, q) = 1$, where e is the identity element of the group G and q in Q.

Proof:

If A is normalized, then there exists x in G such that $A\,(x, q) = \ 1$, but by properties of a Q-fuzzy subgroup A of G,

$(x, q) \leq (e, q)$, for every x in G and q in Q.

Since, $(x, q) = 1$ and

$A(x, q) \leq A(e, q)$,

$1 \leq A\,(e, q)$,

But $1 \geq A(e, q)$.

Hence $A(e, q) = 1$.

Conversely, if $A(e, q) = 1$, then by the definition of normalized Q-fuzzy subset, A is normalized.

Theorem: 6.10

Let A and B be Q-fuzzy subgroups of the groups G and H, respectively. If A and$B$ are Q-fuzzy normal subgroups, then $A \times B$ is a Q-fuzzy normal subgroup of $G \times H$.

Proof:

Let A and B be Q-fuzzy normal subgroups of the groups G and H respectively.

Clearly, $A \times B$ is a Q-fuzzy subgroup of $G \times H$.

Since, A and B are Q-fuzzy subgroups of G and H.

Let x_1 and x_2 be in G, y_1 and y_2 be in H and q in Q.

Then (x_1, y_1) and (x_2, y_2) are in $G \times H$.

Now , $A \times B[(x_1, y_1)(x_2, y_2), q] = A \times B((x_1 x_2, y_1 y_2), q)$

$= \min\{(x_1 x_2, q), (y_1 y_2, q)\}$

$= \min\{A(x_2 x_1, q), B(y_2 y_1, q)\}$

$= A \times B((x_2 x_1, y_2 y_1), q)$

$= A \times B[(x_2, _2)(x_1, y_1), q]$

Therefore, $A \times B[(x_1, y_1)(x_2, y_2), q] = A \times B[(x_2, y_2)(x_1, y_1), q]$

Hence, $A \times B$ is a Q-fuzzy normal subgroup of $G \times H$.

Theorem: 6.11

Let a Q-fuzzy normal subgroup A of a group G be conjugate to a Q-fuzzy normal subgroup M of G and a Q-fuzzy normal subgroup B of a group H be conjugate to a Q-

fuzzy normal subgroup N of H. Then a Q-fuzzy normal subgroup $A \times B$ of a group$G\times H$ is conjugate to a Q-fuzzy normal subgroup $M \times N$ of $G \times H$.

Proof:

It is trivial.

Theorem: 6.12

Let A be a Q-fuzzy subgroup of a finite group G, then (A) / (G).

Proof:

Let A be a Q-fuzzy subgroup of a finite group G with e as its identity element. Clearly, $H = \{ \in G$ / $A(x, q) = A(e, q)\}$ is a subgroup of G for H is a α −level subset of G where $\alpha = A(e, q)$.

By Lagrange's theorem (H) / (G).

Hence, by the definition of the order of the Q-fuzzy subgroup of G, we have (A) /(G).

Theorem: 6.13

Let A and B be two Q-fuzzy subsets of an abelian group G. Then A and B are conjugate Q-fuzzy subsets of the abelian group G if and only if $A = B$.

Proof:

Let A and B be conjugate Q-fuzzy subsets of abelian group G, then for some

y in G,

we have, $A(x, q) = B(y^{-1}xy, q)$, for every x in G and q in Q

$= B(yy^{-1}x,)$,since G is an abelian group,

$= B(ex, q)$

$= B(x, q)$

Therefore, $A(x, q) = (x, q)$, for every x in G and q in Q.

Hence, $A = B$.

Conversely, if $A =$, then for the identity element e of G,

we have, $A(x, q) = B(e^{-1}xe, q)$, for every x in G and q in Q.

Hence, A and B are conjugate Q-fuzzy subsets of G.

Theorem: 6.14

If A and B are conjugate Q-fuzzy subgroups of the normal group G, then $O(A) = O(B)$.

Proof

Let A and B be conjugate Q-fuzzy subgroups of G.

Now, (A) = order of $\{ \in G / A(x, q) = A(e, q) \}$

= order of $\{ \in G / B(y^{-1}xy, q) = B(y^{-1}ey, q) \}$

= order of $\{ \in G / B(x, q) = B(e, q)\}$

$= O(B)$

Hence, $(A) = (B)$.

Theorem: 6.15

Let A be a Q-fuzzy subgroup of a group G, then the pseudo Q-fuzzy coset (aA) is a Q-fuzzy subgroup of a group G, for every a in G.

Proof:

Let A be a Q-fuzzy subgroup of a group G. For every x and y in G and q in Q,

we have,$((aA)p)(xy^{-1}, q) = p(a)A(xy^{-1}, q)$

$\geq p(a)\, min\{A(x, q), A(y, q)\}$

$= min\{ p(a)A(x, q), p(a)A(y, q) \}$

$= min\{ ((aA)p)(x, q), ((aA)p)(y, q) \}$.

Therefore, $((aA)p)(xy^{-1}, q) \geq min \{ ((aA)p)(x, q), ((aA)p)(y, q) \}$, for x and y in G and q in$Q$.

Hence, (aA) is a Q-fuzzy subgroup of a group G.

Theorem: 6.16

If A is a Q-fuzzy subgroup of a group G, then for any a in G the Q-fuzzy middle coset aAa^{-1} of G is also a Q-fuzzy subgroup of G.

Proof:

Let A be a Q-fuzzy subgroup of G and a in G.

To prove aAa^{-1} is a Q-fuzzy subgroup of G.

Let x and y in G and q in Q.

Then $(a\,Aa^{-1})(xy^{-1}, q) = A(a^{-1}xy^{-1}a, q)$,

$= (a^{-1}xaa^{-1}y^{-1}a, q)$

$= A((a^{-1}xa)(a^{-1}ya)^{-1}, q)$

$\geq min \{ (a^{-1}xa, q), A((a^{-1}ya)^{-1}, q) \}$

$\geq min \{ (a^{-1}xa, q), A(a^{-1}ya, q)$, since A is a QFSG of G

$= min \{(aAa^{-1})(x, q), (aAa^{-1})(y, q)\}$.

Therefore, $(aAa^{-1})(xy^{-1}, q) \geq min\{(aAa^{-1})(x, q), (aAa^{-1})(y, q)\}$.

Hence, aAa^{-1} is a Q-fuzzy subgroup of a group G.

Theorem: 6.17

Let A be a Q-fuzzy subgroup of a group G and aAa^{-1} be a Q-fuzzy middle coset of G, then $(aAa^{-1}) = (A)$, for any a in G.

Proof:

Let A be a Q-fuzzy subgroup of G and a in G. By Theorem 6.16, the Q-fuzzy middle coset aAa^{-1} is a Q −fuzzy subgroup of G.

Further by the definition of a Q-fuzzy middle coset of G,

we have,$(aAa^{-1})(x, q) = A(a^{-1}xa, q)$ for every $x\ in\ G$ and $q\ in\ Q$.

Hence for any $a\ in\ G$, $A\ and\ aAa^{-1}$ are conjugate Q-fuzzy subgroups of a group G

as there exists $a\ in\ G$ such that $(aAa^{-1})(x, q) = A(a^{-1}xa, q)$ for every $x\ in\ G\ and\ q\ in\ Q$.

By Theorem 6.14, $(aAa^{-1}) = (A)$ for any $a\ in\ G$.

Theorem: 6.18

Let A be a Q-fuzzy subgroup of a group G and be a Q-fuzzy subset of a group G. If A and B are conjugate Q- fuzzy subsets of the group G, then B is a Q-fuzzy subgroup of a group G.

Proof:

Let A be a Q-fuzzy subgroup of a group G and B be a Q- fuzzy subset of G. And, let A and B be conjugate Q-fuzzy subsets of G.

To prove B is a Q-fuzzy subgroup of G. Let x and y in G and q in Q.Then xy^{-1} in G.

Now, (xy^{-1}, q) $= (g^{-1}xy^{-1}g, q)$, for some g in G

$= A\ (g^{-1}xgg^{-1}y^{-1}g, q)$

$= A\ ((g^{-1}xg)\ (g^{-1}yg)^{-1}, q)$

$\geq m\{\ A(g^{-1}xg, q\ ), A((g^{-1}yg\ )^{-1}, q)\ \}$

$\geq m\{\ A(g^{-1}xg, q), A(g^{-1}yg, q)$, since A is a QFSG of G

$= min\ \{\ B(x, q), B(y, q)\ \}$.

Therefore, (xy^{-1}, q) $\geq min\ \{\ (x, q\ ), B(y, q\ )\ \}$, for x and y in G and q in Q.

Hence, B is a Q-fuzzy subgroup of the group G.

Theorem: 6.19

Let A be a Q-fuzzy subgroup of a group G. Then $(x, q)\, A = (y, q)A$, for x, y in G if and only if $A(x^{-1}y, q) = A(y^{-1}x, q) = A(e, q)$.

Proof:

Let A be a Q-fuzzy subgroup of a group G.

Let $(x, q)A = (y, q)A$, for x and y in G and q in Q.

Then, $(x, q)(x, q) = (y, q)A(x, q)$ and

$(x, q)A(y, q) = (y, q)A(y, q)$,

$\Rightarrow A\, (x^{-1}x, \ ) = A(y^{-1}x, q)$ and

$(x^{-1}y, q) = (y^{-1}y, q)$.

Hence, $(e, q) = A(y^{-1}x, q)$ and $A(x^{-1}y, q) = A(e, q)$.

Therefore, $(x^{-1}y, q) = A(y^{-1}x, q)$

$= A\, (e, q)$, for x and y in G and q in Q.

Conversely, let $(x^{-1}\, y, q) = (y^{-1}x, q)$

$= A\, (e, q)$, for x and y in G and q in Q.

For every g in G and

we have, $(x, q)A(g, q) = A(x^{-1}g, q)$

$= (x^{-1}yy^{-1}g, \ )$

$\geq min\ \{ (x^{-1}y, q), A(y^{-1}g, q) \}$

$= min\ \{ (e, q), A(y^{-1}g, q) \}$

$= A\, (y^{-1}g, \ )$

$= (y, q)A(g, q)$.

Therefore, $(x, q)(g, q) \geq (y, q)A(g, q)$ ------------ (1)

And , $(y, q)A(g, q) = A(y^{-1}g, q)$

$= A\ (y^{-1}xx^{-1}g,\ )$

$\geq min\ \{\ (y^{-1}x,\ q\ ), A(x^{-1}g\ ,\ q)\ \}$

$= min\ \{\ (e, q)\ , A(x^{-1}g, q\ )\ \}$

$= A\ (x^{-1}g,\ )$

$=\ (x, q)A(g, q).$

Therefore, $(y, q)\ A\ (g, q) \ \geq\ (x, q)A(g, q)$ ------------- (2)

From (1) and (2) we get,

$(x, q)\ A\ (g, q) =\ (y, q)A(g, q)$-----(3)

We get, $(x, q)\ A\ =\ (y, q)A$, for all x and y in G and q in.

CONCLUSION

In this project I have concentrated on anti-Q-fuzzy normal subgroups, anti-Q-fuzzy normaliser and anti-Q-fuzzy normal subgroups under anti-Q- homomorphism. The anti-group Q- homomorphism and cartesian product of anti-Q-fuzzy normal subgroups and some properties of Q-fuzzy normal subgroups have been explained. Some results on various Q-fuzzy groups have been discussed. I have taken through Intuitionistic Q-fuzzy normal subgroups, n-generated Q-fuzzy Normalizer and Intuitionistic Q-fuzzy subgroups under homomorphism. I have also examined generalized characteristic Q-fuzzy subgroups and pseudo-Q-fuzzy coset and Q-fuzzy middle coset. Interestingly, it has been observed that Q-fuzzy concept adds another dimension to the defined fuzzy normal subgroups. This concept can further be extended for new results.

REFERENCES

[1] Akgul. M, “Some properties of fuzzy groups”, Journal of Mathematical Analysis and Applications, 133(1988), 93-100.

[2] Akram.M, Shum.K.P, “Fuzzy Lie Ideals over a Fuzzy Field”, Italian Journal of Pure and Applied Mathematics – n. 27−2010 (281−292) 281

[3] Akram.M and Dar.K.H, “On fuzzy d-algebras”, Punjab university journal of mathematics, 37(2005), 61-76.

[4] Anthony.J.M. and Sherwood.H, “Fuzzy groups Redefined”, Journal of mathematical analysis and applications, 69,124 -130 (1979)

[5] Atanassov. K.T “Intuitionistic fuzzy sets”, Fuzzy sets and Systems 20(1986) no.1, 87-96.

[6] Azriel Rosenfeld, “Fuzzy Groups”, Journal of mathematical analysis and applications, 35, 512-517 (1971).

[7] Asaad Mohamed, “Groups and Fuzzy subgroups”, Fuzzy sets and systems 39(1991) 323- 328.

[8] Bhattacharya . P, “Fuzzy Subgroups: Some Characterizations”, J.Math. Anal. Appl.128 (1987),241 – 252.

[9] Biswas. R, “Fuzzy subgroups and anti-fuzzy subgroups”, Fuzzy sets and systems, 35(1990), 121-124.

[10] Choudhury.F.P. and Chakraborty.A.B. and Khare.S.S., "A note on fuzzy subgroups and Fuzzy homomorphism" ,Journal of mathematical analysis and applications 131, 537 - 553 (1988).

[11] Das. P.S, "Fuzzy groups and level subgroups", J.Math.Anal. Appl, 84(1981) 264-269.

[12] Davvaz.B and Wieslaw.A.Dudek, "Fuzzy n-ary groups as a generalization of rosenfield fuzzy groups", ARXIV- 0710.3884VI (MATH.RA) 20 OCT 2007, 1-16.

[13] Dixit.V.N., Rajesh Kumar, Naseem Ajmal., "Level subgroups and union of fuzzy subgroups", Fuzzy Sets and Systems, 37, (1990),359-371.

[14] Eroglu. M.S, "The homomorphic image of a fuzzy subgroup is always a fuzzy subgroup", Fuzzy Sets and Systems, 33(1989), 255-256.

[15] "Gopalakrishnamoorthy.G., Ph.D Thesis", Alagappa university, Karaikudi, Tamilnadu, India, May (2000).

[16] Kandasamy. W.B.V, "Smarandache Fuzzy Algebra", American Research Press, (2003).

[17] Majumder. S.K and Sardar. S.K, "On some properties of anti fuzzy subgroups", J. Mech. Cont. and Math. Sci., 3(2) (2008), 337-343.

[18] Malik D.S. and Mordeson J.N. "Fuzzy subgroups of abelian groups", Chinese Journal of Math., vol. 19 (2), 1991, 21-28. 19.

[19] Mashour. A.S, Ghanim M.H. and Sidky. F.I, "Normal Fuzzy Subgroups",Univ.u Novom Sadu Zb.Rad.Prirod.-Mat.Fak.Ser.Mat. 20(2) ,1990, 53-59

[20] Mustafa Akgul, “Some properties of fuzzy groups”, Journal of mathematical analysis and applications 133, 93-100 (1988).

[21] Muthuraj.R., Sithar Selvam.P.M., Muthuraman.M.S., “Anti Q- fuzzy group and its lower-Level subgroups”, International journal of Computer Applications 3(3),2010,16-20.

[22] Palaniappan. N, Muthuraj. R, “Anti Fuzzy group and Lower Level Subgroups”, Antarctica J.Math.1 (1) (2004), 71-76

[23] Priya. T, Ramachandran. T, Nagalakshmi. K.T “On Q-Fuzzy Normal Subgroups” International Journal of Computer and Organization Trends- Volume 3 (11) (2013), 574-578.

[24] Rajesh kumar, “Homomorphism and fuzzy (fuzzy normal) subgroups”, Fuzzy sets and Systems, 44(191) 165 – 168.

[25] Rajesh Kumar, “Fuzzy Algebra”, Volume 1, University of Delhi Publication Division, July -1993.

[26] Salah Abou-Zaid, “On generalized characteristic fuzzy subgroups of a finite group”, Fuzzy sets and systems, 235-241 (1991).

[27] Samit kumar majumder and Sujit kumar sardar, “On some properties of anti-fuzzy subgroups”,Mech.cont.and Math.Sci.,3(2),2008,337-343.

[28] Sherwood. H, “Product of fuzzy subgroups”, Fuzzy sets and systems 11(1983) ,79-89.

[29] Sitharselvam. P.M, Priya. T and Ramachandran. T, “Anti Q-fuzzy KU- ideals in KU-algebras and its lower-level cuts”, International Journal of Engineering Research & Applications, 2(4) (July) (2012), 1286-1289.

[30] Solairaju. A andNagarajan. R“ Q- fuzzy left R- subgroups of near rings w.r.t T-norms”, Antarctica journal of mathematics, vol 5,2008, 21-28

[31] Solairaju. A and Nagarajan. R, “A New Structure and Construction of Q-Fuzzy Groups”, Advances in Fuzzymathematics,Vol 4 (1) (2009) pp.23-29.

[32] Solairaju.A and Nagarajan.R, “Lattice Valued Q-fuzzy left R-submodules of near rings with respect to T-norms”, Advances in fuzzy mathematics,Vol 4, Num. 2, 137-145(2009).

[33] Vasantha Kandasamy.W.B., Smarandache, “Fuzzy Algebra”, American Research Press,2003.

[34] Vethamanickam.A, Balasubramanian K.R, andMuruganantha. K.L, Prasad, “Q-homomorphism in q-fuzzy subgroups” Elixir Appl. Math. 41 (2011) 5668-5670.

[35] Yunjie Zhand and Dong Yu, “On fuzzy Abelian subgroups”, Zadeh.L.A. Fuzzy sets, Information and Control,8(1965),338-353.

[36] Zadeh. L.A, “Fuzzy sets”, Information and control, 8(1965), 338-353

A STUDY ON PRIME IDEALS IN GAMMA NEAR - RINGS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

W. MARIA INFANT MOUNISHA

Reg. No: 19SPMT16

Under the guidance of

MS. M. PARVATHI BANU M.Sc., M.Phil.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON PRIME IDEALS IN GAMMA NEAR - RINGS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by

W. MARIA INFANT MOUNISHA (Reg. No: 19SPMT16)

M. Parvathi

Signature of the Guide

Signature of the Coordinator

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rosa

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON PRIME IDEALS IN GAMMA NEAR – RINGS "** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. M. Parvathi Banu M.Sc., M.Phil.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

W. Maria infant Mourisha

Station: Thoothukudi **Signature of the Student**

Date: 10.4.2021

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Place: Thoothukudi

Date: 10. 4. 2021

CONTENT

1. PRELIMINARIES

1.1 Γ- near rings

Definition: 1.1.1

A non-empty set N with two binary operations **+** (addition) and • (multiplication) is called a near - ring if it satisfies the following axioms,

(i) (N, +) is a group(not necessarily abelian).

(ii) (N, •) is a semigroup.

(iii) (a+b)c = ac + bc for all a,b,c ∈ N.

Precisely speaking, it is a right near - ring. Moreover, a near - ring *N*

is said to be zero-symmetric near - ring if n0 = 0 for all n ∈ N where 0 is the

additive identity in N.

Definition: 1.1.2

A Γ- near ring is a triple (N, +, Γ), where

(i) (N,+) is a (not necessarily abelian) group.

(ii) Γ is a non-empty set of binary operations on N such that for each γ ∈ Γ, (N,+,γ) is a right near –ring.

(iii) (xγy)μz = xγ(yμz) for all x,y,z ∈ N and γ,μ ∈ Γ.

Definition: 1.1.3

Let N be a Γ- near ring, then a normal subgroup I of (N, +) is said

to be

(i) A left ideal if $a\alpha(b+i) - a\alpha b \in I$ for all $a,b \in N$, $\alpha \in \Gamma$ and $i \in I$.

(ii) A right ideal if $i\alpha a \in I$ for all $a \in N$, $\alpha \in \Gamma$ and $i \in I$.

(iii) An ideal, if it is both left and right ideal.

Definition: 1.1.4

Let N be a Γ- near ring. An element $e \in N$ is said to be left unity (respectively right unity) in *N* if $e\alpha m = m$ (respectively $m\alpha e = m$) $\forall m \in N$ and $\alpha \in \Gamma$.

Definition: 1.1.5

Let N be a near -ring. A subgroup I of N said to be N - subgroup if $NI \subseteq I$.

Definition: 1.1.6

A Γ- near ring N is said to be zero- symmetric if $a\alpha 0 = 0$ $\forall a \in N$ and $\alpha \in \Gamma$ where 0 is the additive identity in N.

Definition: 1.1.7

A Γ- near ring N is said to be simple if $N\Gamma N \neq 0$ and *N* has no nontrivial ideals.

Definition: 1.1.8

A Γ- near ring N is said to be integral if $a\alpha b = 0$ where $a,b \in N$ and

$\alpha \in \Gamma$ implies that either $a = 0$ or $b = 0$.

Definition: 1.1.9

A Γ- near ring N is said to be regular if for all $a \in N$, there exists $x \in N$ such that $a = a\gamma_1 x\gamma_2 a$ for all γ_1 and $\gamma_2 \in \Gamma$.

Definition: 1.1.10

A Γ- near ring N is said to be left strongly regular if for all $a \in N$, there exists $x \in N$such that $a = x\alpha a\beta a$ for all $\alpha,\beta \in \Gamma$.

Lemma: 1.1.11

If N is a left strongly regular Γ- near ring, then $a = a\gamma_1 x\gamma_2 a$ and $a\gamma x = x\gamma a$ for all $\gamma_1,\gamma_2,\gamma \in \Gamma$.

Definition: 1.1.12

A Γ- near ring *N* is said to fulfill the insertion of factors property (IFP) provided that for any $a,b,r \in N, \gamma \in \Gamma$, $a\gamma b = 0$ implies $a\alpha r\beta b = 0$ for all $\alpha,\beta \in \Gamma$.

Definition: 1.1.13

A Γ- near ring N is said to be 3-prime if $a,b \in N, a\Gamma N\Gamma b = 0$ implies $a = 0$ or $b = 0$.

Definition: 1.1.14

An ideal I of a Γ- near ring *N* is called completely prime (Completely semiprime) if $a,b \in N$, $\gamma \in \Gamma$, $a\gamma b \in I$ implies $a \in I$ or $b \in I$ ($a\gamma a \in I$ implies $a \in I$).

An ideal I of N is said to be prime if for any two ideals A, B of N, $A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

An ideal I of N is called semiprime if for any ideal A of N, $A\Gamma A \subseteq I$ implies $A \subseteq I$.

Definition: 1.1.15

An element $0 \neq a \in N$ is called nilpotent if there exists a positive integer $n \geq 1$ such that $(a\gamma)^n a = 0$ for each $\gamma \in \Gamma$. N is said to be reduced if it has no nonzero nilpotent elements.

Proposition: 1.1.16

Let N be a Γ- near ring with a strong left unity. If Q is a prime ideal of L, then Q^+ is a prime ideal of N.

Theorem: 1.1.17

Suppose that a Γ- near ring N has a right unity and a strong left unity. Then the mapping $A \rightarrow A^+$ defines an isomorphism between the lattices

of two sided ideals of N and L.

1.2 Γ- rings

Definition: 1.2.1

Let *M* and Γ be additive abelian groups. If for all a,b,c ∈ M and α,β ∈ Γ, the following conditions are satisfied

(i) $a\alpha b \in M$,

(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$ $a(\alpha+\beta)c$

$= a\alpha c + a\beta c$ $a\alpha(b+c) = a\alpha b + a\alpha c$,

(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then *M* is called a Γ- ring. If these conditions are strengthened to

(i') $a\alpha b \in M$, $\alpha a\beta \in \Gamma$,

(ii') same as (ii),

(iii') $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$,

(iv') $a\gamma b = 0$ for all $a,b \in M$ implies $\gamma = 0$,

then *M* is called a Γ- ring in the sense of Nobusawa.

Definition: 1.2.2

A right (left) ideal of a Γ- ring M is additive subgroup of a Γ-ring M such that $I\Gamma M \subseteq I$ ($M\Gamma I \subseteq I$). If I is both a right and a left ideal, then we say

that I is an ideal of M.

An ideal I of a Γ- ring M is said to be prime if for any ideals

$U, V \subseteq M$, $U\Gamma V \subseteq I$ implies $U \subseteq I$ or $V \subseteq I$.

Definition: 1.2.3

An ideal Q in a Γ- ring *M* is said to be semiprime ideal if for any

ideal *U* of M, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$.

Definition: 1.2.4

Let *S* and *T* be arbitrary associative rings with unity.

By Mod-T (T-Mod) we denote the category of all right (left) T-modules.

Then a module *M* is said to be a generator (in Mod-T) if for every T-

module *K* there is a set I such that the sequence $M^I \to K \to 0$ is exact. M

is said to be progenerator if it is finitely generated, projective and is a

generator. The rings *S* and *T* are said to be Morita equivalent if S-Mod

(Mod-S) and T-Mod

(Mod-T) are equivalent categories. Equivalently *S* and *T* are Mortia

equivalent if there exists a progenerator M_T with $S \cong End_T(M)$.

Theorem: 1.2.5

Let M be a weakly semiprime Γ- ring, L and R be its operator

rings. Then L and R are Morita Equivalent.

Lemma: 1.2.6

Let P, Q and S be a prime ideal of a Γ- ring M, a prime ideal of

the

right operator ring R and a primal ideal of the left operator ring L respectively.

Then $P^{*'}$ is a prime ideal of R, $P^{+'}$ is a prime ideal of L, Q^* and S^+ are

prime

ideals of M.

Theorem: 1.2.7

If Q is an ideal in a Γ- ring M, all the following conditions are equivalent,

(i) Q is a semiprime ideal,

(ii) If $a \in Q$ such that $a\Gamma M\Gamma a \subseteq Q$, then $a \in Q$,

(iii) If $\langle a \rangle$ is a principal ideal in M such that $\langle a \rangle\Gamma\langle a \rangle \subseteq Q$, then $a \in Q$,

(iv) If U is a right ideal in M such that $U\Gamma U \subseteq Q$, then $U \subseteq Q$,

(v) If V is a left ideal in M such that $V\Gamma V \subseteq Q$, then $V \subseteq Q$.

Theorem: 1.2.8

If M is Γ- ring, the following conditions are equivalent

(i) M is prime Γ- ring.

(ii) If $a,b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$,

(iii) If $\langle a \rangle$ and $\langle b \rangle$ are principal ideals in M such that $\langle a \rangle\Gamma\langle b \rangle = (0)$, then $a = 0$ or $b = 0$,

(iv) If A and B are right ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$,

(v) If A and B are left ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.

2. STRONGLY REGULAR GAMMA – NEAR RINGS

2.1 Weakly Regular Γ –Near Rings

Definition: 2.1.1

A Γ- near ring N is said to be left (respectively right) weakly regular if $a \in \langle a \rangle \Gamma a$ (respectively $a \in \Gamma a \langle a \rangle$) for all $a \in N$. N is said

to be weakly regular if it is both left and right weakly regular.

Definition: 2.1.2

A Γ- near ring N is said to be left (respectively right) pseudo π -regular if for every $\in N$, $\gamma \in \Gamma$, there exists a natural number $n = n(x)$

such that $x^n = x\gamma x\gamma x...\gamma x \in \langle x \rangle \Gamma x^n$(respectively $x^n = x\gamma x\gamma x...\gamma x \in \Gamma x^n \langle x \rangle$).

Proposition: 2.1.3

Let N be a Γ- near ring, then

(i) $a^k \in \langle a \rangle \Gamma a^{k+1}$ for some *k* if and only if the descending chain $\langle a \rangle \Gamma a \supseteq \langle a \rangle \Gamma a^2 \supseteq \ldots$ stabilizes after a finite number of steps,

(ii) If N has descending chain condition on left Γ subgroups, then N is left pseudo π – regular,

(iii) If N is finite, then N is left and right pseudo π – regular.

Proof:

(i) Suppose that $a^k \in < a > \Gamma a^{k+1}$. Now

$$< a > \Gamma a^k \subseteq < a > \Gamma(< a > \Gamma a^{k+1})$$

$$= (<a>\Gamma<a>)\Gamma a^{k+1}$$

$$\subseteq < a > \Gamma a^{k+1}$$

$$= < a > \Gamma(a\Gamma a^k)$$

$$= (<a>\Gamma a)\Gamma a^k$$

$$\subseteq < a > \Gamma a^k.$$

Hence $< a > \Gamma a^k = < a > \Gamma a^{k+1}$. Therefore the descending chain $< a > \Gamma a \supseteq < a > \Gamma a^2 \supseteq \ldots$ stabilizes after a finite number of steps.

Conversely, assume that $< a > \Gamma a^m = < a > \Gamma a^{m+1}$, then for each $\alpha \in \Gamma$, $a^{m+1} = a\alpha a^m \in < a > \Gamma a^m$ implies that $a^{m+1} \in < a > \Gamma a^{m+1}$ by assumption. Now

$$a^{m+1} \in < a > \Gamma a^{m+1}$$

$$= (< a > \Gamma a^m)\Gamma < a >$$

$$=(< a > \Gamma a^{m+1}) \Gamma < a >$$

$$=< a > \Gamma a^{m+2}.$$

Thus, $a^{m+1} \in< a > \Gamma a^{m+2}$. Take k = m+1. Hence $a^k \in< a > \Gamma a^{k+1}$.

(ii) Clearly $< a > \Gamma a^i$, $\forall i = 1,2,\cdots$ are left Γ – subgroup and by hypothesis $< a > \Gamma a \supseteq < a > \Gamma a^2 \supseteq \ldots$ stabilizes after a finite number of steps.

Hence from (i) for every $a \in N$,

$$a^k \in< a > \Gamma a^{k+1}$$

$$=< a > \Gamma(a\Gamma a^k)$$

$$= (<a>\Gamma a)\Gamma a^k$$

$$\subseteq< a > \Gamma a^k$$

i.e., $a^k \in< a > \Gamma a^{k+1}$.

Hence N is left pseudo π – regular.

(iii) If N is finite, then $< a > \Gamma a \supseteq < a > \Gamma a^2 \supseteq \ldots$ stabilizes after a finite number of steps.

Therefore by (i) there exists a positive integer k such that $a^k \in< a > \Gamma a^{k+1}$.

Since $< a > \Gamma a^{k+1} \subseteq< a > \Gamma a^k$, $a^k \in< a > \Gamma a^{k+1}$.

Thus N is left pseudo π – regular.

Similarly N is right pseudo π – regular.

Definition: 2.1.4

A Γ- near ring N is said to be left quasi duo if every maximal left

ideal is a two sided ideal and strict left quasi duo if every maximal left ideal is

closed under right multiplication.

Proposition: 2.1.5

If N is a left quasi duo Γ – near rings with left unity e, and k, n are natural numbers, then $a^n \in < a^k > \Gamma a^n$ if and only if $N = < a^k > + (0:a^n)\ \forall a \in N$.

Proof:

Let $a^n \in < a^k > \Gamma a^n$ for $a \in N$. Then

$$N\Gamma a^n \subseteq N\Gamma < a^k > \Gamma a^n$$
$$\subseteq < a^k > \Gamma a^n$$
$$\subseteq N\Gamma a^n.$$

Consequently,

$$N\Gamma a^n = < a^k > \Gamma a^n.$$

We claim $N = < a^k > + (0:a^n)\ \forall a \in N$.

If not, there exists a maximal left ideal M such that $< a^k > + (0:a^n) \subseteq M$.

Since N is left quasi duo, M is also two sided ideal. Since $< a^k > \subseteq M$.

We have $< a^k > \Gamma a^n \subseteq M\Gamma a^n \subseteq N\Gamma a^n = < a^k > \Gamma a^n$.

$\therefore\ M\Gamma a^n = \langle a^k \rangle \Gamma a^n$.

Hence, there exists $x \in M$ such that $a^n = e\Gamma a^n = x\Gamma a^n$. From this,

We have $(e\text{-}x)\Gamma a^n = 0$, and therefore $(e\text{-}x) \in (0:a^n) \subseteq M$.

Hence $e = (e\text{-}x) + x \in M$. This is not possible.

Hence $N = \langle a^k \rangle + (0:a^n)$.

Conversely, suppose that $N = \langle a^k \rangle + (0:a^n)\ \forall a \in N$.

We shall prove that there exists natural numbers k and n such that

$a^n \in \langle a^k \rangle \Gamma a^n$.

Since $e \in N$, there exists $t \in \langle a^k \rangle$ and $l \in (0:a^n)$ such that $e = t + l$.

Hence for each $\alpha \in \Gamma$,

$$a^n = e\alpha a^n = (t+l)\alpha a^n = t\alpha a^n + l\alpha a^n = t\alpha a^n \in \langle a^k \rangle \Gamma a^n.$$

Definition: 2.1.6

A Γ- near ring N is said to be left (respectively right) weakly

π – regular if for every $x \in N$, $\gamma \in \Gamma$, there positive a natural number n such

that $x^n = x\alpha x\alpha x\ldots\alpha x \in \langle x^n \rangle \Gamma x^n$.

Corollary: 2.1.7

If N is a left quasi duo Γ- near ring with left unity then

(i) N is left weakly π – regular, if and only if

$N = \langle a^k \rangle + (0:a^k)\ \forall a \in N$ and some natural number k.

(ii) N is left weakly regular if and only if $= \langle a \rangle + (0:a)$

$\forall a \in N$.

Proof:

This is an easy consequence of proposition 2.1.5.

Definition: 2.1.8

A Γ- near ring N is said to be strict left weakly regular if

$a \in (N\Gamma a)\Gamma(N\Gamma a)\ \forall a \in N$.

Definition: 2.1.9

A Γ- near ring N is said to be strict left weakly π- regular if

$a^n \in (N\Gamma a^n)\Gamma(N\Gamma a^n)\ \forall a \in N$.

Proposition: 2.1.10

If N is a zero – symmetric and strict left quasi duo Γ- near ring

with

left unity *e*, then

(i) N is strict left weakly regular if and only if

$N = N\Gamma a + (0:a)$

$\forall a \in N$.

(ii) N is strict left weakly π- regular if and only if

$N = N\Gamma a + (0:a^n)\ \forall a \in N$ and some natural number n.

Proof:

(i) Suppose N is strict left weakly regular and let $a \in N$.

We have to prove that $N = N\Gamma a + (0:a)$.

If not, there is a maximal left Γ- subgroup M of N such that

$N\Gamma a + (0:a) \subseteq M$.

Since N is strict left weakly regular $a \in (N\Gamma a)\Gamma(N\Gamma a)$.

Hence $a = x\Gamma a$ for some $x \in N\Gamma a\Gamma N$. Since M is closed under

multiplication from the right, $N\Gamma a\, \Gamma N \subseteq M$ and consequently $x \in M$.

Since $a = e\Gamma a$, it follows that $(e\text{-}x) \in (0:a^n)$.

Hence $e = (e\text{-}x) + x \in M$. This is not possible.

Hence $N = N\Gamma a + (0:a)$.

Consequently, suppose that $N = N\Gamma a + (0:a)$ for every $a \in N$.

We shall prove that $a \in (N\Gamma a)\Gamma(N\Gamma a)$. Now,

$$N = N\Gamma N = (N\Gamma a)\Gamma N + (0:a)\Gamma N$$

$$N = N\Gamma a\Gamma N.$$

Then $(N\Gamma a)\Gamma(N\Gamma a) = N\Gamma a$. Since N has left unity e, $a = e\alpha a \in N\Gamma a$,

$\forall a \in \Gamma$.

Hence $a \in (N\Gamma a)\Gamma(N\Gamma a)$.

(ii) Suppose N is strict left weakly π- regular and $\forall a \in N$.

We shall prove that $N = N\Gamma a + (0{:}a^n)$ where n is natural number.

If not, If not, there is a maximal Γ- subgroup M of N such that

$N\Gamma a + (0{:}a^n) \subseteq M$.

By similar argument as in (i), we can show that $e \in M$ and

consequently

$N = N\Gamma a + (0{:}a^n)$.

Conversely, suppose that $N = N\Gamma a + (0{:}a^n)$ for every $a \in N$ and

some

natural number n.

We have $N\Gamma a^n = N\Gamma a^{n+1}$ $\forall$ $a \in N$.

Let $b \in N$ and $b^n = x\Gamma b^{n+1}$ for some $x \in N$.

Now $b^n = x\alpha b^n \alpha b = x\alpha(x\alpha b^{n+1})\alpha b = x^2 \alpha b^n \alpha b^2 = \cdots =$

$x^{n+1}\alpha b^n \alpha b^{n+1} \in N\Gamma b^n \Gamma b^n, \alpha \in \Gamma$,

i.e., $b^n \in N\Gamma b^n \Gamma N\Gamma b^n$ $\forall b \in N$ and consequently N is strict left weakly

π- regular.

2.2 Strongly Regular Γ- Near Rings

In this section we shall prove that the characterisation of

strongly

regular Γ-near ring. Throughout this section N stands for zero

symmetric

Γ-near ring.

Proposition: 2.2.1

N is left strongly regular if and only if it is regular and has the IFP.

Proof:

From the definition of left strongly regular it follows that N is regular.

First we have to prove that N is reduced. Let $a\gamma a = 0$, for all $\gamma \in \Gamma$.

Since N is left strongly regular, there exists $x \in \Gamma$ such that

$a = x\gamma a^2 = x\gamma 0 = 0 \ \forall \gamma \in \Gamma$.

Now to prove that IFP holds, let $a,b \in N$ such that $a\gamma b = 0$.

Our claim is that $a\gamma m\gamma b = 0 \ \forall m \in N$.

Now,

$$
\begin{aligned}
(a\gamma m\gamma b)^2 &= (a\gamma m\gamma b)\gamma(a\gamma m\gamma b) \\
&= a\gamma m\gamma(b\gamma a)\gamma m\gamma b \\
&= a\gamma m\gamma 0\gamma m\gamma b = 0
\end{aligned}
$$

Since N is reduced, $a\gamma m\gamma b = 0$. Hence IFP holds.

Conversely, suppose that N is regular and has the IFP. For any idempotent f of N and any $a \in N$, $\gamma \in \Gamma$, we have

$$
\begin{aligned}
(a-a\gamma f)\gamma f &= a\gamma f-(a\gamma f)\gamma f \\
&= a\gamma f-a\gamma(f\gamma f)
\end{aligned}
$$

$= a\gamma f - a\gamma f = 0$

Since N has the IFP, for any $m \in \Gamma$, we have $(a - a\gamma f)\gamma m\gamma f = 0$. Then

$$a\gamma m\gamma f = (a\gamma f)\gamma(m\gamma f) \qquad \text{........ (*)}$$

Since $x\gamma a, a\gamma x$ are idempotent, we have

$$\begin{aligned} x\gamma a &= (x\gamma a)\gamma x\gamma a \\ &= [x\gamma(x\gamma a)]\gamma[a\gamma(x\gamma a)] \\ &= [(x\gamma x)\gamma a]\gamma a \\ &= (x\gamma x)\gamma(a\gamma a) = a^2\gamma x^2 \quad \text{........(**)} \end{aligned}$$

Since N is regular, there exists, $x \in N$ such that $a = a\gamma_1 x\gamma_2 a$ for every

pair of non zero elements γ_1 and γ_2 in Γ. It follows from (**) that

$$a = a\gamma_1 x\gamma_2 a = a\gamma_1 x^2\gamma_2 a^2 = y\gamma_2 a^2, \qquad \text{Where } y = a\gamma_1 x^2$$

And $a\gamma_1 y\gamma_2 a = a\gamma_1 a\gamma_1 x^2\gamma_2 a = a\gamma_1 x\gamma_2 a = a$

Thus N is left strongly regular.

Corollary: 2.2.2

N is left strongly regular if and only if it is regular and reduced.

Proof:

This is clear, since any reduced Γ-near ring has the IFP.

Definition: 2.2.3

A Γ- near ring is called a weakly left duo if for every $a \in N$ there is a

positive integer $n = n(a)$ such that $N\Gamma a^n$ is an ideal of N.

Proposition: 2.2.4

Let N be a weakly left duo and strict left weakly π- regular. Then N is

left strongly π- regular.

Proof:

Let $a \in N$. Then there exists positive integer m and n such that

$N\Gamma a^n = N\Gamma a^n\Gamma N$ and $N\Gamma a^m = N\Gamma a^m\Gamma N\Gamma a^m$. Observe that

$$N\Gamma a^{2n} = N\Gamma a^n\Gamma a^n = N\Gamma a^n\Gamma N\Gamma a^n = N\Gamma a^n\Gamma N\Gamma a^n\Gamma N$$

$$= N\Gamma a^n\Gamma a^n N\Gamma = N\Gamma a^{2n}\Gamma N.$$

An induction argument yields $N\Gamma a^{kn} = N\Gamma a^{kn}\Gamma N$ for any positive integer k.

Also $N\Gamma a^{2m} = (N\Gamma a^m)\Gamma a^m = (N\Gamma a^m\Gamma N\Gamma a^m)\Gamma a^m = N\Gamma a^m\Gamma a^{2m}$.

Again an induction arguments yields $N\Gamma a^{km} = N\Gamma a^m\Gamma N\Gamma a^{km}$ for any

positive integer k.

Now using the above observation, we have that

$$N\Gamma a^{mn}\Gamma N\Gamma a^{mn} = N\Gamma a^{mn}\Gamma a^{mn} = N\Gamma a^{2m}.$$

Also we have that

$$N\Gamma a^{mn}\Gamma N\Gamma a^{mn} = N\Gamma a^{mn}\Gamma(N\Gamma a^m\Gamma N\Gamma a^{mn})$$

$$= N\Gamma a^{mn}\Gamma a^{m}\Gamma N\Gamma a^{mn}$$

$$= N\Gamma a^{mn+m}\Gamma N\Gamma a^{mn}$$

$$= N\Gamma a^{mn+m}\Gamma(N\Gamma a^{m}\Gamma N\Gamma a^{mn})$$

$$= \cdots$$

$$= N\Gamma a^{mn+2m}\Gamma N\Gamma a^{mn}$$

$$= \cdots$$

$$= N\Gamma a^{mn+mn}\Gamma N\Gamma a^{mn}$$

$$= (N\Gamma a^{mn}\Gamma N\Gamma a^{mn})\Gamma N\Gamma a^{mn}$$

$$= (N\Gamma a^{mn}\Gamma N\Gamma a^{mn}\Gamma N\Gamma a^{mn})\Gamma N\Gamma a^{mn}$$

$$= (N\Gamma a^{mn}\Gamma N\Gamma a^{mn})\Gamma(N\Gamma a^{mn}\Gamma N\Gamma a^{mn})$$

$$= N\Gamma a^{2mn}\Gamma N\Gamma a^{2mn}$$

$$= N\Gamma a^{4mn} \subseteq N\Gamma a^{2mn+1} \subseteq N\Gamma a^{2mn}.$$

Hence $N\Gamma a^{2mn} = N\Gamma a^{2mn+1}$.

Therefore N is left strongly π- regular Γ- near ring.

Since left strongly regular Γ- near ring are strict left weakly π-regular.

We have the following corollary.

Corollary: 2.2.5

Let N be weakly left duo Γ- near ring. Then the following statements are equivalent.

(i) N is strict left weakly π- regular.

(ii) N is left strongly π- regular.

3. STRONGLY PRIME GAMMA – NEAR RINGS

3.1 Strongly prime Γ- near rings

Definition: 3.1.1

Let N be a Γ- near ring, then the right α- annihilator of subset A of N is $r_\alpha(A) = \{x \in N / A\alpha x = 0\}$.

Definition: 3.1.2

A Γ- near ring N is said to be strongly prime if for each $a \neq 0 \in N$,

there exists a finite subset F of N such that $r_\alpha(\alpha \Gamma N) = 0 \ \forall \alpha \in \Gamma$. F is called an

insulator for *a* in N.

Lemma: 3.1.3

If *a* Γ- near ring N is strongly prime, then N is prime.

Proof:

Let $0 \neq A$, B is ideal of N. We shall show that $A\Gamma B \neq 0$.

Since $A \neq 0$ there exists a finite subset F of A such that $r_\alpha(F) = 0$, for each $\alpha \in \Gamma$.

Hence for each $0 \neq b \in B$ we have $F\Gamma B \neq 0$.

Therefore $A\Gamma B \neq 0$.

Definition: 3.1.4

A Γ- near ring N is said to be left (right) weakly semiprime if

$[x,\Gamma] \neq 0([\Gamma,x]\neq 0)$ $\forall x \neq 0 \in N$.

N is said to be weakly semi prime if it is both left and right

semiprime.

Proposition: 3.1.5

If N is strongly prime Γ- near ring, then N is weakly semiprime

Γ- near ring.

Proof:

Suppose that N is a strongly prime Γ- near ring.

We shall prove that N is weakly semiprime Γ- near ring. Let $x \neq 0 \in N$.

It is enough to prove that $[x,\Gamma] \neq 0$ and $[\Gamma,x] \neq 0$.

Suppose that $[x,\Gamma] = 0$. Since N is strongly prime Γ- near ring, for every

$\beta \in \Gamma$ there exists a finite $S_\beta(x)$ such that for $b \in N$,

$\left\{x\beta c\alpha b / c \in S_\beta(x)\right\} = 0, \forall \alpha \in \Gamma$ implies that $b = 0$.

Now $x\beta c\alpha x = [x,\beta]c\alpha x = 0c\alpha x = 0, \ \forall \beta,\alpha \in \Gamma, c \in S_\beta(x)$.

Hence $x = 0$, a contradiction.

Thus N is a weakly semiprime Γ- near ring.

Proposition: 3.1.6

If *a* Γ- near ring N is strongly prime then, the left operator near

ring L

is left strongly prime.

Proof:

Let $\sum_i [x_i,\alpha_i] \neq 0 \in L$, then there exists $x \in N$ such that $\sum_i [x_i,\alpha_i]x \neq 0$, i.e., $\sum_i x_i \alpha_i x \neq 0$.

Since N is strongly prime, there exists a finite subset $F = \{a_1, a_2 \cdots, a_n\}$ (say) such that for any $b \in N$,

$$\sum_i x_i \alpha_i x \Gamma F \Gamma b = 0 \text{ implies } b = 0. \qquad \text{……… (1)}$$

Fix $\alpha,\beta \in \Gamma$. Consider $G = \{[x\alpha a_1,\beta], \cdots [x\alpha a_2,\beta]\}$.

Our claim is that G is an insulator for $\sum_i [x_i,\alpha_i]$.

Let $\sum_j [y_j,\beta_j] \in L$ such that $\sum_i [x_i,\alpha_i] G \sum_j [y_j,\beta_j] = 0$.

We shall prove that $\sum_j [y_j,\beta_j] = 0$.

Now,

$$\sum_i [x_i,\alpha_i] G \sum_j [y_j,\beta_j] = 0$$

Implies $\sum_i [x_i,\alpha_i][x\alpha a_k,\beta]\sum_j [y_j,\beta_j] = 0 \quad \forall k = 1,2,\cdots n$.

Hence $\left(\sum_i [x_i,\alpha_i][x\alpha a_k,\beta]\sum_j [y_j,\beta_j]\right)z = 0 \quad \forall z \in N; k = 1,2,\cdots n$.

This implies that $\sum_i [x_i,\alpha_i][x\alpha a_k,\beta]\sum_j [y_j,\beta_j]z = 0 \quad \forall z \in N; k = 1,2,\cdots n$.

Hence $\sum_i x_i \alpha_i x \alpha a_k \beta \sum_j y_j \beta_j z = 0 \quad \forall z \in N; k = 1,2,\cdots n$.

By (1) $\sum_j y_j \beta_j z = 0 \quad \forall z \in N$. Therefore $\sum_j [y_j,\beta_j] = 0$.

Thus L is strongly prime.

Theorem: 3.1.7

Let N be a left weakly semiprime Γ- near ring having no zero divisor,

Then N is strongly prime if and only if L is strongly prime.

Proof:

Suppose that L is strongly prime.

To prove N is strongly prime, let $x \neq 0 \in N$.

Since N is left weakly semiprime, $[x,L] \neq 0$ and since L is strongly prime, there exists a finite subset $F = \left\{\sum_{j=1}^{n}\left[y_{j_k},\beta_{j_k}\right] \big/ k=1,2,\cdots m\right\}$(say) such that for any $\sum_l\left[z_l,\delta_l\right] \in [x,\Gamma]F\sum_l\left[z_l,\delta_l\right]=0$ implies $\sum_l\left[z_l,\delta_l\right]=0$ ……. (1)

consider $F' = \left\{y_{j_k}\beta_{j_k}x \big/ j=1,2,\cdots n;k=1,2\cdots m\right\}$.

Our claim is that F' is an insulator for x. let $y \in N$ such that $x\Gamma F'\Gamma y = 0$.

We shall prove that $y = 0$.

Now $x\Gamma F'\Gamma y = 0$ implies $x\alpha y_{j_k}\beta_{j_k}x\beta y = 0 \quad \forall j = 1,2,\cdots n;k = 1,2\cdots m$,

for all $\alpha,\beta \in \Gamma$. Therefore

$$\left[x\alpha y_{j_k}\beta_{j_k}x\beta y,\Gamma\right] = 0 \quad \forall j = 1,2,\cdots n;k = 1,2\cdots m.$$

Hence $[x,\alpha]\left[y_{j_k},\beta_{j_k}\right][x\beta y,\Gamma] = 0 \quad \forall k = 1,2,\cdots m$.

By (1) $[x\beta y,\Gamma] = 0$. Therefore $x\beta y = 0$.

Since N is weakly semiprime and N has no zero divisor, $y = 0$ and

Consequently F' is an insulator for *x*. Therefore N is strongly prime.

Converse part follows from proposition 3.1.6.

Rules: 3.1.8

We recall that for $X \subseteq N$, $\langle X \rangle$ is connected by the following

recursive

Rules.

(i) $a \in \langle X \rangle \quad \forall a \in X$

(ii) If $b, c \in \langle X \rangle$, then $b + c \in \langle X \rangle$

(iii) If $b \in \langle X \rangle$ and $x, y \in N$, $\alpha \in \Gamma$, then $x\alpha(b+y) - x\alpha y \in \langle X \rangle$

(iv) If $b \in \langle X \rangle$ and $x \in N$, $\alpha \in \Gamma$, then $b\alpha x \in \langle X \rangle$

(v) If $b \in \langle X \rangle$ and $x \in N$, then $x - b \in \langle X \rangle$

(vi) Nothing else is in $\langle X \rangle$.

Definition: 3.1.9

Suppose $X \subseteq N$ and $d \in \langle X \rangle$. We call a sequence $s_1, s_2, \cdots s_n$ of

element of N, a generating sequence of length m for d with respect to X.

If

$s_1 \in X, s_m = d$, $\alpha \in \Gamma$ and for each $i = 2,3 \cdots m$, one of the following

applies

(i) $s_i \in X$

(ii) $s_i = s_j + s_l$, $1 \le j,l < i$

(iii) $s_i = s_j \alpha x$, $1 \le j < i$ and $x \in N$

(iv) $s_i = x\alpha(s_j + y) - x\alpha y$, $1 \le j < i$ and $x,y \in N$

(v) $s_i = x + s_j - x$, $1 \le j < i$ and $x \in N$

The complexity of d with respect to X denoted by $C_X(d)$, is the length of a generating sequence of least length for d with respect to X.

Lemma: 3.1.10

Let N be a Γ- near ring. If $X \neq 0$ and $X\Gamma N = 0$, then $\langle X\rangle \Gamma N = 0$.

Proof:

Let $X\Gamma N = 0$ and suppose $x \in \langle X\rangle$ arbitrary.

We use induction on $C_X(x) = 1$, then $x \in X$ and from our assumption we have $X\Gamma N = 0$.

Suppose $y\Gamma N = 0$ $\forall y \in \langle X\rangle$ such that $C_X(y) < n$ and let $C_X(x) = n$.

We have the following possibilities

(i) $x = a + b$ where $a,b \in \langle X\rangle$ and $C_X(a), C_X(b) < n$. Hence

$$\begin{aligned} x\Gamma N &= (a + b)\Gamma N \\ &= a\Gamma N + b\Gamma N \\ &= 0 \end{aligned}$$

(ii) $x = a\alpha n$ where $\in \langle X\rangle$, $n \in N$, $\alpha \in \Gamma$ and $C_x(a) < n$. Hence

$$x\Gamma N = (a\alpha n)\Gamma N$$

$$\subseteq a\Gamma N$$

$$= 0$$

(iii) $x = a\alpha(d+b) - a\alpha b$ where $d \in \langle X\rangle$ and $a,b \in N$, $\alpha,\beta \in \Gamma$ with $C_x(d) < n$. If m is arbitrary element of N, then

$$x\beta m = (a\alpha(d+b) - a\alpha b)\beta m$$

$$= a\alpha(d\beta m + b\beta m) - (a\alpha b)\beta m$$

$$= a\alpha b\beta m - a\alpha b\beta m = 0$$

Hence $x\Gamma N = 0$.

(iv) If $x = a + b - a$ where $b \in \langle X\rangle$, $a \in N$, $\alpha \in \Gamma$ and $C_x(b) < n$.

Let $m \in N$, then

$$x\alpha m = (a+b-a)\alpha m$$

$$= a\alpha m + b\alpha m - a\alpha m$$

$$= 0$$

This completes the proof.

Corollary: 3.1.11

If every non zero ideal of a Γ- near ring N contains a subset F with

$r_\alpha(F) = 0$, $\forall\alpha \in \Gamma$, then for each $a \in N, a \neq 0, \beta \in \Gamma$, there is a $y \in N$ with

$\alpha\beta y \neq 0$.

Proof:

Let $a \neq 0 \in N$ and suppose F is a subset of $\langle a\rangle$ such that

$r_\alpha(F) = 0$

$\forall\alpha \in \Gamma$. For every $n \neq 0 \in N$, we have $F\Gamma n \neq 0$ and therefore $\langle a\rangle\Gamma N \neq 0$.

From lemma 3.1.10 there exists $y \neq 0 \in N$ such that $a\beta y \neq 0$, for all

$\beta \in \Gamma$.

Theorem: 3.1.12

Let N be a Γ- near ring, then N is strongly prime if and only if

every

non zero ideal of N contains a finite subset F with $r_\alpha(F) = 0$, $\forall\alpha \in \Gamma$.

Proof:

Let $I \neq 0$ be an ideal in N and $a \neq 0 \in I$.

Since N is strongly prime, there exists a finite subset $F \subseteq N$ such

that

$r_\alpha(a\,\Gamma F) = 0$, $\forall\alpha \in \Gamma$. Put $F_1 = a\Gamma F$.

Hence F_1 is a finite subset of I with $r_\alpha(F_1) = 0$, $\forall\alpha \in \Gamma$.

Conversely, let $a \neq 0 \in N$, then $\langle a\rangle \neq 0$.

From our assumption, there exists a finite subset F of $\langle a\rangle$ such that

$r_\alpha(F) = 0, \forall\alpha \in \Gamma$.

It follows from the corollary 3.1.11, that there exists $y \in N$ with $\alpha\beta y \neq 0$

for

all $\beta \in \Gamma$.

Again we use our assumption, we can find a finite subset

$G_1 = \{g_1, g_2, \cdots g_n\} \subseteq \langle a\beta y\rangle$ with $r_\alpha(G) = 0, \forall\alpha,\beta \in \Gamma$.

For each *i*, let $s_{i_1}, s_{i_2}, \cdots s_{i_{m_i}}$ be the corresponding generating sequence of g_i.

Each of these sequence involve a finite number of terms of the form

$a\beta y$ or

$(a\beta y)\gamma t_k, t_k \in N, \forall\alpha,\beta,\gamma \in \Gamma$.

Let $G_1 = \{a\beta y, (a\beta y)\gamma t_k /$ these occur in the generating sequence of

an

element of G}.

Clearly G_1 is finite and $r_\alpha(G_1) \subseteq r_\alpha(G) = 0, \forall\alpha \in \Gamma$.

Take $H = \{x / a\beta x \in G_1, \forall\beta \in \Gamma\}$.

Our claim is that H is an insulator for a.

Now $r_\alpha(G_1) = 0$ implies that for any $n \in N$, $G_1\alpha n = 0, \forall\alpha \in \Gamma$ implies

$n = 0$

Since $a\Gamma H \subseteq G_1$, we have H is an insulator for a and consequently N is

strongly prime.

Proposition: 3.1.13

Let N be zero symmetry Γ- near ring then the following are equivalent.

(i) N is strongly prime Γ- near ring.

(ii) Every non zero right Γ- subgroup of N contains a finite subset F such that $r_{\alpha}(F) = 0, \forall \alpha \in \Gamma$.

(iii) Every non zero right ideal of N contains a finite subset F such that $r_{\alpha}(F) = 0, \forall \alpha \in \Gamma$.

(iv) Every non zero ideal of N contains a finite subset F such that

$r_{\alpha}(F) = 0, \forall \alpha \in \Gamma$.

Proof:

(i) $\Rightarrow$ (ii):

Let $I \neq 0$ be a right Γ- subgroup of N and let $a \neq 0 \in I$.

Since N is strongly prime, a has an insulator F such that $r_{\alpha}(a\Gamma F) = 0, \forall \alpha \in \Gamma$.

Let $G = a\Gamma F$. Then $G \subseteq I$ and $r_{\alpha}(G) = 0, \forall \alpha \in \Gamma$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obivious.

(iv) $\Rightarrow$ (i) follows from theorem 3.1.12.

Proposition: 3.1.14

Let N be a zero symmetric Γ- near ring with descending chain

condition on right annihilators, then N is 3-prime if and only if N is strongly prime.

Proof:

Suppose N is strongly prime.

To prove N is 3-prime, let $a,b \in N$ such that $a \neq 0$ and $b \neq 0$.

Since N is strongly prime, there exists a finite subset F of N such that

$a\Gamma F\Gamma b \neq 0$. Hence $a\Gamma N\Gamma b \neq 0$.

Conversely, let $I \neq 0$ be an ideal in N and for each $\alpha \in \Gamma$, consider the

collection of right α- annihilators $\{r_\alpha(F)\}$ where F runs over all finite subset

of I.

From our hypothesis, there exists a minimal element $M = r_\alpha(F_0)$.

If $M \neq 0$, let $m \neq 0 \in M$ and $a \neq 0 \in I$.

Since N is 3-prime, there exists $n \neq 0 \in N$ such that $a\beta n\gamma m \neq 0$ for all

$\beta,\gamma \in \Gamma$. Hence $a\gamma n \neq 0$.

Let $S_\alpha = r_\alpha(F_0 \cup \{a\gamma n\})\ \forall \alpha \in F$.

Now $m \in M$ but $M \notin S_\alpha$ implies that S_α is smaller than M, a

contradiction.

This force that M = (0).

Hence for every non zero ideal I of N, there exists a finite subset F such

that $r_\alpha(F) = 0$ $\forall \alpha \in \Gamma$ and consequently N is strongly prime.

3.2 Radicals of strongly prime Γ- near rings.

In this section we shall prove that the strongly prime radical $P_s(N)$

of N coincides with $P_s(L)^+$ where $P_s(L)$ is the strongly prime radical of the

left operator near - ring L of N.

Definition: 3.2.1

An ideal I of a Γ- near ring N is said to be strongly prime if for each

$a \notin I$, there exists a finite subset F such that for any $b \in N$, $a\Gamma F\Gamma b \subseteq I$ implies

that $b \in I$. F is called an insulator for a.

Proposition: 3.2.2

Let N be a Γ- near ring. If P is a strongly prime ideal of N, then

$P^{+'} = \{l \in L/lx \in P \ \ \forall x \in N\}$ is a strongly prime ideal of L.

Proof:

Suppose that P is a strongly prime ideal of N.

We shall prove that $P^{+'}$ is a strongly prime ideal of L.

Let $\sum_i[x_i,\alpha_i] \notin P^{+'}$, then there exists $x \in N$ such that $\sum_i[x_i,\alpha_i]x \notin P$, that is

$\sum_i x_i \alpha_i x \notin P$. Since P is strongly prime in N, there exists a finite subset

$F = \{f_1, f_2 \cdots, f_n\}$ of N such that for any $b \in N$,

$$\sum_i x_i \alpha_i x \Gamma F \Gamma b \subseteq P \text{ implies } b \in P. \qquad \ldots\ldots\ldots(1)$$

Fix $\alpha, \beta \in \Gamma$.

Consider the collection $F' = \{[x\alpha f_1,\beta], \cdots, [x\alpha f_n,\beta]\}$.

Our claim is that F' is an insulator for $\sum_i[x_i,\alpha_i]$.

Let $\sum_j[y_j,\beta_j] \in L$ such that $\sum_i[x_i,\alpha_i]F'\sum_j[y_j,\beta_j] \subseteq P^{+'}$.

To prove $\sum_j[y_j,\beta_j] \in P^{+'}$. Now

$$\sum_i[x_i,\alpha_i]F'\sum_j[y_j,\beta_j] \subseteq P^{+'}$$

Implies $\sum_i[x_i,\alpha_i][x\alpha f_k,\beta]\sum_j[y_j,\beta_j] \in P^{+'} \quad \forall k = 1,2,\cdots,n$.

i.e., $\left(\sum_i[x_i,\alpha_i][x\alpha f_k,\beta]\sum_j[y_j,\beta_j]\right)z \in P \ \ \forall z \in N; k = 1,2,\cdots n$.

Hence $\sum_i x_i \alpha_i x \Gamma F \Gamma \sum_j y_j \beta_j z \subseteq P \ \forall z \in N$.

By (1) $\sum_j y_j \beta_j z \in P \ \ \forall z \in N$. i.e., $\sum_j[y_j,\beta_j]z \in P \ \ \forall z \in N$.

Hence $\sum_j[y_j,\beta_j]z \in P^{+'}$ and therefore F' is an insulator for $\sum_i[x_i,\alpha_i]$ and consequently $P^{+'}$ is a strongly prime ideal of ideal of L.

Proposition: 3.2.3

Let N be a distributive strongly 2-primal Γ- near ring with strong left unity. If Q is a strongly prime ideal of L, then

$Q^+ = \{x \in N/[x,\alpha] \in Q \forall \alpha \in \Gamma\}$ is a strongly prime ideal of N.

Proof:

Suppose Q is a strongly prime ideal of L.

We shall prove that Q^+ is a strongly prime ideal of N.

Let $x \notin Q^+$, then there exists $\alpha \in \Gamma$ such that $[x,\alpha] \notin Q$.

Since Q is a strongly prime ideal of L, then there exists a finite subset

$F = \left\{\sum_j^n[y_{j_k},\beta_{j_k}]\big/k=1,2,\cdots,m\right\}$ (say) such that for any $\sum_l[z_l,\delta_l] \in L$,

$$[x,\alpha]F\sum_l[z_l,\delta_l] \subseteq Q \text{ implies that } \sum_l[z_l,\delta_l] \in Q. \qquad \text{....... (1)}$$

Consider $F' = \left\{y_{j_k}\beta_{j_k}x\big/j=1,2,\cdots,n;k=1,2,\cdots,m\right\}$.

Our claim is that F' is an insulator for x. Let $a \in N$ such that $x\Gamma F'\Gamma a \subseteq Q^+$.

To prove $a \in Q^+$. Now $x\Gamma F'\Gamma a \subseteq Q^+$ implies

$$\left[x\Gamma F'\Gamma a,\Gamma\right] \subseteq Q,$$

$$\text{i.e., } \left[x\alpha y_{j_k}\beta_{j_k}x\beta a,\gamma\right] \in Q,$$

$\forall j = 1,2,\cdots,n;k = 1,2,\cdots,m$ and $\forall \alpha,\beta,\gamma \in \Gamma$. This implies that

$$[x,\alpha]F[x\beta a,\gamma] \subseteq Q. \qquad \ldots\ldots\ldots (2)$$

By (1) $[x\beta a,\gamma] \in Q$. Now since Q is strongly prime in L, Q is prime in L.

By Proposition 1.1.16, Q^+ is prime ideal of N. Since N is strongly 2-primal,

Q^+ is completely prime in N. Hence $x\gamma a \in Q^+$ and $x \notin Q^+$ implies $a \in Q^+$.

Thus Q^+ is strongly prime in N.

Proposition: 3.2.4

Let N be a distributive strongly 2-primal Γ- near ring with strong left unity and L, *a* left operator near- ring of N. Then $P_s(N) = P_s(L)^+$.

Proof:

Let P be a strongly prime ideal of L. Then by proposition 3.2.3, P^+ is a strongly prime ideal of N. Moreover $(P^+)^{+'} = P$ by Theorem 1.1.17.

Suppose Q is a strongly prime ideal in N, then by Proposition 3.2.2,

$Q^{+'}$ is a strongly prime in L and $(Q^{+'})^+ = Q$ by Theorem 1.1.17 Thus the mapping $P \rightarrow P^+$defines a 1-1 correspondence between the set of strongly

prime ideals of L and N.

4. STRONGLY PRIME GAMMA RINGS

4.1 Prime and Semiprime ideals of Γ - Rings

In this section, we shall give the basic connection between

prime ideals and semiprime ideals of Γ- ring.

Definition: 4.1.1

A subset N of a Γ- ring M is said to be an n-system if N = ϕ or

if

a ∈ N implies ⟨ a ⟩ Γ ⟨ a ⟩ ∩ N ≠ ϕ.

Lemma: 4.1.2

Let M be a Γ- ring. An ideal Q in M is semiprime if and only if

Q^C

is an n-system.

Proof:

Suppose that Q is a semiprime ideal and let a ∈ Q^C, then

a ∉ Q.

Since Q is semiprime, it follows from Theorem 1.2.7 that ⟨ a ⟩ Γ

⟨ a ⟩⊄Q.

This implies that ⟨ a ⟩Γ⟨ a ⟩ ∩ Q^C ≠ ϕ, so that Q^C is an n-system.

Conversely, suppose Q^C is an n-system and let a ∉ Q.

We shall prove that ⟨ a ⟩ Γ ⟨ a ⟩⊄Q.

Since Q^C is an n-system, $\langle a \rangle \Gamma \langle a \rangle \cap Q^C \neq \phi$. Take $z \in \langle a \rangle \Gamma$

$\langle a \rangle \cap Q^C$

so that $z \in \langle a \rangle \Gamma \langle a \rangle$ and $z \notin Q$.

Hence $\langle a \rangle \Gamma \langle a \rangle \not\subset Q$. Thus Q is a semiprime ideal.

Definition: 4.1.3

For any ideal U of a Γ- ring M, we define n(U) to be set of all

element x of M such that every n-system containing x contains an

element

of U.

Lemma: 4.1.1

Let M be a Γ- ring in the sense of Nobusawa and let $N \subseteq M$

be an

n-system and P be an ideal maximal with respect to the property that P

is

disjoint from N. then P is semiprime ideal.

Proof:

Suppose that $\langle a \rangle \Gamma \langle a \rangle \subset P$ and $a \notin P$.

By the maximal property of P, there exists $x \in N$ such that $x \in P + \langle a \rangle$.

Since N is an n-system $\langle x \rangle \Gamma \langle x \rangle \cap N \neq \phi$. Let $z \in \langle x \rangle \Gamma \langle x \rangle \cap N$.

Then z is the finite sum of element of the form

$$(nx + c\alpha x + x\beta d + e\gamma x\delta f)\rho(mx + g\mu x + xvh + j\xi x\eta k)$$

Where m and n are integers, c,d,e,f,g,h,j,x and k are in M and

$\alpha,\beta,\delta,\rho,\mu,\gamma,\xi,\eta,v$ in Γ. But every element in such a product is in P by

condition (i'),(iii') of definition 1.2.1 and the assumption that $\langle a \rangle \Gamma \langle a \rangle \subset P$.

For example,

$$\begin{aligned}(c\alpha x)\rho(g\mu x) &= c\alpha(x\rho(g\mu x)) \\ &= c\alpha(x\rho(g\mu x)) \\ &= c\alpha(x(\rho g\mu)x) \\ &\in c\alpha(x\Gamma M\Gamma x) \\ &\subseteq c\alpha[(P + \langle a \rangle\Gamma M\Gamma(P+\langle a \rangle)] \\ &\subseteq c\alpha[(P + \langle a \rangle\Gamma M\Gamma\langle a \rangle] \\ &\subseteq P.\end{aligned}$$

Hence $z \in P$, which is contradiction.

Thus P must be a semiprime ideal.

Lemma: 4.1.5

Let M be a Γ- ring in the sense of Nobusawa. If U is any ideal in M,

then n(U) equals the intersection of all semiprime ideal containing U. In particular n(U) is an ideal in M.

Proof:

We first prove that the inclusion '$\subseteq$'.

Let $x \in n(U)$ and P be any semiprime ideal containing U.

Since P is semiprime ideal, P^C is an n-system . This n-system can not contain

x, for otherwise it meets U and hence also P.

Therefore, we have $x \in P$.

Conversely, suppose that x belongs to the intersection of all semiprime

ideals containing U. We show that $x \in n(U)$.

If $x \notin n(U)$, then by definition there exists an n-system N containing x which is disjoint from U.

By Zorn's lemma, there exists an ideal P containing U which is maximal with

respect to being disjoint from N.

By lemma 4.1.4, P is a semiprime ideal and we have $x \notin P$, which is

contradiction and hence $x \in n(U)$.

Next we need the following lemma relating m-system and n-systems.

Lemma: 4.1.6

Let S be an m-system in a Γ-ring M and let $a \in S$. Then there exists

an n-system $N \subseteq S$ such that $a \in N$.

Proof:

We define $N = \{a_1, a_2, \cdots\}$ inductively as follows, $a_1 = a$,

Since S is an m-system, let $a_2 \in \langle a_1 \rangle \Gamma \langle a_1 \rangle \cap S$, then a_2 is the finite sums of

the form

$$(n_1 a_1 + c_1 \alpha_1 a_1 + a_1 \beta_1 d_1 + e_1 \gamma_1 a_1 \delta_1 f_1)$$

$$\rho(m_1 a_1 + g_1 \mu_1 a_1 + a_1 v_1 h_1 + j_1 \xi_1 a_1 \eta_1 k_1)$$

Where $a_1, c_1, d_1, e_1, f_1, g_1, h_1, j_1, k_1$ are element in M and m_1, n_1 are integers,

$\alpha_1, \beta_1, \gamma_1, \delta_1, \mu_1, v_1, \xi_1, \eta_1$ are element in Γ. Again use S is an m-system, take

$a_3 \in \langle a_2 \rangle \Gamma \langle a_2 \rangle \cap S$.

We continue the similar fashion we can have the element $a_3, a_4, \cdots$ of N.

Now for any *i*, $\langle a_i \rangle \Gamma \langle a_i \rangle$ contains a_{i+1}, an element of N.

Hence $\langle a_i \rangle \Gamma \langle a_i \rangle \cap N \neq \phi$ and $N \subseteq S$ such that $a \in N$.

Definition: 4.1.7

An ideal Q in a Γ- ring M is said to be right primary if for any ideal

U and V, $U \Gamma V \subseteq Q$ implies $U \subseteq m(Q)$ or $V \subseteq Q$.

Theorem: 4.1.8

Let M be a Γ- ring in the sense of Nobusawa. For any right primary

ideal Q in M, the following are equivalent.

(i) Q is a prime ideal;

(ii) Q = n(Q);

(iii) Q is a semiprime ideal;

Proof:

(i) ⇒ (ii): Let Q be a prime ideal, then Q ⊆ n(Q) is obvious.

On other hand, let x ∈ n(Q) and suppose that x ∉ Q. Since Q is prime, Q^C is

an m-system and x ∈ Q^C.

By lemma 4.1.6, there exists an n-system N ⊆ Q^C such that x ∈ N. But N is

disjoint from Q, therefore x ∉ n(Q), which is contradiction.

Hence x ∈ Q, so that n(Q) ⊆ Q.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i): Suppose that Q is a semiprime ideal.

We have to prove that Q is a prime ideal.

Let U and V be any ideal in M with UΓV ⊆ Q.

Since Q is primary, $U\Gamma V \subseteq Q$ implies that $U \subseteq m(Q)$ or $V \subseteq Q$.

Thus Q is a prime ideal in M.

Theorem: 4.1.9

For any ideal Q in M, Q is prime if and only if Q is primary and semiprime.

Proof:

Suppose that Q is a prime ideal.

We have to prove that Q is primary.

Let U and V be any ideal in M such that $U\Gamma V \subseteq Q$. Since Q is a prime ideal,

$U \subseteq n(Q)$ or $V \subseteq Q$ by theorem 4.1.8

Now our claim is that $n(Q) \subseteq m(Q)$.

Let $x \in n(Q)$ and S be any m-system containing x. Since any m-system is an

n-system, S is an n-system containing x. Since $x \in n(Q)$, S meet Q.

Hence $x \in m(Q)$ and therefore $U \subseteq n(Q)$ or $V \subseteq Q$ implies that $U \subseteq m(Q)$ or

$V \subseteq Q$.

Hence Q is primary ideal. Since every prime ideal is a semiprime ideal, Q is

Semiprime.

Thus Q is semiprime and hence primary ideal.

Conversely, suppose that Q is primary and semiprime ideal.

By theorem 4.1.8, Q is prime ideal.

4.2 Semiprime Γ- rings

In this section we shall relate semiprime Γ- rings to

semisimple

Γ- rings.

Definition: 4.2.1

Let M be a Γ- ring. M is said to be semiprime if (0) is a

semiprime

ideal. M is said to be prime if (0) is a prime ideal.

Definition: 4.2.2

Let M be a Γ- ring. If for any non zero element a of M there

exists

an element γ(depending on a) in Γ such that $a\gamma a \neq 0$, we say that M is semisimple. If for any non zero element a and b of M, there exists γ (depending on a and b) in Γ such that $a\gamma b \neq 0$, we say that M is simple.

Theorem: 4.2.3

Let M be a Γ- ring in the sense of Nobusawa. Then M is semisimple if and only if M is semiprime.

Proof:

Suppose that $\langle a \rangle \Gamma \langle a \rangle = 0$ for any $a \in M$.

Since $a\Gamma a \subseteq \langle a \rangle \Gamma \langle a \rangle$, $a\Gamma a = 0$. Since M is semisimple, $a\Gamma a = 0$ implies that $a = 0$.

Hence $\langle a \rangle = 0$, so that M is semiprime by theorem 1.2.7.

Conversely, suppose $a\Gamma a = 0$ for any $a \in M$.

Since $a\Gamma M\Gamma a \subseteq a\Gamma a$, $a\Gamma M\Gamma a = 0$. Since M is semiprime, it follows from Theorem 1.2.7, That $a = 0$.

Hence M is semisimple.

Corollary: 4.2.4

M is semiprime if and only if for any ideal U,V in M, $U\Gamma V = 0$ implies that $U \cap V = 0$.

Proof:

Suppose M is semiprime.

Let U,V be ideals in M such that $U\Gamma V = 0$ and let $x \in U \cap V$.

Since $x\Gamma x \subseteq U\Gamma V$, $x\Gamma x = 0$.

Since M is semiprime, M is semisimple by theorem 4.2.3.

Hence $x\Gamma x = 0$ implies that $x = 0$ and consequently $U \cap V = 0$.

Conversely, suppose $U\Gamma V = 0$ implies $U \cap U = 0$ by hypothesis.

Hence $U = 0$, so that M is semiprime.

4.3 Strongly prime Γ- rings

In this section we shall prove that left (right) operator ring of a right (left) strongly prime Γ- ring is right (left) strongly prime and also we shall

prove that if M is strongly prime Γ- rings then their left and right operator rings are Morita equivalent.

Definition: 4.3.1

Let M be a Γ- ring. If A is a subset of M, we define a right (left) α- annihilator of A to be a right (left) ideal $r_\alpha(A) = \{m \in M / A\alpha m = 0\}$

$(l_\alpha(A) = \{m \in M / m\alpha A = 0\})$.

We adopt the symbol M^* to denote the non zero element of M.

Definition: 4.3.2

A right (left) β – insulator for $a \in M^*$ is a finite subset of M, $S_\beta(a)$, such that $r_\alpha(\{a\beta c / c \in S_\beta(a)\}) = (0)$ $\left(l_\alpha(\{c\beta a / c \in S_\beta(a)\}) = (0)\right)$,

$\forall \alpha \in \Gamma$.

Definition: 4.3.3

A Γ- ring M is said to be right (left) strongly prime if for every $\beta \in \Gamma$, each non zero element of M has a right (left) β- insulator, that is for

every $\beta \in \Gamma$ and $a \in M^*$, there is a finite subset $S_\beta(a)$ such that $b \in M$,

$\{a\beta c\alpha b / c \in S_\beta(a)\} = 0$ $\left(\{b\alpha c\beta a / c \in S_\beta(a)\} = 0\right)$, $\forall \alpha \in \Gamma$ implies $b = 0$.

A Γ- ring M is said to be strongly prime if it is both left and

right

strongly prime.

Theorem: 4.3.4

Let M be a Γ- ring with descending chain condition on

annihilators

then M is prime if and only if M is strongly prime.

Proof:

Suppose that M is right strongly prime.

To prove M is prime, let $a, b \in M$ such that $a \neq 0$ and $b \neq 0$.

Since M is right strongly prime, for every $\beta \in \Gamma$, there exists a right

β- insulator $S_\beta(a)$. Then $r_\alpha\left(\{a\beta c / c \in S_\beta(a)\}\right)$, $\forall \alpha, \beta \in \Gamma$.

Since $b \neq 0$, $b \notin r_\alpha\left(\{a\beta c / c \in S_\beta(a)\}\right)$, $\forall \alpha, \beta \in \Gamma$, there exists $\alpha, \beta \in \Gamma$, such

that $a\beta c\alpha b \neq 0$ where $c \in S_\beta(a)$.

Hence M is prime.

Conversely, suppose that M is prime.

We have to prove that M is right strongly prime.

Let $m \in M^*$ and consider the collection of right α- annihilator ideals of the

form $r_\alpha(\{m\beta n/n\in I\})$, $\forall \alpha,\beta \in \Gamma$ where I runs over all finite subset of M

containing the identity.

Since M satisfies the descending chain condition on right annihilator, choose a

minimal element K. If $K \neq \{0\}$, we can find an element $a \in K$ such that $a \neq 0$.

Since M is a prime Γ- ring, it follows from Theorem 1.2.8, That there exists

$b \in M$, such that $m\gamma b\delta a \neq 0$ for $\gamma,\delta \in \Gamma$.

Let I' be a finite subset of M containing the identity and b.

Since $m\gamma b\delta a \neq 0$, $r_\delta(\{m\beta n/n\in I'\})$, a contradiction.

This forces that $K = \{0\}$. Thus m has a right β- insulator $\forall \beta \in \Gamma$.

Since $m \in M^*$ is arbitrary, every element of M^* has a right β- insulator for

all $\beta \in \Gamma$.

Similarly every element of M^* has a left β- insulator for all $\beta \in \Gamma$.

Hence M is a strongly prime Γ- ring.

Theorem: 4.3.5

If M is a right (left) strongly prime Γ- ring, then the left (right) operator ring L(R) is right (left) strongly prime ring.

Proof:

Suppose M is a right strongly prime Γ- ring.

To prove L is right strongly prime ring, it is enough to prove that every non

zero element in L has a right insulator.

Let $\sum_i [x_i,\alpha_i]x \neq 0 \in L$. Then there exists x ∈ M such that $\sum_i [x_i,\alpha_i]x \neq 0$,

that is $\sum_i x_i \alpha_i x \neq 0$.

Since M is right strongly prime, for every β ∈ Γ, there exists an β- insulator

for $\sum_i x_i \alpha_i x$, say it $S_\beta = \{a_1, a_2, \cdots a_n\}$.

$$r_\alpha\left(\left\{\sum_i x_i \alpha_i x \beta c \big/ c \in S_\beta\right\}\right) = \{0\}, \quad \forall \alpha,\beta \in \Gamma.$$

Hence for any m ∈ M,

$$\left(\sum_i x_i \alpha_i x\right)\beta a_k \alpha m = 0,\ \forall \alpha,\beta \in \Gamma,\ a_k \in S_\beta \Rightarrow m = 0. \qquad (1)$$

Now fix α,β ∈ Γ, consider the collection

$$S_{\beta'} = \{[x\beta a_1,\alpha],[x\beta a_2,\alpha]\cdots[x\beta a_n,\alpha]\}$$

We shall prove that $S_{\beta'}$ is an insulator for $\sum_i [x_i,\alpha_i]$.

It is enough to prove that $\mathrm{Ann}\left(\left\{\sum_i [x_i,\alpha_i]c' / c' \in S_{\beta'}\right\}\right) = \{0\}$.

Let $\sum_j[y_j,\beta_j] \in Ann(\{\sum_i[x_i,\alpha_i]c' / c' \in S_{\beta'}\})$.

Then $\sum_i[x_i,\alpha_i][x\beta a_k,\alpha]\sum_j[y_j,\beta_j] = 0, \quad \forall k$.

We claim that $\sum_j[y_j,\beta_j] = 0$. Now

$$\sum_i[x_i,\alpha_i][x\beta a_k,\alpha]\sum_j[y_j,\beta_j] = 0, \quad \forall k$$

Implies that

$$\left(\sum_i[x_i,\alpha_i][x\beta a_k,\alpha]\sum_j[y_j,\beta_j]\right)m = 0, \quad \forall m \in M.$$

$$\therefore \sum_i[x_i,\alpha_i][x\beta a_k,\alpha]\sum_j[y_j,\beta_j](m) = 0,$$

$$\text{i.e.,} \sum_i[x_i\alpha_i x\beta a_k\alpha]\sum_j y_j\beta_j m = 0.$$

$$\text{i.e.,} \sum_i x_i\alpha_i x\beta a_k\alpha\sum_j y_j\beta_j m = 0.$$

By (1), $\sum_j y_j\beta_j m = 0$, i.e., $\sum_j[y_j,\beta_j](m) = 0, \ \forall m \in M$.

Hence $\sum_j[y_j,\beta_j] = 0$. Since $\sum_i[x_i,\alpha_i]\neq 0$ is arbitrary, every non zero element in L has a right β- insulator.

Similarly if M is left strongly prime, then every non zero element of R has a left β- insulator.

Thus L is right strongly prime and R is a left strongly prime ring.

Theorem: 4.3.6

If a Γ- ring M is weakly semiprime then M is strongly prime if

and

only if its left operator ring L is right strongly prime and its right

operator ring

R is left strongly prime.

Proof:

Suppose that L is a right strongly prime Γ- ring.

In order to prove that M is a strongly prime Γ- ring, we shall prove that

for

every $\beta \in \Gamma$, every non zero element in M has a right β- insulator.

Let $x \neq 0 \in M$, $\beta \in \Gamma$.

Since M is a left weakly semiprime Γ- ring, $[x,\beta] \neq 0$.

Since L is right strongly prime, there exists a right insulator

$$S([x,\beta]) = \left\{ \sum_{j=1}^{n} \left[y_{j_k},\beta_{j_k}\right] \big/ k=1,2,\cdots m \right\}$$

For $[x,\beta]$. Then $Ann(\{[x,\beta]c \,/\, c \in S([x,\Gamma])\}) = \{0\}$.

Therefore any $\sum_l [z_l,\delta_l] \in L$,

$$[x,\beta]\sum_{j=1}^{n}\left[y_{j_k},\beta_{j_k}\right]\sum_{l}[z_l,\delta_l]=\{0\}, \quad \forall k1,2,\cdots m$$

implies that $\sum_l [z_l,\delta_l] = 0$ (1)

Consider $S'_\beta = \left\{ y_{j_k}\beta_{j_k}x \big/ j=1,2,\cdots,n; k=1,2,\cdots,m \right\}$.

We claim that S'_β is a right β- insulator for x.

It is enough to prove that for each $\alpha \in \Gamma$, $r_\alpha\left(\left\{x\beta c / c \in S'_\beta\right\}\right) = \{0\}$.

Let $y \in r_\alpha\left(\left\{x\beta c / c \in S'_\beta\right\}\right)$, $\forall \alpha \in \Gamma$. Then

$$\left(x\beta y_{j_k}\beta_{j_k}x\right)\alpha y = 0,\ \forall \alpha \in \Gamma \text{ and } k = 1,2,\cdots,m.$$

$$\therefore \left[x\beta y_{j_k}\beta_{j_k}x\alpha y,\Gamma\right] = 0,\ \forall \alpha \in \Gamma \text{ and } k = 1,2,\cdots,m.$$

Hence $\left[x\beta y_{j_k},\beta_{j_k}\right][x\alpha y,\Gamma] = 0$, $\forall \alpha \in \Gamma$ and $k = 1,2,\cdots,m$, that is

$[x,\beta]\left[y_{j_k},\beta_{j_k}\right][x\alpha y,\Gamma] = 0$, $\forall \alpha \in \Gamma$ and $k = 1,2,\cdots,m$, so that

$[x,\beta]\sum_{j=1}^{n}\left[y_{j_k},\beta_{j_k}\right][x\alpha y,\Gamma]=0$, $\forall \alpha \in \Gamma$ and $k = 1,2,\cdots,m$.

From (1), $[x\alpha y,\Gamma] = 0$, $\forall \alpha \in \Gamma$, so that $x\alpha y = 0$, $\forall \alpha \in \Gamma$.

Since M is faithful L-R bimodule, we have $y = 0$. Since $x \neq 0 \in M$ is arbitrary, for every $\beta \in \Gamma$, every non zero element in M has a right β- insulator

Hence M is right strongly prime Γ- ring.

Similarly if R is a left strongly prime Γ- ring then M is a left strongly prime

Γ- ring. Converse part follows from theorem 4.3.5.

Proposition: 4.3.7

If M is strongly prime Γ- ring, then M is weakly semiprime Γ-ring.

Proof:

Suppose that M is strongly prime Γ- ring.

We shall prove that M is weakly semiprime Γ- ring.

Let $x \neq 0 \in M$.

It is enough to prove that $[x,\Gamma] \neq 0$ and $[\Gamma,x] \neq 0$.

Suppose $[x,\Gamma] = 0$.

Since M is strongly prime Γ- ring, for every $\beta \in \Gamma$ there exists a

finite

subset $S_\beta(x)$ such that for $\beta \in M$, $\left\{x\beta c\alpha b \middle/ c\in S_\beta(x)\right\} = 0$, $\forall \alpha \in \Gamma$ implies

that $b = 0$.

Now $x\beta c\alpha x = [x,\beta]c\alpha x = 0c\alpha b = 0$, $\forall \alpha,\beta \in \Gamma$, $c \in S_\beta(x)$.

Hence $x = 0$, a contradiction.

Thus M is weakly semiprime Γ- ring.

Theorem: 4.3.8

Let M be a strongly prime Γ- ring, L and R be its operator

rings. Then L and R are Morita equivalent.

Proof:

It follows from proposition 4.3.7 and theorem 1.2.5.

A STUDY ON PENDANT DOMINATION IN GRAPHS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

P. MARIA JOEL

Reg. No: 19SPMT17

Under the guidance of

Ms. A. FERDINA M.Sc., M.Phil., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON PENDANT DOMINATION IN GRAPHS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the **Degree of Master of Science in Mathematics** and is the work done during the year **2020-2021** by

P. MARIA JOEL (REG. NO: 19SPMT17)

Ferdina
Signature of the Guide

Signature of the Coordinator

Joyce
Signature of the Director
Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose
Signature of the Principal
Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON PENDANT DOMINATION IN GRAPHS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. A. Ferdina M.Sc., M.Phil., SET.,** Assistant Professor, Department of Mathematics (SSC), **St.Mary's College(Autonomous), Thoothukudi.**

P. Maria Joel

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER - I

PRELIMINARIES

Definition: 1.1

A **Graph $\boldsymbol{G}$** is an order triple $\left(V(G), E(G), X(G)\right)$ where, $V(G)$ is a non-empty set of verties, $E(G)$ is a set of edges disjoint from $V(G)$, $X(G)$ is a function from $E(G)$ to the set of all unorder pairs of elements of V.

Definition: 1.2

A graph G is called a **planar graph** if it has a digraph in which no two edges intersect at a vertex or a point other than a vertex.

Definition: 1.3

An edge starting and ending with the same vertex is called a **loop.**

An edge with distinct ends is called a **link.**

Definition: 1.4

A graph G is called a **simple graph** if

i. It has no loops.

ii. No two links join the same pair of vertices.

Definition: 1.5

A simple graph G is said to be a **complete graph** if every vertex is adjacent to all the other vertices. A complete graph with γ vertices is denoted by K_{γ}.

Definition: 1.6

A graph G is said to be **Bipartite graph** if $V(G)$ is partitioned into two sets X and Y such that every edge of G has one end in X and another end in Y.

The pair (X,Y) is called a Bipartition of V.

Definition:1.7

If (X,Y) is a bipartition of a graph G such that every vertex in X is adjacent to every vertex in Y. Then the graph G is called **complete bipartite graph.**

If $|X| = m$ and $|Y| = n$ then the complete bipartite graph is denoted by $K_{m,n}$.

Definition: 1.8

Two graphs G and H are said to be **isomorphic** if there are two bijection $\theta: V(G) \rightarrow V(H)$ and $\varphi: E(G) \rightarrow E(H)$ such that $\chi_G(e) = uv \Leftrightarrow \chi_H(\varphi(e)) = \theta(u)\theta(v)\ \forall e \in E(G)$.

The pair (θ, φ) is called an isomorphism.

If G is isomorphic to H, then its denoted by $G \cong H$.

Definition: 1.9

The **compliment** G^c of a simple graph G is the simple graph with vertex set V, two vertices being adjacent in G^c iff they are not adjacent in G.

Definition: 1.10

Let $G = (V, E, \chi_G)$ be a graph. A graph $H = (V', E', \chi_H)$ is a **subgraph** of G if

i. $V' \subseteq V$
ii. $E' \subseteq E$
iii. χ_H is a restriction of χ_G to E'.

Definition: 1.11

A subgraph H of G is a **proper subgraph** if $V(H) \subseteq V(G)$.

A subgraph H of G is a called a **spanning subgraph** of G if $V(H) = V(G)$.

Definition: 1.12

The **degree** or **valency** of a vertex in a graph G is the number of edges of G incident with V, counting each loop twice.

Remark: 1.13

i. A vertex of degree 0 is called an **isolated vertex.**

ii. A vertex of degree 1 is called a **'n' vertex** or a **pendent vertex.**

Definition: 1.14

A graph G is **regular** if degree of each vertex is the same.

A graph G is **k-regular graph** if the degree of each vertex is k.

ie.)d(v)=k $\forall v \in V(G)$.

Definition: 1.15

For any graph G,

$\boldsymbol{\delta}(\boldsymbol{G}) = min\{d(v)\backslash v \in V(G)\}$

$\Delta(\boldsymbol{G}) = max\{d(v)\backslash v \in V(G)\}$

Remark: 1.16

i. In any graph G, $\delta(G) \leq d(v) \leq \Delta(G)$.

ii. A graph G is **regular** iff $\delta(G) = \Delta(G)$.

Definition: 1.17

A finite sequence in which vertices and edges alternatively and which begins and end with vertices is called a **walk.**

Definition: 1.18

The **length** of a walk is the number of edges occurring in the walk.

Definition: 1.19

A walk in which edges are not repeated is called a **trail**.

A walk in which vertices are not repeated is called a **path.**

Definition: 1.20

A non-trivial closed path of a graph G is called the **cycle** of G.

Definition: 1.21

A cycle of length k is called a **k-cycle.** It is denoted by C_k.

Definition: 1.22

A vertex v of a graph G is called a **cut vertex** if its removal increases the number of components.

Definition: 1.23

A graph that contains no cycles is called an **acyclic graph**. A connected acyclic graph is called a **tree.**

A collection of tree is called a **forest**.

Definition: 1.24

A connected graph that has no cut vertices is called a **block**.

Definition: 1.25

A closed trail containing all points and lines is called an **eulerian trail**. A graph having an eulerian trail is called an **eulerian graph.**

Definition: 1.26

A spanning cycle in a graph is called a **Hamiltonian cycle.**

A graph having a Hamiltonian cycle is called a **Hamiltonian graph**

Definition: 1.27

Any graph G with atleast one bridge is called a **bridge graph.**

Definition: 1.28

A **dominating set** for a graph $G = (V(G), E(G))$ is a subset D of $V(G)$ such that every vertex not in D is adjacent to atleast one member of D.

Definition: 1.29

The least cardinality of a dominating set in G is called the **domination number** of G and is usually denoted by $\gamma(G)$.

CHAPTER - II

PENDANT DOMINATION IN SOME GENERALIZED GRAPHS

Introduction:

In this chapter, we see the application and significance of pendant domination in graphs and also find the pendant domination number $\gamma_{pe}(G)$ in graphs such as Crown graph, Helm graph, Cocktail party graph, Banana tree graph, Fire cracker graph, Stacked graph, Octahedral graph, Jahangir graph. The symbol $\lceil x \rceil$ stands for smallest integer greater than or equal to x.

Definition: 2.1

A dominating set S in G is called a **pendant domination** set if $\langle S \rangle$ contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by $\gamma_{pe}(G)$.

The pendant domination parameter is defined for all non-trivial connected graphs of order at least two. Hence, throughout the chapter we assume that by a graph we mean a connected graph of order atleast two.

Results: 2.2

(i) Let G be a complete graph. Then $\gamma_{pe}(G) = 2$.

(ii) Let G be a wheel W_n or a star $K_{1,n-1}$. Then $\gamma_{pe}(G) = 2$.

(iii) Let C_n be a cycle with $n \geq 3$ vertices and let P_n be a path with $n \geq 2$ vertices. Then

$$\gamma_{pe}(C_n) = \gamma_{pe}(P_n) = \begin{cases} \frac{n}{3} + 1, & if\ n \equiv 0(mod3) \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n \equiv 1(mod3) \\ \left\lceil \frac{n}{3} \right\rceil + 1, & if\ n \equiv 2(mod3) \end{cases}$$

Definition: 2.3

The **crown graph** S_n for $n \geq 3$ is the graph with vertex set $V = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and an edge from $V = \{u_1, u_2, / 1 \leq i, j \leq n, i \neq j\}$. Therefore S_n coincides with the complete bipartite graph S_n with the horizontal edges removed. Crown graph S_6 is shown in Fig.1.

Example: 2.4

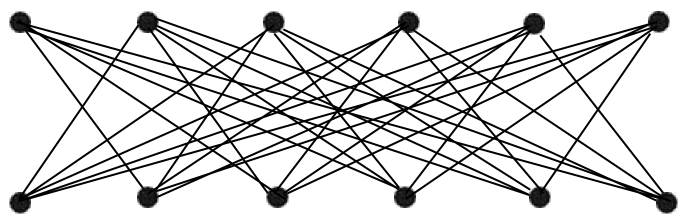

Fig. 1. $\boldsymbol{S_6}$

Theorem: 2.5

Let G be a crown graph with 2n vertices. Then $\gamma_{pe}(G) = \gamma_{pe}(G) + 1$.

Proof:

Let G be a crown graph with the vertex set $V = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$.

Clearly, the set $V = \{u_1, v_1\}$ is a dominating set of G.

Choose any one vertex u_i or v_i where $i > 1$, then the set $S' = \{u_1, v_1\} \cup \left\{\{u_i\} or \{v_i\}\right\}$ will be a pendant dominating set of G.

Therefore $\gamma_{pe}(G) = \gamma(G) + 1$.

Hence Proved.

Definition: 2.6

The **helm graph** H_n is the graph obtained from a n-wheel graph by adjoining a pendant edge at each node of the cycle. The helm graph H_n has $2n + 1$ vertices and $3n$ edges. Helm graph H_6 is shown in Fig. 2.

Example: 2.7

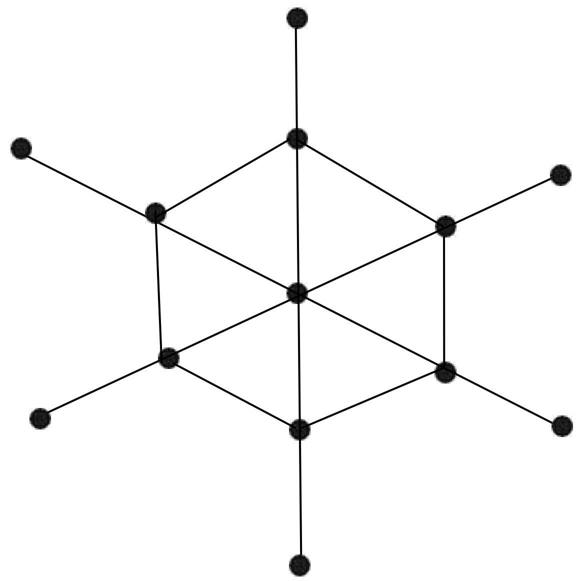

Fig. 2. H_6

Theorem: 2.8

For any helm graph H_n, then $\gamma_{pe}(H_n) = n$.

Proof:

Let (XY) be a partition of H_n, with $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{u_1, u_2, \ldots, u_n\} \cup \{v\}$.

Let u_1, u_2 are the adjacent vertices of the graph H_n and S is the set of collection of all leaves of H_n except the leaves of u_1 and u_2, then the set $S' = |S| \cup \{u_1, u_2\}$ will be a pendant dominating set of H_n.

Hence $\gamma_{pe}(H_n) = (n - 2) + 2 = n$.

Definition: 2.9

The **cocktail party graph** $K_{n\times 2}$ is a graph of order $2n$ with the vertex set $V = \{u_i v_i / 1 \leq i, j \leq n\}$ and the edge set $V = \{u_i u_j, v_i v_j, u_i v_j, v_i u_j \backslash 1 \leq i \leq j \leq n\}$.

Cocktail graph of order 8 is shown in Fig. 3.

Example: 2.10

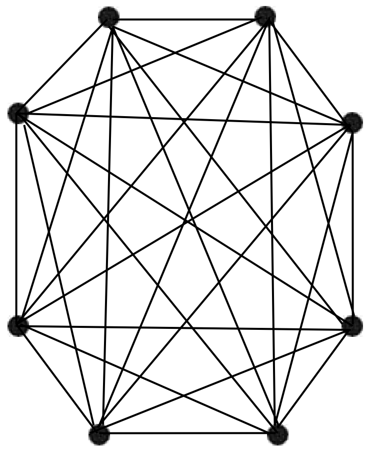

Fig. 3. Cocktail graph of order 8

Theorem: 2.11

Let G be a cocktail party graph of order 2n, then $\gamma_{pe}(\, H_n) = 2$.

Proof:

Let G be a cocktail party graph and $\{v_1, v_2, \dots, v_{2n}\}$ are vertices of G.

Let us choose the set $S = \{v_1, v_2\}$, where v_1 and v_2 are two adjacent vertices in G.

Then the set S will be a minimal pendant dominating set of G.

Hence $\gamma_{pe}(\, H_n) = 2$.

Hence Proved.

Definition: 2.12

A **banana tree graph** (n, k) is a graph is obtained by connecting one leaf of each n copies of a K star graph with a single root vertex that is distinct from all the stars. Banana tree graph $B_{3,4}$ is shown in the Fig. 4.

Example: 2.13

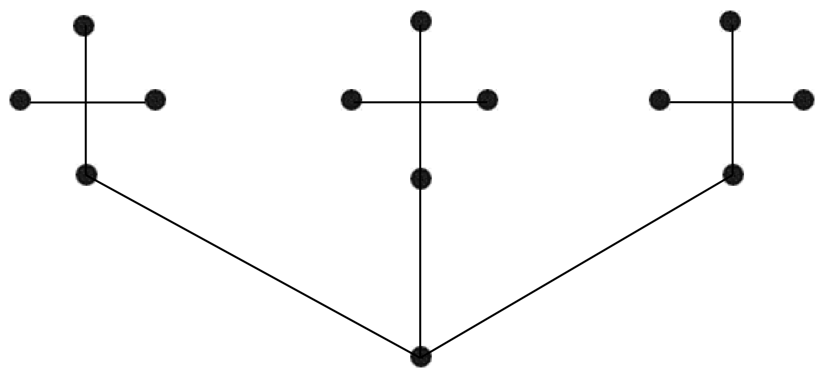

Fig. 4. $B_{3,4}$

Theorem: 2.14

Let G be a banana tree graph, then $\gamma_{pe}(\mathrm{G}) = \gamma(G)$.

Proof:

Let G be a banana tree graph

Clearly $\gamma(G) = n$.

The set $S = \{V\} \cup \{v\}$ is a dominating set of G and S is itself a pendant dominating set of G, where $\{V\}$ is the collection of all centre vertices of n copies of a star graph and v is a vertex in any one copy of a star graph and $deg(v) = (n + 1)$.

Therefore $\gamma_{pe}(\mathrm{G}) = \gamma(G)$.

Hence Proved.

Definition: 2.15

An **fire cracker** $F(m, n)$ is a graph obtained by the series of interconnected m –copies of n stars by linking one leaf from each. Fire cracker graph $F_{4,7}$ is shown in Fig.5.

Examle: 2.16

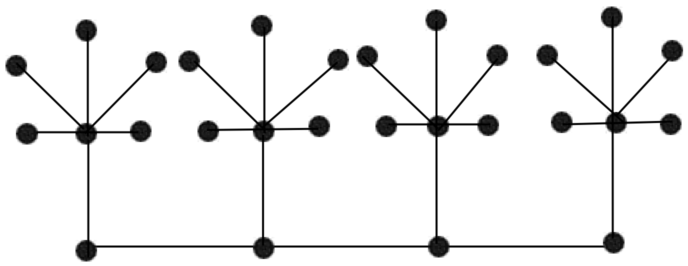

Fig. 5. $\boldsymbol{F_{4,7}}$

Observation: 2.17

For any firecracker graph, $F_{n,k}$ where $n \geq 2, k \geq 3$, then $\gamma_{pe}(\text{ G}) = (n + 1)$.

Theorem: 2.18

Let G be an octahedral graph, then $\gamma_{pe}(\text{ G}) = \gamma(G)$.

Proof:

Let G be an octahedral graph with 6-nodes and 12-nodes and is isomorphic to circulant graph.

The set $S = \{u, v\}$ is a dominating set of G and S is itself a pendant dominating set if u,v are adjacent vertices of G, and then the set S will be a minimal pendant dominating set of G.

Therefore $\gamma_{pe}(\text{ G}) = \gamma(G)$.

Theorem: 2.19

For any stacked book graph $B_{n,m}$ where $m \geq 3, n \geq 2$, then $\gamma(G) = n$.

Proof:

Let $B_{n,m}$ be the stacked book graph with $V(B_{n,m}) = v_1, v_2, \ldots, v_{2n+2}$. Which is obtained by the cartesian product of $S_{m+1} \otimes P_n$, where S_m is the star graph and P_n is the path graph of order n.

Let $\{v_1, v_2, \ldots, v_n\}$ are vertices of the path and these vertices are dominates all other vertices of $B_{n,m}$.

Then the set $S = \{v_1, v_2, \ldots, v_n\}$ is a dominating set of $B_{n,m}$ and $\langle S \rangle$ contains a pendant vertex,

Therefore S will be a minimal pendant dominating set of $B_{n,m}$.

Hence $\gamma_{pe}(\mathrm{G}) = n$.

Definition: 2.20

For $m \geq 2$, **Jahangir graph** $J_{n,m}$ is a graph of order $mn + 1$, consisting of a cycle of order nm with one vertex adjacent to exactly m vertices of $C_{n,m}$ at a distance n to each other. Jahangir graph $J_{2,8}$ is shown in Fig. 6.

Example: 2.21

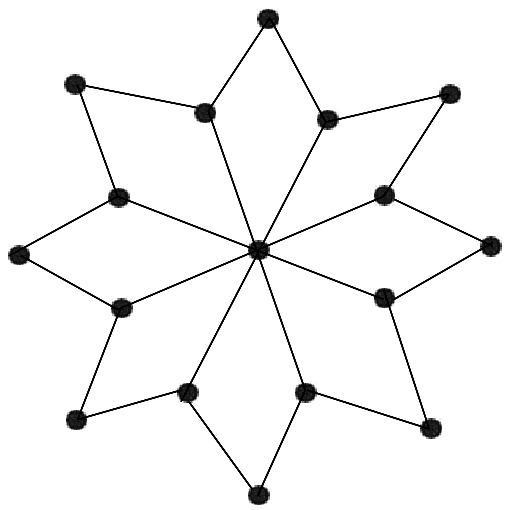

Fig. 6. $\boldsymbol{J_{2,8}}$

Theorem: 2.22

Let $G = J_{n,m}$ be a Jahangir graph with $m, n \geq 3$. Then

$$\gamma_{pe}(P_n) = \begin{cases} \frac{m(n-1)}{3} + 2, & if\ n \equiv 0(mod3) \\ \left\lceil \frac{mn}{3} \right\rceil + 1, & if\ \ n \equiv 0\ or\ 2(mod3) \end{cases}$$

Proof:

Let $G \cong J_{n,m}$ be a Jahangir graph with $m, n \geq 3$ and let $V(G) = \{v_1, v_2, \dots, v_{nm}, v_{nm+1}\}$, where v_{nm+1} is the vertex at the centre, adjacent to vertices of $C_{n,m}$.

First assume $n \equiv 1(mod3)$

i.e., $n = 3k + 1$, for some positive integer k.

From the definition, the vertex v_{mn+1} is adjacent to m vertices of $C_{n,m}$ at a distance $3k + 1$.

Removing the vertex v_{mn+1} and its neighbourhood vertices from G, the graph induced by $V(G) - \{N[v]\}$ splits into m components each component isomorphic to P_{3k}.

Therefore, the minimum pendant dominating set of G is obtained by taking dominating set from each component togeather with v_{mn+1} and one of its neighbourhood vertex.

That is, if $S = \bigcup_{i=1}^{m} s_i$, where s_i denotes γ set of i th component, then $S \cup \{v_{mn+1}, v_1\}$, where v_1 is the vertex adjacent to v_{mn+1}.

Then the set S will be a minimal pendant dominating set of G.

Hence $\gamma(G) = \frac{m(n-1)}{3} + 2$.

Next, suppose $n \equiv 2(mod3)$.

Here, we may consider two possible cases.

First, assume $m \equiv 0(mod3)$. Then $\{v_1, v_m, v_{2m}, v_{3m}, \dots, v_{nm}\}$ will be a dominating set of cardinality $\frac{nm}{3} + 1$.

Next, suppose $m \equiv 1(mod3)$.

In this case $\{v_1, v_3, v_6, \dots, v_{nm}\}$ will be a dominating set of size $\frac{nm+4}{3}$

i.e., $\left\lceil\frac{mn}{3}\right\rceil + 1$

Finally, assume $n \equiv 0(mod3)$.

For any integer $m \geq 3$, clearly nm will be a multiple of 3.

Further, no dominating set contains the centre vertex v_{nm}.

Let S be the dominating set of $C_{n,m}$ and $\langle S \rangle$ contains only isolated vertices.

For the purpose of the pendant vertex choose any vertex in $C_{n,m}$ is adjacent to any vertex in the dominating set.

Hence, $\gamma_{pe}(\mathrm{G}) = \gamma(C_{n,m}) + 1$ i.e., , $\gamma_{pe}(\mathrm{G}) = \left\lceil\frac{mn}{3} + 1\right\rceil$.

Hence Proved.

CHAPTER - III

BI-PENDANT DOMINATION NUMBER IN GRAPHS

Introduction:

In this chapter, we see the application and significance of Bi-Pendant domination number $\gamma_{bpe}(G)$ in graphs such as Helm graph, Wheel graph, Crown graph, Barbell graph, Pan graph, Connected graph and Triangle free graph.

Definition: 3.1

A Pendant dominating set S of a graph G is a **bi-pendant dominating set** if $\langle V - S \rangle$ also contains pendant vertex. The least cardinality of the bi-pendant dominating set in G is called the bi-pendant domination number of G, denoted by $\gamma_{bpe}(G)$.

The bi-pendant domination number is not defined for the complete graph and bistar graph. In complete graph bi-pendant domination is defined only when $n = 4$, in all other cases γ_{bpe} is not defined.

Example: 3.2

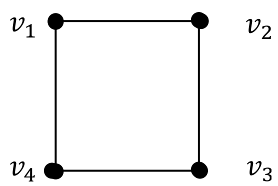

Fig. 7. $G = K_4$

The possible minimum bi-pendant domination sets for the following graph G are:

(i) $D_1 = \{v_1, v_2\}$

(ii) $D_2 = \{v_2, v_3\}$

(iii) $D_3 = \{v_3, v_4\}$

(iv) $D_3 = \{v_4, v_1\}$

Theorem: 3.3

Let P_n be a path with $n \geq 5$ vertices. Then

$$\gamma_{bpe}(P_n) = \begin{cases} \frac{n}{3} + 1, & if\ n \equiv 0(mod3) \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n \equiv 1(mod3) \\ \left\lceil \frac{n}{3} \right\rceil + 1, & if\ n \equiv 2(mod3) \end{cases}$$

Proof:

Let $G \cong P_n$ be a path and let $V(G) = \{v_1, v_2, \dots, v_n\}$. We consider the following possible cases here:

Case 1:

Suppose $n \equiv 0(mod3)$.

Then $n = 3k$, for some integer $k > 1$.

Then the set $S = \{v_2, v_3, v_{3i}/2 \leq i \leq k\}$ is a bi-pendant dominating set of G.

Hence $\gamma_{bpe}(\mathrm{G}) = |S|$. i.e., $\gamma_{pe}(\mathrm{G}) = \frac{n}{3} + 1$.

On the other hand, we have $\gamma(G) = \frac{n}{3}$ and any minimum dominating set of G contains only isolated vertices.

Thus $\gamma_{bpe}(\mathrm{G}) \geq \frac{n}{3} + 1$.

Therefore $\gamma_{bpe}(\mathrm{G}) = \frac{n}{3} + 1$.

Case 2:

Suppose $n \equiv 0(mod3)$.

Then $n = 3k + 1$, for some integer $k > 1$.

Then it is easy to check that any γ-set in G contains a pendant vertex and $\langle V - S \rangle$ also contains a pendant vertex.

Hence any γ-set S in G itself a bipendant dominating set in G.

Therefore $\gamma_{bpe}(\mathrm{G}) = \gamma(G) = \left\lceil \frac{n}{3} \right\rceil$.

Case 3:

Proof of this case is similar to case 1.

Observation: 3.4

i. Let C_n be a cycle with $n \geq 4$ vertices. Then

$$\gamma_{bpe}(\mathrm{G}) = \begin{cases} \frac{n}{3} + 1, & if\ n \equiv 0(mod3) \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n \equiv 1(mod3) \\ \left\lceil \frac{n}{3} \right\rceil + 1, & if\ n \equiv 2(mod3) \end{cases}$$

ii. Let G be a ladder graph with $2n$ vertices. Then $\gamma_{bpe}(\mathrm{G}) = \left\lceil \frac{n}{2} \right\rceil + 1$

Theorem: 3.5

For any helm graph H_n with $2n + 1$ vertices. Then $\gamma_{bpe}(H_n) = n + 1$.

Proof:

Let (XY) be a partion of H_n with $X = \{v_1, v_2, \dots, v_n\}$ and

$Y = \{u_1, u_2, \dots, u_n\} \cup \{v\}$.

Where $\{v\}$ is the vertex attached to all the vertices in the set Y.

Let v, u_1 are the two adjacent vertices of the graph H_n and S is the set of all collection of leaves of H_n, except the leaf of u_1.

Then the set $S' = |S| \cup \{v, u_1\}$ will be a bi-pendant dominating set of H_n.

Therefore $\gamma_{bpe}(H_n) = |S'| = (n-1) + 2 = (n+1)$.

Hence Proved.

Theorem: 3.6

Let G be wheel graph with n vertices and $n \geq 3$. Then $\gamma_{bpe}(W_n) = 2$.

Proof:

Let G be a wheel graph of order $n \geq 3$.

Then $G \cong C_{n-1} + K_1$.

The set $S = \{u, v\}$ is a pendant dominating set of G where v is the vertex in K_1 and $u \in C_{n-1}$.

Therefore S is itself a bi-pendant dominating set of G.

Hence $\gamma_{bpe}(W_n) = |S| = 2$.

Observation: 3.7

Let G be a crown graph with $2n$ vertices. Then $\gamma_{bpe}(G) = n$.

Theorem: 3.8

Let $G \cong K_{m,n}$ be a complete bipartite graph with $m \leq n$. Then $\gamma_{bpe}(K_{m,n}) = m$.

Proof:

Let $G \cong K_{m,n}$ be a complete bipartite graph with $V_1 = \{v_1, v_2, \dots, v_n\}$ and $V_2 = \{u_1, u_2, \dots, u_m\}$ are two partie set in G.

The bi-pendant dominating set of G is obtained by taking the one vertex in partite set V_1 and $m - 1$ vertices in the another partite set V_2.

Therefore $\gamma_{bpe}(G) = 1 + (m - 1) = m$.

Theorem: 3.9

Let G be a barbell graph of order n. Then $\gamma_{bpe}(G) = n - 1$.

Proof:

Let G be a barbell graph and let $V(G) = \{v_1, v_2, \dots, v_{2n}\}$. Let v_1 and v_2 be the adjacent vertices of G is attached to the copies of complete graph.

The bi-pendant dominating set of G is obtained by taking the vertices v_1, v_2 and $(n - 3)$ vertices in any one copies of complete graph.

Therefore $\gamma_{bpe}(G) = 2 + (n - 3) = n - 1$.

Hence Proved.

Definition: 3.10

The **pan graph** is the graph obtained by joining a cycle C_n to singleton graph K_1 with a bridge. It is denoted by P_n. Pan graph P_3 is shown in the Fig. 8.

Example: 3.11

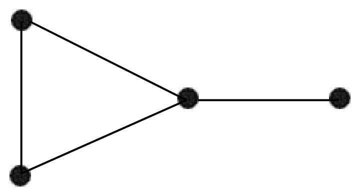

Fig. 8. $\boldsymbol{P_3}$

Theorem: 3.12

Let G be a pan graph. Then $\gamma_{bpe}(G) = 2 + \left\lceil \frac{n-3}{3} \right\rceil$.

Proof:

Let G be a pan graph with vertices $\{v_1, v_2, \dots, v_n\}$ where v_n is the vertex attached to the vertex v_1 of C_n.

Fix an edge $e = v_1 v_n$.

Then $\gamma_{bpe}(G) = \{u, v\} \cup \gamma(H)$ where H is the graph obtained by removing the vertices v_1, v_n and its neighbour from G.

Clearly $H \cong P_{n-3}$.

Hence $\gamma_{bpe}(G) = 2 + \gamma(P_{n-3}) = 2 + \left\lceil \frac{n-3}{3} \right\rceil$.

Hence Proved.

Theorem: 3.13

If G is a graph then $\gamma_{bpe}(G) = 2$ if and only if $G \cong T + K_1$. Where T is a tree of order $n \geq 3$.

Proof:

Assume that $G \cong T + K_1$, then clearly the set $S = \{u, v\}$ will be a bi-pendant dominating set of G, where u and v are vertices in T and K_1 respectively.

Conversely, if $\gamma_{bpe}(G) = 2$ then there exist a bi-pendant dominating set of G with $|S| = 2$. Such that $\langle V - S \rangle$ is a tree.

Since each vertex in $\langle V - S \rangle$ is adjacent to the vertex in S.

Let ς be the collection of graphs of following types. A cycle, complete graph of order 4, cycle, path and wheel of order 5 and $K_{2,2}$.

Theorem: 3.14

Let G be a connected graph of order n. Then $\gamma_{bpe}(G) = n - 2$ if and only if $G \in \varsigma$.

Theorem: 3.15

For any integer $a > 0$, there exist a connected graph G such that $\gamma(G) = \gamma_{bpe}(G) = a + 1$.

Proof:

Let P_j: $\{u_j, v_j, w_j, x_j, y_j\}(1 \leq j \leq a)$ be a path of order 5.

We show that $(G) = \gamma_{bpe}(G) = a + 1$.

$H_j = \{v_j, u_j, x_j\}(1 \leq j \leq a)$

Its easily observed X belongs to every minimum bi-pendant dominating set of G and so $\gamma_{bpe}(G) \geq 1$.

Also its easily seen that every dominating set of G contains at least one element of $H_j(1 \leq j \leq a)$ and so $\gamma_{bpe}(G) \geq a + 1$.

Now the set $S = \{X\} \cup \{v_1, v_2, v_3, \dots, v_a\}$ will be a bi-pendant dominating set of G.

So that $\gamma(G) = \gamma_{bpe}(G) = a + 1$.

Theorem: 3.16

Let G be any graph with n vertices. Then $\gamma(G) \leq \gamma_{pe}(G) \leq \gamma_{bpe}(G)$. Equality holds if G is a cycle of order G.

Proof:

Since every bi-pendant dominating set ia a pendant dominating set and every pendant dominating set is a dominating set of G, it follows that $\gamma(G) \leq \gamma_{pe}(G) \leq \gamma_{bpe}(G)$.

Suppose G is a cycle with 4 vertices. Then $\gamma(G) = \gamma_{pe}(G) = \gamma_{bpe}(G) = 2$.

Theorem: 3.17

For any graph G, we have $\gamma(G) \leq \gamma_{pe}(G) \leq \gamma(G) + \delta(G)$.

Proof:

Since a bi-pendant dominating set of G is a dominating set, it follows that $\gamma(G) \leq \gamma_{bpe}(G)$.

Now let v be a vertex in G with $\deg(v) = \delta(G)$ and let S be a dominating set in G and every dominating set of G contains $N[v]$ so that the set $S' = S \cup N[v]$ will be a bi-pendant dominating set of G,it follows that $\gamma_{bpe}(G) \leq \gamma(G) + \delta(G)$ and hence the right inequality follows.

Theorem: 3.18

Let G be a graph with n vertices. Then $\gamma(G) + \gamma_{bpe}(G) \leq n$.

Proof:

Let S be a bi-pendant dominating set. Then S is a dominating set and $\langle V - S \rangle$ contains a pendant vertex.

Obviously, $\gamma_{bpe}(G) \leq |S|$.

Since S is dominating $\langle V - S \rangle$ is also a dominating.

Thus $\gamma(G) \leq |V - S|$. Hence $\gamma(G) + \gamma_{bpe}(G) \leq |S| + |V - S|$ proving the result.

Theorem: 3.19

Let G be a connected graph with n vertices and H be any graph. Then

$$\gamma_{bpe}(G \circ H) = \begin{cases} n, & if \ \gamma(H) = 1 \\ n+1, & otherwise \end{cases}$$

Proof:

For any connected graph G with n vertices and H be any graph, we have $\gamma_{bpe}(G \circ H) = n$ and hence $\gamma_{bpe}(G \circ H) \leq n + 1$.

First, suppose H has a pendant vertex, then clearly the set $S = |V(G)|$ is a bi-pendant dominating set in $(G \circ H)$.

If $\delta(G) \geq 2$, then the set $S = |V(G)| \cup \{u\}$ will be a bi-pendant dominating set of $G \circ H$, where u is a vertex in H is adjacent to any one vertex in G.

Therefore $\gamma_{bpe}(G \circ H) = |S| = n + 1$.

Theorem: 3.20

Let G be any graph. If $diam(G) \geq 3$ then $\gamma_{bpe}(\overline{G}) = 2 \ or \ 3$

Proof:

If G has a pendant vertex then clearly $\gamma_{bpe}(G) = 2$. Let G be a connected graph of diameter at least 3.

If $u, v \in V(G)$ with $diam(u, v) \geq 3$ then the set $S = \{u, v\}$ is a pendant dominating set of $(\overline{G})$.

Therefore $\gamma_{bpe}(\overline{G}) = 3$.

Theorem: 3.21

Let G be a triangle free graph order at least 3. Then $\gamma_{bpe}(\overline{G}) = 2 \ or \ 3$.

Proof:

Let G be a triangle free graph.

If G contains a pendant and an isolated vertex then clearly $\gamma_{bpe}(\overline{G}) = 2$.

Suppose G has no pendant and an isolated vertex, then G contain atleast one edge say $e = uv$.

As G is triangle free no vertex in G can be adjacent to both u and v.

Thus $S = \{u, v\}$ will be a γ_{bpe}-set in $\overline{G}$.

Now, for any vertex $w \in V(G)$, the set $S \cup \{w\}$ will be a γ_{bpe}-set in $\overline{G}$.

Hence $\gamma_{bpe}(\overline{G}) = 3$.

Hence Proved.

CHAPTER - IV

UPPER PENDANT DOMINATION IN GRAPHS

Introduction:

In this chapter, we see the application and significance of Upper Pendant domination in graphs and we also find the Upper Pendant Domination number $\Gamma_{pe}(G)$ for graphs such as Pan graph, Wheel graph, Grid graph, Ladder graph, Stacked book graph and Book graph.

Definition: 4.1

The minimal pendant dominating set with maximum cardinality is called the **Upper pendant dominating set.** The cardinality of an upper pendant dominating set is called an upper domination number, denoted by $\Gamma_{pe}(G)$. Any upper pendant dominating set of cardinality $\Gamma_{pe}(G)$ is called the Γ_{pe}- set.

Result: 4.2

1. Let G be a completely graph. Then $\Gamma_{pe}(G) = 2$

2. Let $G \cong K_{m_1,m_2,\dots,m_k}$ be a complete multipartite graph. Then $\Gamma_{pe}(G) = 2$.

3. Let be a barbell graph. Then $\Gamma_{pe}(G) = 3$.

Observation: 4.3

The upper pendant domination is not defined for totally disconnected graph. From the definition of the pendant domination, it is clear that the parameter $\Gamma_{pe}(G)$ is defined if G has atleast one edge. Thus, hereafter by a graph, we mean a graph having atleast one edge.

Observation: 4.4

1. $\gamma_{pe}(K_n) = \Gamma_{pe}(K_n)$ for all n.
2. $\gamma_{pe}(P_n) = \Gamma_{pe}(P_n)$ if and only if n=2 or 3 and $\gamma_{pe}(C_n) = \Gamma_{pe}(C_n)$ if and only if n≤6.
3. $\gamma(K_{m,n}) = \gamma_{pe}(K_{m,n}) = \Gamma(K_{m,n})$ if and only if m,n = 2 and $\gamma_{pe}(K_{m,n}) = \Gamma_{pe}(K_{m,n})$ for all m,n≥ 1

Theorem: 4.5

Let $G \cong K_m\ (a_1, a_2, \ldots. a_m\ )$ be a multi star graph. Then

$$\Gamma_{pe}(G) = 2 + \max_{1 \le i \le m} \sum_{j=1, j \ne i}^{m} a_j$$

Proof.

Let $G \cong K_m(a_1, a_2, \ldots. a_m)$ be a multi star of order a_1 +a_2+…..+ a_m + m.

Assume that $a_1 \le a_2 \le \ldots \le a_m$

Then, the collection S of all leaves will be an upper dominating set in G and

Hence $\Gamma(G) = a_1 + a_2 + \ldots\ a_m$

Picking an edge uv from the star k_{a1} and taking the leaves of not in K_{a1} ,

the set $S' = (S - V(k_{a1})) \cup \{u, v\}$ will be a pendant dominating set in G.

As the vertices in S' are leave and contains exactly one edge, S' will be a

Minimal pendant dominating set of maximum cardinality.

Therefore $\Gamma_{pe}(G) = |S'| = 2 + \sum_{i=2}^{m} a_i$

In general, by the maximality, we have $\Gamma_{pe}(G) = 2 + \max_{1 \le i \le m} \sum_{j=1. j \ne i}^{m} a_j$.

Hence Proved.

Corollary: 4.6

For any integer $k \geq 3$, there exists a graph G such that $\Gamma_{pe}(G) = k$

Theorem: 4.7

Let $G \cong P_n$ be a path of order $n \geq 2$. Then

$$\Gamma_{pe}(G) = \begin{cases} 2, & if\ n = 2 \\ 2 + \left\lceil \frac{n-3}{2} \right\rceil, & otherwise \end{cases}$$

Proof.

Let $G \cong P_n$ be a path and let $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Clearly, $\Gamma_{pe}(P_2) = 2$.

Suppose $n \geq 3$. Since any upper pendant dominating set should contain a

Pendant vertex, we may fix an edge $\{v_1, v_2\}$ in G and let $H = V(G) - N[v_1, v_2]$

Then $\Gamma_{pe}(G) = 2 + \Gamma(H)$ where $H \cong P_{n-3}$.

Therefore, $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-3}{2} \right\rceil$.

Theorem: 4.8

Let C_n be a cycle of order $n \geq 3$. Then $\Gamma_{pe}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof.

Let C_n be a cycle and $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n.

Fix an arbitrary edge, say uv in C_n and let $H = V(G) - N[\mathrm{u}, \mathrm{v}]$.

Then, $\Gamma_{pe}(G) = 2 + \Gamma(H)$ where $H \cong P_n - 4$.

Therefore, $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-4}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$.

Theorem: 4.9

Let G be a Pan graph with $n \geq 5$ vertices. Then $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-3}{2} \right\rceil$.

Proof. .

Let G be a pan graph and $V(G) = \{v_1,v_2,\ldots,v_n\}$ where v_n is the vertex attached to the vertex v_1 of C_{n-1}

Fix the edge $\{v_1,v_n\}$ then for Γ -set of H, where H is the graph obtained by removing $\{v_1,v_n\}$ and its neighbours from G.

Clearly, $H \cong P_n$-3 and so $\Gamma_{pe}(G) = 2 + \left\lceil \frac{n-3}{2} \right\rceil$.

Theorem: 4.10

Let G_1 and G_2 be any two graphs. Then

$\Gamma_{pe}(G_1 \vee G_2) = \max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}$.

Proof.

Let G_1 and G_2 be any two graphs and let S_1, S_2 be the Γ_{pe}-sets of G_1 and G_2 respectively.

By the definition of join of graphs, S_1 and S_2 are minimal pendant dominating Sets of $G_1 \vee G_2$ and so $\Gamma_{pe}(G_1 \vee G_2) \geq \max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}$.

Let v be any vertex of $G_1 \vee G_2$.

Assume $v \in V(G_1)$.

Then $S_1 \cup \{ v\}$ fails to be a minimal pendant dominating set.

On the other hand, $S_2 \cup \{ v\}$ fails to be dominating set in G_1 v G_2.

Thus, $\Gamma_{pe}(G_1 \vee G_2) \leq \max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}$.

Therefore, we have $\Gamma_{pe}(G_1 \vee G_2) = \max\{\Gamma_{pe}(G_1), \Gamma_{pe}(G_2)\}$.

Theorem: 4.11

Let G be any graph of size at least one. Then $\Gamma_{pe}(G \vee \overline{k_n}) = \Gamma_{pe}(G)$.

Proof.

Let G be any graph of size at least one and let S be the Γ_{pe} -set of G.

Then S is also a minimal dominating set of $G \vee \overline{k_n}$ and so $\Gamma\,(G \vee \overline{k_n}) \geq |s|$.

On the other hand, for any vertex v of $(G \vee \overline{k_n})$, the set $S \cup \{ v\}$ will not be minimal.

This proves that, $\Gamma\,(G \vee \overline{k_n}) \leq |s|$ and hence $\Gamma_{pe}(G \vee \overline{k_n}) = |S| = \Gamma_{pe}(G)$.

Corollary: 4.12

Let G be an m-gonal n-cone graph. Then $\Gamma_{pe}(G) = \left\lceil \frac{m}{2} \right\rceil$.

Proof.

Let G be an m-gonal n-cone graph.

Then G is the graph join of the cycle graph C_m with $\overline{k_n}$.

Taking G to be the cycle graph on m vertices in the above theorem, we get

$\Gamma_{pe}(G) = \left\lceil \frac{m}{2} \right\rceil$.

Definition: 4.13

A **wheel graph** is a graph formed by connecting a single universal vertex to all vertices of a cycle. Wheel graph W_7 is shown in the Fig. 9.

Example: 4.14

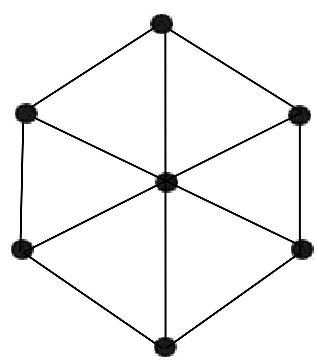

Fig. 9. W_7

Corollary: 4.15

For a wheel W_n of order n≥ 4, $\Gamma_{pe}(W_n) = \left\lceil \frac{n-1}{2} \right\rceil$.

Proof.

Let G be a wheel graph of order n≥ 4.

Then $G \cong C_{n-1} + \overline{k_n}$

Therefore, by taking G to be the cycle on n-1 vertices in above proposition, We obtain that $\Gamma_{pe}(W_n) = \left\lceil \frac{n-1}{2} \right\rceil$

For a wheel W_n of order n, the line graph $G \cong L(W_n)$ is a bi-regular graph on 2(n-1) vertices such that degree of any vertex in G belongs to the set {n – 1, n}.

Remark: 4.16

For a wheel W_{n+1} of order n ≥ 4, Γ_{pe} (L(W_{n+1})) = $\left\lceil \frac{n+1}{2} \right\rceil$.

Theorem: 4.17

Let G be a disconnected graph with components $G_1, G_2,...,G_m$. Then $\Gamma_{pe}(G) = \min_{1\le i\le m}\{ \Gamma_{pe}(G_i) + \sum_{j=1, j\neq i}^{m} \Gamma(G_j)\}$.

Proof.

We prove this result by using mathematical induction.

The result is trivially true for m=1.

Suppose m=2.

Then $G = G_1 \cup G_2$.

Let S_1, S_2 be the Γ_{pe} -sets of G_1 and G_2 respectively.

Then $S_1 \cup S_2{}'$ and $S_2 \cup S_2{}'$ are pendant dominating sets in G , where $S_i{}'$ denotes the Γ -set of G_i, i = 1,2.

Therefore $\Gamma_{pe}(G) \le \min\{\Gamma_{pe}(G_1) + \Gamma(G_2),\ \Gamma_{pe}(G_2) + \Gamma(G_1)\}$.

On the other hand, let S be any pendant dominating set in G.

Then S has to dominate both $V(G_1)$ and $V(G_2)$ and $\langle S \rangle$ should contain at least one pendant vertex.

Moreover, the set S should the pendant dominating set of G_1 or G_2.

Otherwise $\langle S \rangle$ contains no pendant vertex which is a contradiction.

This contradiction shows that $| S | \geq \min\{\Gamma_{pe}(G_1) + \Gamma(G_2),\ \Gamma_{pe}(G_2) + \Gamma(G_1)\}$.

Hence, $| S | = \min\{\Gamma_{pe}(G_1) + \Gamma(G_2), + \Gamma_{pe}(G_2) + \Gamma(G_1)\}$, providing the result for m=2.

Next, suppose m $\geq$ 3 and assume that result is true for m = k-1.

Let G be any graph with the components $G_1, G_2, \ldots, G_{k-1}, G_k$.

Let G' be a graph with k-1 components, say $G_1, G_2, \ldots, G_{k-1}$.

Then from the induction hypothesis we have

$\Gamma_{pe}(G') = \min_{1 \leq i \leq k-1}\{\Gamma_{pe}(G_i) + \sum_{j=1, j\neq 1}^{k-1} \Gamma(G_j)\}$.

Now, we have $G = G' \cup G_m$.

That is, G is the graph having only two components namely G and G_m .

Hence from the case m = 2, we obtain that

$\Gamma_{pe}(G) = \min_{1 \leq i \leq k}\{\Gamma_{pe}(G_i) + \sum_{j=1, j\neq 1}^{k-1} \Gamma(G_j)\}$

Therefore the result is true for m = k and hence true for any positive integer m.

Thus we have $\Gamma_{pe}(G) = \min_{1 \leq i \leq r}\{\Gamma_{pe}(G_i) + \sum_{j=1, j\neq 1}^{m} \Gamma(G_j)\}$

Let G_1 and G_2 any two graphs. Then the cartesian product of G_1 and G_2 is Denoted by $G_1 \blacksquare G_2$ and defined to be the graphs G where the vertices where u = (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 = v_1$ and u_2 adjacent to v_2 or $u_2 = v_2$ and u_1 adjacent to v_1 .

The graph $P_m \blacksquare P_n$ is called a grid graph and $C_n \blacksquare P_2$ is called a prism graph.

Definition: 4.18

A two-dimensional grid graph, also known as a rectangular grid graph or two-dimentional lattice graph that is the graph Cartesian product of path graphs on vertices.

Example: 4.19

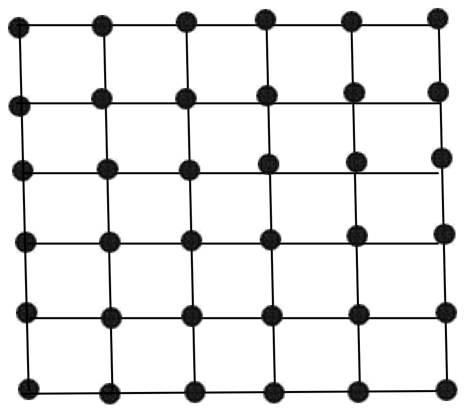

Fig. 10.

Theorem: 4.20

Let $G \cong P_m \blacksquare P_n$ be a grid graph. Then $\Gamma_{pe}(G) = n\left\lceil\frac{m}{2}\right\rceil$.

Proof.

Let $G \cong P_m \blacksquare P_n$ be a grid graph and let

$V(G) = \{u_{ij}/1 \leq i \leq m,\ 1 \leq j \leq n\}$.

Choose the minimum dominating set S' in one copy of P_m and let S be the set of all vertices in the row to which the vertex of S' belongs to.

Then, S is a minimal pendantdominating set in G and further, for no vertex in $V - S$, the set $S \cup \{v\}$ will be a minimal pendant dominating set.

Therefore, $\Gamma_{pe}(G) = |S| = n\left\lceil\frac{m}{2}\right\rceil$.

Definition: 4.21

The **Ladder graph** $L_n = P_n \times K_2$ where P_n is a path with n vertices and x denotes the Cartesian product and K_2 is a complete graph with two vertices. Ladder graph L_7 is shown in the Fig. 10.

Example: 4.22

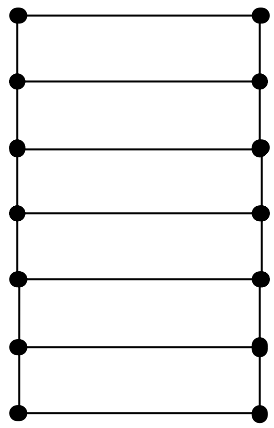

Fig. 11. L_7

Corollary: 4.23

Let G be Ladder Graph. Then $\Gamma_{pe}(G)$ = n.

Proof.

Let $G \cong \mathrm{P}_2 \blacksquare P_n$ be a ladder graph and let $V(G) = \{(u_i, v_i) / 1 \leq \mathrm{i} \leq \mathrm{n}\}$. Fix an edge e = $u_1 v_1$ of G and let H be the graph obtained by removing the vertices u_1 , $v_1$and its neighbours from G.

Then, $H \cong \mathrm{P}_2 \blacksquare P_{n-2}$ and so Γ_{pe} $(\mathrm{P}_2 \blacksquare P_n)$ = 2 + Γ_{pe} $(\mathrm{P}_2 \blacksquare P_{n-2})$ = (n-2) + 2 = n.

Theorem: 4.24

Let $G \cong P_n \blacksquare K_m$. Then $\Gamma_{pe}(G)$ = n+1.

Proof.

Let $G \cong P_n \blacksquare K_m$ be a graph of order of 2n where K_n be a complete graph of Order n and $\gamma(K_n) = 1$.

Choosing two vertices from one copy of K_n and exactly one vertex from other copies of K_n, we obtain the minimal pendant dominating set of G.

In fact, this set would be a minimal pendant dominating set of maximal cardinality.

Therefore, $\Gamma_{pe}(G)$ = n+1 .

Definition: 4.25

The **Cartesian product** $G \times H$ of graphs G and H is a graph such that

- The vertex set $G \times H$ is the Cartesian product $V(G) \times V(H)$, and
- Any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either

➢ $u = v$ and u is adjacent to v in H, or

➢ $u = v$ and u is adjacent to v in G.

Definition: 4.26

Book graph is a Cartesian product of a star and single edge, denoted by B_m. The m-book graph is defined as the graph Cartesian product $S_{m+1} \times P_2$, where S_{m+1} is a star graph and P_2 is the path graph.

Definition: 4.27

The **Stacked book graph** of order (m, n) is defined as the Cartesian product of $S_{m+1} \times P_n$ where S_{m+1} is the star graph and P_n is the path graph on n nodes. It is therefore the graph corresponding to the edges of n copies of an m-page "book" stacked one on top of another and is the generalization of a book graph. Stacked book graph $B_{3,4}$ is shown in the Fig. 12

Example: 4.22

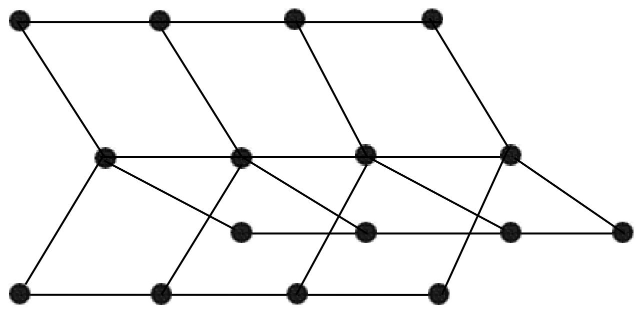

Fig. 12. $\boldsymbol{B_{3,4}}$

Theorem: 4.23

Let G be a stacked book graph. Then,

$$\Gamma_{pe}(G) = \begin{cases} 2, & if\ n = 1 \\ \frac{m(n+1)}{2}, & \text{if n} \geq 3 \text{ and odd} \\ \frac{mn}{2}, & if n\ is\ even \end{cases}$$

Proof.

Let G be a stacked book graph.

Then G is the product graph of $K_{1,m}$ with P_n, hence G contains m copies of the path P_n attached to one copy of P_n obtained by joining the centers of the star $K_{1,m}$ and call it as the graph H.

Suppose n=1, then G is a star and so Γ_{pe}(G) = 2.

Assume n ≥ 2 .

Let G' be the graph obtained by deleting vertices of H from G.

Then G' is the union of m copies of the path P_n .

Moreover Γ_{pe} (G) = $\Gamma_{pe}(G')$ from G.

It is clear that the upper pendant dominating set is obtained by choosing upper pendant dominating set in one copy of P_n and upper dominating set from other copies of path .

Therefore, $\Gamma_{pe}(G)$ = (m – 1) $\left\lceil\frac{n}{2}\right\rceil$ + 2 + $\left\lceil\frac{n-3}{2}\right\rceil$.

Suppose n even, then n =2k, for some integer K.

Substituting for n, we get $\Gamma_{pe}(G) = \frac{mn}{2}$.

Similarly, whenever n odd, we obtain that $\frac{m(n+1)}{2}$.

Corollary: 4.24

Let G be a book of order 2m. Then $\Gamma_{pe}(G)$ = m.

Proof.

Let G be a book graph.

Then G is the graph Cartesian product of the star $K_{1,m}$ with P_2.

Hence, taking n = 2 in the above theorem, we obtain that Γ_{pe} (G) = m.

Definition: 4.25

A **Prism graph** of n-layers, Y_m^n is a simple graph given by the Cartesian product of the cycle C_m and P_n. The graph consists of mn vertices and $m(2n-1)$ edges.

Example: 4.25

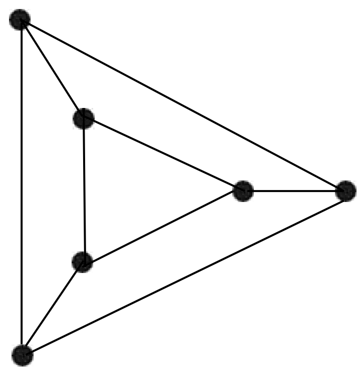

Fig. 13.

Theorem: 4.26

Let G be a prism graph of order 2n. Then

$$\Gamma_{pe}(G) = \begin{cases} n-1, if\ n\ is\ even \\ n, \quad\quad if\ n\ is\ odd \end{cases}$$

Proof.

Let G be a prism graph of order 2n, then $G \cong C_n$ ■ P_2.

Let (u,v) be any pair of adjacent vertices in G.

Assume that u and v are from outside cycle.

Suppose n odd, select $\frac{n-3}{2}$ non-adjacent vertices from outside cycle not in the neighborhood of u and v.

Then, exactly $\frac{n-1}{2}$ in the cycle inside not dominated by any of the vertices in G.

Therefore, the collection of these vertices will be a minimal dominating set in G.

Since, we are selecting alternative vertices, the collection will be a minimal dominating set of maximum cardinality.

Hence, $\Gamma_{pe}(G)$ = n.

Next, suppose n even.

As in the above case, let S and S' be the upper dominating sets of the inner and outer cycles on removing the vertices u,v and its neighbor from G.

Then, $\Gamma_{pe}(G) = 2 + |S| + |S'|$.

That is, $\Gamma_{pe}(G) = 2 + \frac{n-4}{2} + \frac{n-2}{2} = n - 1$.

Since the prism graph $G \cong C_n$ ■ P_2 consists of two cycles, upper pendant domination set may be choose by taking upper pendant dominating set in one copy of C_n and upper dominating set from another copy.

Therefore, $\Gamma_{pe}(G) = 2\,\Gamma_{pe}(c_n) - 1$.

Generally, the graph $C_n \blacksquare P_m$ consists of m – cycles each of order n.

CHAPTER - V

THE COMPLEMENTARY PENDANT DOMINATION NUMBER IN GRAPHS

Introduction:

In this chapter, we see the application and significance of The Complementary Pendant domination number in graphs such as Bistar graph, Multi star graph, Barbell graph, Ladder graph, Triangle free graph.

Definition: 5.1

A dominating set S in G is called a **complementary pendant dominating set** if $\langle V - S \rangle$ contains atleast one pendant vertex. The minimum cardinality of a complementary pendant dominating set is called he complementary pendant domination number of G, denoted by $\gamma_{cpe}(G)$.

Example: 5.2

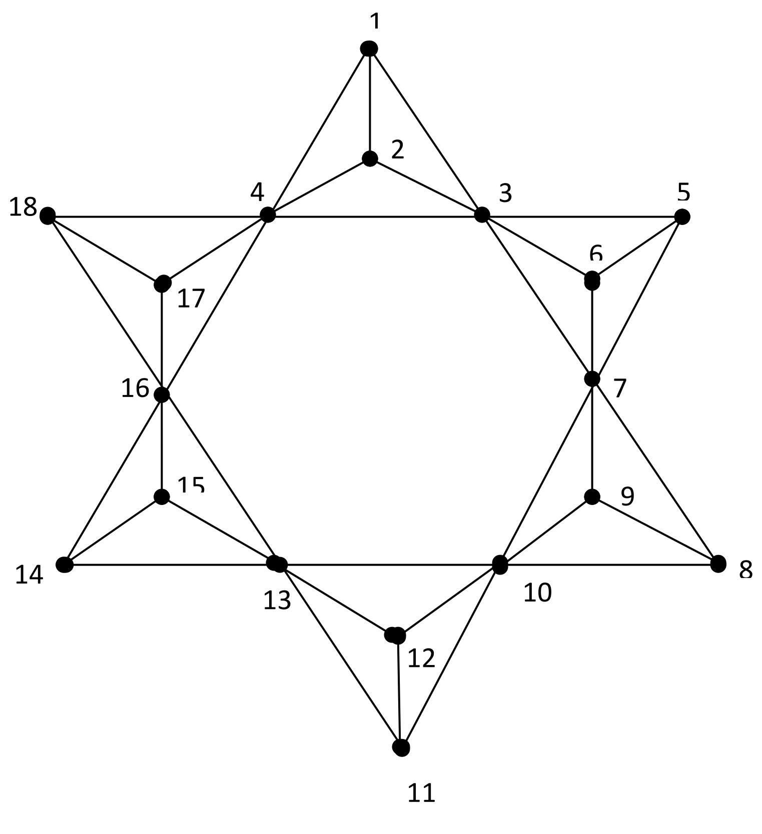

Fig.14.

For the graph G in Fig. 14, one can verify that $S = \{3,4,9,12,15\}$ is a dominating set with $\{1,2\}$ as a pendant vertex in $\langle V - S \rangle$. Hence S is a $\gamma_{cpe}(G)$-set. Further $\{3,4,10,15\}$ is the minimum $\gamma_{cpe}(G)$-set and so $\gamma_{cpe}(G) = 4$.

Observation: 5.3

Let G be totally disconnected or G a star. Then complementary pendant domination number is not defined for G.

Hence throughout this chapter, a graph G we assume $m \geq 1$.

- $\gamma(P_n) = \gamma_{pe}(P_n) = \Gamma(P_n) = \gamma_{pe}(P_n)$ if and only in $n = 4$
- $\gamma(K_{m,n}) = \gamma_{pe}(K_{m,n}) = \Gamma(K_{m,n}) = \Gamma_{pe}(K_{m,n}) = \gamma_{cpe}(K_{n,m})$ if and if $m, n = 2$.
- Let G be bistar, then $\Gamma(G) = \gamma_{cpe}(G)$.

Lemma: 5.4

The following are true

(i) $\gamma_{cpe}(K_n) = n - 2, n \geq 3$.

(ii) $\gamma_{cpe}(K_{n,m}) = m + n - 3, m, n \geq 2$.

(iii) $\gamma_{cpe}(P_n) = \gamma_{cpe}(C_n) = \left\lceil \frac{n}{3} \right\rceil, n \geq 4$.

(iv) $\gamma_{cpe}(W_n) = 2,\ n \geq 4$.

(v) $\gamma_{cpe}(\overline{C}_n) = \gamma_{cpe}(\overline{P}_n) = n - 3, n \geq 3$.

(vi) For any graph $G, \gamma(G) \leq \gamma_{cpe}(G)$.

Proof:

(i) Every induced sub graph of k_n is complete.

For any two adjacent vertices $\{u, v\}$ in k_n, the set $S = \{V(k_n) - \{u, v\}\}$ will be a complementary pendant dominating set of k_n.

Hence $\gamma_{cpe}(K_{m,n}) = |S| = n - 2$.

(ii) Let $\{V_1, V_2\}$ are two parties set in $K_{m,n}$.

Choose an arbitrary path $P_3 = \{v_1, v_2, v_3\}$ in $K_{m,n}$.

Then, the set $S = V - \{v_1, v_2, v_3\}$ will be a complementary pendant dominating set of $k_{m,n}$.

Hence, $\gamma_{cpe}(K_{m,n}) \leq |S| = m + n - 2$.

On the other hand, it may be noted that any subset S' of size at least $m + n - 4$, the set $V - S$ has minimum degree at least 2.

Thus, we must have $\gamma_{cpe}(K_{m,n}) \geq m + n - 3$, proving (ii).

(iii) Let G be a Cycle or a path with $n \geq 4$ vertices.

Then $S = \{v_1, v_2, v_3\}$ will be a γ −set of G and $< V - S >$ contains a pendant vertex and so S itself a complementary pendant dominating set of G.

Therefore $\gamma_{cpe}(G) = |S| = \left\lceil \frac{n}{3} \right\rceil$.

(iv) Let W_n be a wheel with $n \geq 2$ vertices and let u be a vertex at the centre of W_n.

Clearly $\{u\}$ will be a dominating set but $V - \{u\}$ is a cycle $C_{n-1.}$

Hence $\gamma_{cpe}(W_{,n}) \geq 2$

Next, choosing an arbitrary vertex v on cycle C_{n-1}, the set $S = \{u, v\}$ will be a minimum complementary pendant dominating set.

Therefore $\gamma_{cpe}(W_n) = 2$.

(v) Clearly $\delta(\overline{C_n}) = n - 3$ and so there exist a vertex v_1 in $\overline{C_n}$ which is not adjacent to two vertices v_1 and v_n.

Now the set $S = V - \{v_1, v_2, v_n\}$ is a complementary pendant dominating set.

So $\gamma(\overline{C_n}) \leq |V - \{v_1, v_2, v_n\}| = n - 3$.

Therefore $\gamma_{cpe} = n - 3$.

(vi) Since every complementary pendant dominating set is also a dominating set of a graph G, it follows that $(G) \leq \gamma_{cpe}(G)$.

Theorem: 5.5

Let G be a n-pan Graph, of order $n \geq 4$. Then $\gamma_{cpe}(G) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem 5.6

Let $G \cong K_m(a_1, a_2, \dots, a_m)$ be a multi star graph, with $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$. Then $\gamma_{cpe}(G) = |a_1| + |a_2| + m - 2$.

Proof:

Let $G \cong K_m(a_1, a_2, \dots, a_m)$ be a multi star of order $a_1 + a_2 + \cdots + a_m + m$.

Assume that $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_m$.

Let u and v be a two adjacent supported vertices of G.

The set S contains leaves of u,v and all the supported vertices of a multi star graph G except u,v.

Therefore $\gamma_{cpe}(G) = |S| = |a_1| + |a_2| + m - 2$.

Theorem: 5.7

Let G be a bistar graph then $\gamma_{cpe}(G) = m + n$.

Proof:

Let G be a bistar graph.

The set $S = \{u, v\}$ is the dominating set of G.

Then $< V - S >$ contains a pendant vertex, therefore S is a complementary pendant dominating set of G.

So $\gamma_{cpe}(G) = |S| = \{m + n + 2\} - 2 = m + n$.

Definition: 5.8

A **Barbell graph** $B(p, n)$ is the graph obtained by connecting n-copies of a complete graph K_p by a bridge. Barbell graph B_{16} is shown in the Fig. 15.

Example: 5.9

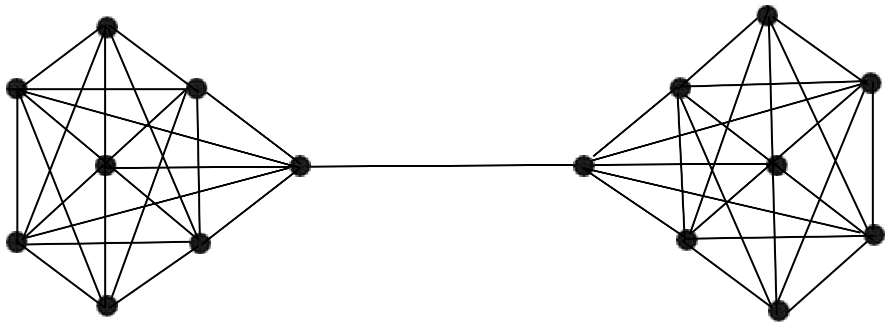

Fig. 15. B_{16}

Theorem: 5.10

Let G be a Barbell graph. Then $\gamma_{cpe}(G) = 2(n-1)$.

Proof:

Let G be a barbell graph and let $V(G) = \{v_1, v_2, v_n\}$.

Let v_1 and v_2 be the adjacent vertices of G attached to the copies of complete graph.

Then, clearly the set $S = \{v_1, v_2\}$ is a dominating set of G.

Now the set $S' = V - S$ is a complementary pendant dominating set of G.

Therefore $\gamma_{cpe}(G) = 2n - 2 = 2(n-1)$.

Hence Proved.

Theorem: 5.11

Let G be a Ladder graph. Then $\gamma_{cpe}(G) = \left\lceil \frac{n}{2} \right\rceil + 2$.

Proof:

Let G be a ladder graph, fix an edge u_2v_2 of G.

Then for any γ −set S of $P_2 \times P_{n-3}$, the set $S = S' \cup \{u_2, v_2\}$ be the minimum complementary pendant dominating set of G.

Hence $\gamma_{cpe}(P_2 \times P_{n-3}) = \gamma(P_2 \times P_{n-3}) + 2 = \left\lceil\frac{n}{2}\right\rceil + 2$.

Theorem: 5.12

If T is a tree of order $n \geq 3$, then $\Delta(T) \leq \gamma_{cpe}(T)$.

Furthermore $\gamma_{cpe}(T) = \Delta(T)$ if and only if T is a wounded spider graph which is not a star.

Theorem: 5.13

For $r \geq 2$, if G is a r −regular graph. Then $\gamma_{cpe}(G) \leq \gamma(G) + r - 2$.

Theorem: 5.14

Let G be a graph with n vertices.

Then $\gamma(G) + \gamma_{cpe}(G) \leq n$.

Proof:

Let S be a complementary pendant dominating set.

Then S is a dominating set and $< V - S >$ contains a pendant vertex.

Obviously, $\gamma_{cpe}(G) \leq |S|$.

Since S is a dominating $< V - S >$ is also a dominating.

Thus $\gamma(G) \leq |V - S|$.

Hence $\gamma(G) + \gamma_{cpe}(G) \leq |S| + |V - S| = n$.

Hence Proved.

Theorem: 5.15

Let G be any graph. Then $\left\lceil\frac{n}{1+\Delta(G)}\right\rceil \leq \gamma_{cpe}(G) \leq n - \Delta(G)$.

Theorem: 5.16

Let G be any graph with n vertices. Then,

$$\gamma_{cpe}(GVP_m) = \begin{cases} min\{m,n\}, if\ G\ contains\ a\ pendant\ vertex. \\ n, \qquad otherwise \end{cases}$$

Proof:

Let G be any graph of order n and let P_m be a path of order m.

Let $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_m\}$ are vertices of G and P_m respectively.

If G contains a pendant vertex then clearly $\gamma_{cpe}(GVP_m) = min\{m,n\}$.

Then $< V(GVP_m) - \{v_1, v_2, \dots, v_n\} >$ contains a pendant vertex.

Therefore $\gamma_{cpe}(GVP_m) = |G| = n$.

Theorem: 5.17

If G is a graph, then $\gamma_{cpe}(G) = 1$ if $G \cong TVK_1$, where T is a tree.

Proof:

Assume $G \cong TVK_1$, then the set $\{v\}$ is a complementary pendant dominating set of G.

Where $V(K_1) = \{v\}$.

Conversely if $\gamma_{cpe}(G) = 1$, then there exist a complementary pendant dominating set S of G with $|S| = 1$.

Such that $< V(G) - S >$ is tree, since each vertex in $< V(G) - S >$ is adjacent to the vertex S.

$G \cong TVK_1$. where $T =< V(G) - S >$

Theorem: 5.18

Let G be a connected graph with n vertices. Then $\gamma_{cpe}(G \circ K_1) = n$.

Proof:

Let us choose u and v be any two leaves of adjacent supported vertices of the graph $(G \circ H)$.

The set $S = \{V - N(u,v)\} \cup \{u,v\}$ will be a complementary pendant dominating set of $(G \circ H)$.

Then $|S| \leq |V - N(u,v)| \cup \{u,v\} = n - 2 + 2 = n.$

Theorem: 5.19

Let T_1 and T_2 be any two trees of order n_1 and n_1 respectively. Then $\gamma_{cpe}(T_1 \circ T_2) = n_1$.

Proof:

Let $V(T_1)$ denotes the vertex set of T_1 and $V(T_2)$ is a dominating set of $T_1 \circ T_2$.

Then $<V(T_1 \circ T_2) - V(T_1)>$ contains a pendant vertex, therefore $V(T_1)$ is a complementary pendant dominating set of $T_1 \circ T_2$.

$\gamma_{cpe}(T_1 \circ T_2) = |V(T_{1)}| = n_1.$

Hence Proved.

Theorem: 5.20

Let G be any graph, if $diam(G) \geq 3$ and G contains a no isolated vertex. Then $\gamma_{cpe}(G) = 2$.

Proof:

Let u and v be two vertices of G such that $d(u,v) = diam(G) \geq 3$.

Obviously u and v dominates $\bar{G}$ since there is no vertex in G adjacent to both u and v.

Hence $\{u,v\}$ dominates $\bar{G}$ and $\gamma_{cpe}(G) \leq 2$.

If $\gamma_{cpe}(G) = 1$,then G has an isolated vertex, contrary to the hypothesis.

Definition: 5.21

A graph is said to be **triangle free** if no two adjacent vertices are adjacent to a common vertex.

Example: 5.22

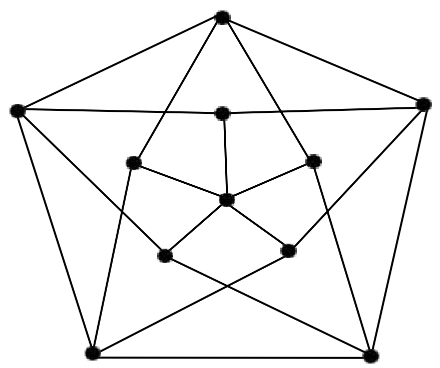

Fig. 15.

Theorem: 5.23

Let G be a triangle free graph of order at least 4. Then $\gamma_{cpe}(\bar{G}) = 2$ or 3.

Proof:

Let G contains an isolated vertex then clearly $\gamma_{cpe}(\bar{G}) = 2$.

Suppose G has no isolated vertex, then G contain at least one edge say $e = uv$.

As G is triangle free no vertex in G can be adjacent to both u and v.

Thus $S = \{u, v\}$ will be a γ-set in G.

Now, for any vertex $w \in V(G)$, the set $S \cup \{w\}$ will be a γ_{cpe}-set in G.

Hence, $\gamma_{cpe}(\bar{G}) = 3$.

Hence Proved.

A STUDY ON SOFT TOPOLOGY

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

A. MARY THANGAM

Reg. No: 19SPMT18

Under the guidance of

Dr. C. REENA M.Sc., B.Ed., M.Phil., SET., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON SOFT TOPOLOGY"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **A. MARY THANGAM (Reg. No: 19SPMT18)**

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Course
St. Mary's College (Autono
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON SOFT TOPOLOGY"** submitted for the degree of Master of Science is my work carried out under the guidance of **Dr. C. Reena M.Sc., B.Ed., M.Phil., SET., Ph.D.,** Assistant Professor , Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

A. Mary Thangam.

Station : Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10-04-2021

CONTENT

CHAPTER 1

PRELIMINARIES

Definition: 1.1

Let X be a non empty set. Then a collection τ of subsets of X is called a **topology** for X if τ satisfies each of the following axioms:

(i) $X \in \tau$

(ii) $\emptyset \in \tau$

(iii) If $\{G_\alpha : \alpha \epsilon A\}$ is any collection of sets in τ, then $\cup \{\{G_\alpha : \alpha \epsilon A\}\} \in \tau$

and

(iv) If G_1 and G_2 are any two members of τ, then $G_1 \cap G_2 \in \tau$.

The set X along with the topology τ is called a **topology space** and is denoted by the ordered pair (X, τ).

The sets in the collection τ are called **τ open set** or simply the open sets of the topological space (X, τ).

Thus in a topological space (X, τ).

(i) X is τ open

(ii) $\emptyset$ is τ open

(iii) The union of any number of τ open sets is a τ open set.

(iv) The intersection of any two τ open set is a τ open set.

Definition: 1.2

Let X be a non empty. Then the collection $\tau = \{\emptyset, X\}$ consisting of the empty set $\emptyset$ and the whole space X is obviously topology, for X and is called the indiscrete topology, while the pair (X, τ) is known as the indiscrete topological space.

Definition: 1.3

Let X be a non empty set and D be the collection of all subsets of X then D is a topology for X, as

(i) $X \subseteq X \Rightarrow X \in D$

(ii) $\emptyset \subseteq X \Rightarrow \emptyset \in D$

(iii) If $\{G_\alpha : \alpha \epsilon A\}$ is any collection of member of D, then $G_\alpha \subseteq X$ for all $\alpha \epsilon A$ and therefore $\cup \{\{G_\alpha : \alpha \epsilon A\}\} \subseteq X$ following that $\cup \{\{G_\alpha : \alpha \epsilon A\}\} \in D$,

(iv) If G_1 and G_2 are any two members of D,

$G_1 \subseteq X$ and $G_2 \subseteq X$ and so $G_1 \cap G_2 \subseteq X$

Consequently, $G_1 \cap G_2 \in D$.

The topology D so defined is called a **discrete topology** and pair (X, D) is called a discrete topological space.

The topologies listed in definition 1.2 and definition 1.3 are known as **Trivial topologies.**

Definition: 1.4

Let X be a topological pace, $A \subseteq X$ an arbitrary subset. The relative or **subspace topology on A** is the collection of intersections with open sets in X.

In other words, a subset $U \subseteq A$ is open in the subspace topology if and only if there exits an open subset $V \subset X$ such that $U = V \cap A$.

Definition: 1.5

Let (X, τ) be a soft topological space. Then a subset F of X is said to be **τ closed**. If and only if its complement F^c is τ is open.

Thus F is τ closed $\Leftrightarrow F^c$ is τ open $\Leftrightarrow F^c \in \tau$.

Definition: 1.6

Let be (X, τ) a soft topological space and A be any subset of X. Then a point $x \in X$ is called an **interior point of A** if there exists a τ open set G such that

$x \in G \subset A$.

In other words, the point $x \in A$ is called its interior point A is the τ neighbourhood of the point x.

The set of all interior points of A is called the interior of the set A and is denoted by Int. (A) or by A°.

Definition: 1.7

Let (X, τ) be a soft topological space and A be any subset of X. Then the

τ closure of A denoted by $\bar{A}$ is the intersection of all τ closed subsets of X containing

the set A.

Definition: 1.8

The point $x \in X$ is called a **limit point** or cluster point or a point of accumulation of the set A if each open set in (X, τ) containing the point x contains at least one point

of A other than the point x.

Definition: 1.9

A topological space (X, τ) is said to be a **Hausdorff space** if for every pair of distinct points x_1 and x_2 there exist disjoint neighbourhoods N_1 and N_2 of x_1 and x_2 respectively.

Definition: 1.10

If (X, τ) and (Y, γ) are two topological spaces, then a one- one bicontinuous mapping is called **homeomorphism.**

Definition: 1.11

Let X and Y be topology spaces. A function $f : X \rightarrow Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Remark: 1.12

1. $f^{-1}(V)$ is the set of all points of x of X for which $f(x) \in V$.
2. $f^{-1}(V)$ is empty if V does not intersect the image set $f(x)$ of f.

CHAPTER 2

SOFT SET

Definition: 2.1

A **soft set** F_A on the universe U is defined by the set of ordered pairs $F_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}$, where $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here, f_A is called an approximate function of the soft set F_A. The value of $f_A(x)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection.

Note that the **set of all soft sets** over U will be denoted by S(U).

Example: 2.2

Suppose that there are six houses in the universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ under consideration, and that $E = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of decision parameters.

The x_i (i = 1 , 2 , 3 , 4 , 5) stand for the parameters " expensive " ," beautiful ",

" wooden " , "cheap" and " in green surroundings" , respectively.

Consider the mapping f_A given by " houses (.) " , where(.) is to be filled in by one of the parameters $x_i \in E$. For instance, $f_A(e_1)$ means " houses (expensive)" and its functional value is the set { h ∈ U : h is an expensive house} . Suppose that

$A = \{x_1 , x_3 , x_4\} \leq E$ and $f_A(x_1) = \{h_2 ,h_4\}$, $f_A(x_3) = U$ and

$f_A(x_4) = \{h_1 ,h_3 ,h_5\}$.

Then , we can view the soft set F_A as consisting of the following collection of approximations :

$F_A = \{ (x_1 , \{h_2 ,h_4\}) , (x_3 , U) , (x_4 , \{h_1 ,h_3 ,h_5\}) \}$.

Definition: 2.3

For two soft sets F_A and F_B over a common universe U ,we say that F_A is a soft subset of F_B if

i) $A \subseteq B$ and

ii) For all $x \in E$ $f_A(x)$ and $f_B(x)$ are identical approximations.

We write $F_A \tilde{\subset} F_B$.

F_A is said to be **soft super set** of F_B. If F_B is a soft subset of F_A we denote it by $F_A \tilde{\supset} F_B$

Definition: 2.4

Let $F_A \in S(U)$. If $f_A(x) = \emptyset$ for all $x \in E$, then F_A is called an

empty set, denoted by $F_{\emptyset}$. $f_A(x) = \emptyset$ means that there is no element in U related to the parameter $x \in E$. Therefore, we do not display such elements in the soft sets, as it is meaningless to consider such parameters.

Definition: 2.5

A soft set F_A over U is said to be absolute soft set denoted by $\tilde{A}$ if for all $\varepsilon \in A$, $F(\varepsilon) = U$. Clearly $\tilde{A}^c = \emptyset$ and $\emptyset^c = \tilde{A}$.

Definition: 2.6

Let $F_A \in S(U)$. If $f_A(x) = U$ for all $x \in A$, then F_A is called an **A – universal soft set**, denoted by $F_{\tilde{A}}$. If $A = E$, then the A – universal soft set is called a **universal soft set**, denoted by $F_{\tilde{E}}$.

Definition: 2.7

Let $F_A, F_B \in S(U)$. Then, F_A is a **soft subset** of F_B, denoted by $F_A \tilde{\leq} F_B$, if $f_A(x) \leq f_B(x)$ for all $x \in E$.

Definition: 2.8

The intersection F_C of two soft sets F_A and F_B over a common universe U, denoted $F_A \cap F_B$ is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all $e \in C$

Definition: 2.9

Let $F_A, F_B \in S(U)$. Then, F_A and F_B are soft equal, denoted by $F_A = F_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$.

Definition: 2.10

Let $F_A, F_B \in S(U)$. Then, the soft union, denoted by $F_A \tilde{\cup} F_B$, the soft intersection denoted by $F_A \tilde{\cap} F_B$, and the soft difference denoted by $F_A \tilde{\setminus} F_B$ of F_A and F_B are defined by the approximation functions

$f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_B(x)$, $f_{A \tilde{\cap} B}(x) = f_A(x) \cap f_B(x)$,

$f_{A \tilde{\setminus} B}(x) = f_A(x) \setminus f_B(x)$,respectively, and the soft complement denoted by $F_{\tilde{A}^c}$ of F_A is defined by the approximate function $f_{\tilde{A}^c}(x)$ = $f_{A^c}(x)$,

where $f_{A^c}(x)$ is the complement of the set $f_A(x)$; that is,

$f_{A^c}(x) = U \setminus f_A(x)$ for all $x \in E$.

It is easy to see that $(F_A{}^{\tilde{c}})^{\tilde{c}} = F_A$ and $F_{\tilde{\emptyset}^c} = F_{\tilde{E}}$.

Proposition: 2.11

Let $F_A \in S(U)$. Then ,

(i) $F_A \tilde{\cup} F_A = F_A$, $F_A \tilde{\cap} F_A = F_A$

(ii) $F_A \tilde{\cup} F_\emptyset = F_A$, $F_A \tilde{\cap} F_\emptyset = F_\emptyset$

(iii) $F_A \tilde{\cup} F_E = F_E$, $F_A \tilde{\cap} F_E = F_A$

(iv) $F_A \tilde{\cup} F_{\tilde{A}^c} = F_E$, $F_A \tilde{\cap} F_{\tilde{A}^c} = F_\emptyset$.

Proof

Let $F_A \in S(U)$.

(i) To prove: $F_A \tilde{\cup} F_A = F_A$

$$f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_A(x)$$
$$= f_A(x)$$

$\to F_A \tilde{\cup} F_A = F_A$.

To prove : $F_A \tilde{\cap} F_A = F_A$

$f_{\tilde{A \cap B}}(x) = f_A(x) \cap f_A(x)$

$= f_A(x)$

$\to F_A \tilde{\cap} F_B = F_A$.

(ii) To prove : $F_A \tilde{\cup} F_\emptyset = F_A$

$f_{\tilde{A \cup \emptyset}}(x) = f_A(x) \cup f_\emptyset(x)$

$= f_A(x)$ (since $f_\emptyset(x) = \emptyset$)

$\to F_A \tilde{\cup} F_\emptyset = F_A$

To prove : $F_A \tilde{\cap} F_\emptyset = F_\emptyset$

$f_{\tilde{A \cap \emptyset}}(x) = f_A(x) \cap f_\emptyset(x)$

$= f_\emptyset(x)$ (since $f_\emptyset(x) = \emptyset$)

$\to F_A \tilde{\cap} F_\emptyset = F_\emptyset$

(iii) To prove : $F_A \tilde{\cup} F_{\tilde{E}} = F_{\tilde{E}}$

$f_{\tilde{A \cup E}}(x) = f_A(x) \cup f_{\tilde{E}}(x)$

$= f_{\tilde{E}}(x)$ (since $f_{\tilde{E}}(x) = U$)

$\to F_A \tilde{\cup} F_{\tilde{E}} = F_{\tilde{E}}$

To prove : $F_A \tilde{\cap} F_{\tilde{E}} = F_A$

$f_{\tilde{A \cap B}}(x) = f_A(x) \cap f_E(x)$

$= f_A(x)$ (since $f_{\tilde{E}}(x) = U$)

$\to F_A \tilde{\cap} F_{\tilde{E}} = F_A$.

(iv) To prove : $F_A \tilde{\cup} F_{\tilde{A}^c} = F_{\tilde{E}}$.

$f_{\tilde{A} \cup \bar{A}^c}(x) = f_A(x) \cup f_{\bar{A}^c}(x)$

$= f_A(x) \cup f_{A^c}(x)$

$= f_A(x) \cup [U \setminus f_A(x)]$

$f_{\tilde{A} \cup \bar{A}^c}(x) = U$

$= f_{\tilde{E}}(x)$

$\rightarrow F_A \tilde{\cup} F_{A^c} = F_{\tilde{E}}$.

To prove : $F_A \tilde{\cap} F_{A^c} = F_{\varnothing}$.

$f_{\tilde{A} \cap \bar{A}^c}(x) = f_A(x) \cap f_{A^c}(x)$

$= f_A(x) \cap f_{A^c}(x)$

$= f_A(x) \cap [U \setminus f_A(x)]$

$= \varnothing$

$= f_{\varnothing}(x)$

$\rightarrow F_A \tilde{\cap} F_{\bar{A}^c} = F_{\varnothing}$.

Proposition: 2.12

Let $F_A, F_B, F_C \in S(U)$ .Then ,

(i) $F_A \tilde{\cup} F_B = F_B \tilde{\cup} F_A$, $F_A \tilde{\cap} F_B = F_B \tilde{\cap} F_A$

(ii) $(F_A \tilde{\cup} F_B)^{\tilde{c}} = F_{\bar{B}^c} \tilde{\cap} F_{\bar{A}^c}$, $(F_A \tilde{\cap} F_B)^{\tilde{c}} = F_{\bar{B}^c} \tilde{\cup} F_{A^c}$

(iii) $(F_A \tilde{\cup} F_B) \tilde{\cup} F_C = F_A \tilde{\cup} (F_B \tilde{\cup} F_C)$, $(F_A \tilde{\cap} F_B) \tilde{\cap} F_C = F_A \tilde{\cap} (F_B \tilde{\cap} F_C)$

(iv) $F_A \tilde{\cup} (F_B \tilde{\cap} F_C) = (F_A \tilde{\cup} F_B) \tilde{\cap} (F_A \tilde{\cup} F_C)$ $F_A \tilde{\ } (F_B F_C) = (F_A \tilde{\cap} F_B) \tilde{\cup} (F_A \tilde{\cap} F_C)$.

Proof

Let $F_A, F_B, F_C \in S(U)$.

(i) To prove : $F_A \tilde{\cup} F_B = F_B \tilde{\cup} F_A$.

$f_{\tilde{A \cup B}}(x) = f_A(x) \cup f_B(x)$

$= f_B(x) \cup f_A(x)$

$= f_{\tilde{B \cup A}}(x)$

$\rightarrow F_A \tilde{\cup} F_B = F_B \tilde{\cup} F_A$.

To prove : $F_A \tilde{\cap} F_B = F_B \tilde{\cap} F_A$.

$f_{\tilde{A \cap B}}(x) = f_A(x) \cap f_B(x)$

$= f_B(x) \cap f_A(x)$

$= f_{\tilde{B \cap A}}(x)$

$\rightarrow F_A \tilde{\cap} F_B = F_B \tilde{\cap} F_A$.

(ii) To prove : $(F_A \tilde{\cup} F_B)^{\tilde{c}} = F_{\tilde{B}^c} \tilde{\cap} F_{\tilde{A}^c}$.

$f_{\tilde{A \cup B}^{\tilde{c}}}(x) = (f_{\tilde{A \cup B}})^c(x)$

$= U \setminus f_{\tilde{A \cup B}}(x)$

$= U \setminus [f_A(x) \cup f_B(x)]$

$= [f_A(x) \cup f_B(x)]^c$

$= f_{B^c}(x) \cap f_{A^c}(x)$

$= f_{\tilde{B}^c}(x) \cap f_{\tilde{A}^c}(x)$

$\rightarrow (F_A \tilde{\cup} F_B)^{\tilde{c}} = F_{\tilde{B}^c} \tilde{\cap} F_{\tilde{A}^c}$.

To prove : $(F_A \tilde{\cap} F_B)^{\tilde{c}} = F_{\tilde{B}^c} \tilde{\cup} F_{\tilde{A}^c}$.

$f_{\tilde{A \cap B}^{\tilde{c}}}(x) = (f_{\tilde{A \cap B}})^c(x)$

$= U \setminus f_{\tilde{A \cap B}}(x)$

$= U \setminus [f_A(x) \cap f_B(x)]$

$= [f_A(x) \cap f_B(x)]^c$

$= f_{B^c}(x) \cup f_{A^c}(x)$

$= f_{\tilde{B}^c}(x) \cup f_{\tilde{A}^c}(x)$

$\rightarrow (F_A \tilde{\cap} F_B)^{\tilde{c}} = F_{\tilde{A}^c} \tilde{\cup} F_{\tilde{A}^c}.$

To prove : $(F_A \tilde{\cup} F_B) \tilde{\cup} F_C = F_A \tilde{\cup} (F_B \tilde{\cup} F_C)$.

$f_{(\tilde{A \cup B}) \cup C}(x) = f_{\tilde{A \cup B}}(x) \cup f_C(x)$

$= (f_A(x) \cup f_B(x)) \cup f_C(x)$

$= f_A(x) \cup (f_B(x) \cup f_C(x))$

$= f_A(x) \cup f_{\tilde{B \cup C}}(x)$

$= f_{(\tilde{A \cup B}) \cup C}(x)$

$\rightarrow (F_A \tilde{\cup} F_B) \tilde{\cup} F_C = F_A \tilde{\cup} (F_B \tilde{\cup} F_C)$.

To prove : $(F_A \tilde{\cap} F_B) \tilde{\cap} F_C = F_A \tilde{\cap} (F_B \tilde{\cap} F_C)$.

$f_{(\tilde{A \cap B}) \cap C}(x) = f_{\tilde{A \cap B}}(x) \cap f_C(x)$

$= (f_A(x) \cap f_B(x)) \cap f_C(x)$

$= f_A(x) \cap (f_B(x) \cap f_C(x))$

$= f_A(x) \cap f_{\tilde{B \cap C}}(x)$

$= f_{(\tilde{A \cap B}) \cap C}(x)$

$\rightarrow (F_A \tilde{\cap} F_B) \tilde{\cap} F_C = F_A \tilde{\cap} (F_B \tilde{\cap} F_C)$.

(iv) To prove : $F_A \tilde{\cup} (F_B \tilde{\cap} F_C) = (F_A \tilde{\cup} F_B) \tilde{\cap} (F_A \tilde{\cup} F_C)$.

$f_{\tilde{A} \cup (\tilde{B} \cup C)}(x) = f_A(x) \cup f_{\tilde{B} \cap C}(x)$

$= f_A(x) \cup (f_B(x) \cap f_C(x))$

$f_{\tilde{A} \cup (\tilde{B} \cup C)}(x) = (f_A(x) \cup f_B(x))) \cap (f_A(x) \cup f_C(x))$

$= f_{\tilde{A} \cup B}(x) \cap f_{\tilde{A} \cup C}$

$= f_{\tilde{A} \cup B} \tilde{\cap} (\tilde{A} \cup C)(x)$

$\rightarrow F_A \tilde{\cup} (F_B \tilde{\cap} F_C) = (F_A \tilde{\cup} F_B) \tilde{\cap} (F_A \tilde{\cup} F_C)$.

To prove : $F_A \tilde{\cap} (F_B \tilde{\cup} F_C) = (F_A \tilde{\cap} F_B) \tilde{\cup} (F_A \tilde{\cap} F_C)$.

$f_{\tilde{A} \cap (\tilde{B} \cup C)}(x) = f_A(x) \cap f_{\tilde{B} \cup C}(x)$

$= f_A(x) \cap (f_B(x) \cup f_C(x))$

$= (f_A(x) \cap f_B(x))) \cup (f_A(x) \cap f_C(x))$

$= f_{\tilde{A} \cup B}(x) \cap f_{\tilde{A} \cup C}$

$= f_{\tilde{A} \cup B} \tilde{\cap} (\tilde{A} \cup C)(x)$

$\rightarrow F_A \tilde{\cap} (F_B \tilde{\cup} F_C) = (F_A \tilde{\cap} F_B) \tilde{\cup} (F_A \tilde{\cap} F_C)$.

CHAPTER 3

SOFT TOPOLOGY

In this Chapter, I have given the definition of soft topology on a soft set and its related properties.

Definition: 3.1

Let $F_A \in S(U)$. The **soft power set** of F_A is defined by

$\widetilde{P}(F_A) = \{F_{A_i} : F_{A_i} \tilde{\leq} F_A, i \in I \leq N\}$ and its **cardinality** is defined

by

$|\widetilde{P}(F_A)| = 2^{\Sigma x \in E |f_A(x)|}$.

where $|f_A(x)|$ is the cardinality of $f_A(x)$.

Example: 3.2

Let $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$, $A = \{x_1, x_2\} \leq E$ and

$F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2, u_3\})\}$. Then

$F_{A_1} = \{(x_1, \{u_1\})\}$.

$F_{A_2} = \{(x_1, \{u_2\})\}$.

$F_{A_3} = \{(x_1, \{u_1, u_2\})\}$.

$F_{A_4} = \{(x_2, \{u_2\})\}$.

$F_{A_5} = \{(x_2, \{u_3\})\}$.

$F_{A_6} = \{(x_2, \{u_2, u_3\})\}$.

$F_{A_7} = \{(x_1, \{u_1\}), (x_2, \{u_2\})\}$.

$F_{A_8} = \{(x_1, \{u_1\}), (x_2, \{u_3\})\}$.

$F_{A_9} = \{(x_1, \{u_1\}), (x_2, \{u_2, u_3\})\}$.

$F_{A_{10}} = \{ (x_1, \{u_2\}), (x_2, \{u_2\}) \}$.

$F_{A_{11}} = \{ (x_1, \{u_2\}), (x_2, \{u_3\}) \}$.

$F_{A_{12}} = \{ (x_1, \{u_2\}), (x_2, \{u_2, u_3\}) \}$.

$F_{A_{13}} = \{ (x_1, \{u_1, u_2\}), (x_2, \{u_2\}) \}$.

$F_{A_{14}} = \{ (x_1, \{u_1, u_2\}), (x_2, \{u_3\}) \}$.

$F_{A_{15}} = F_A$.

$F_{A_{16}} = F_\emptyset$.

all soft subsets of F_A. So $|\tilde{P}(F_A)| = 2^{\Sigma |f_A(x)|} = 2^4 = 16$.

Definition: 3.3

Let $\tilde{\tau}$ be the collection of soft sets over X, then $\tilde{\tau}$ is said to be soft topology on X. If

i) F_A, $F_\emptyset$ belongs to $\tilde{\tau}$.

ii) The union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

iii) The intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The pair $(F_A, \tilde{\tau})$ is called a **soft topological space.**

Example: 3.4

Let us consider the soft subsets of F_A that are given in Example 2.2. Then

$\tilde{\tau}_1 = \{ F_A, F_\emptyset \}$, $\tilde{\tau}_2 = \tilde{\tilde{P}}(F_A)$, and $\tilde{\tau}_3 = \{ F_\emptyset, F_A, F_{A_2}, F_{A_{11}}, F_{A_{13}} \}$ are soft topologies on F_A.

Definition: 3.5

Let $(F_A, \tilde{\tau})$ be a soft space over X, then the members of $\tilde{\tau}$ are said to be soft open sets in X.

Definition: 3.6

Let $(F_A, \tilde{\tau})$ be a soft space over X. A soft set F_A over X is said to be a soft closed set in X, if its relative complement F'_A belongs to $\tilde{\tau}$.

Proposition: 3.7

Let $(F_A, \tilde{\tau})$ be a soft space over X. Then

i) F_A, $F_\varnothing$ are closed soft sets over X

ii) The intersection of any number of soft closed set is a soft closed sets over X.

iii) The union of any two soft closed sets is a soft closed set over X.

Definition: 3.8

Let X be an initial universe set, E be the set of parameters and $\tilde{\tau} = \{F_A, F_\varnothing\}$.

Then $\tilde{\tau}$ is called the soft indiscrete topology on X and $(F_A, \tilde{\tau})$ is said to be a soft indiscrete space over X.

Definition: 3.9

Let X be an initial universe set, E be the set of parameters and let $\tilde{\tau}$ be the collection of all soft sets which can be defined over X. Then $\tilde{\tau}$ is called the soft discrete topology on X and $(F_A, \tilde{\tau})$ is said to be a soft discrete space over X.

Proposition: 3.10

Let $(F_A, \tilde{\tau})$ be a soft space over X. Then the collection $\tilde{\tau}_\alpha = \left\{F(\alpha) \mid F_A \in \tilde{\tau}\right\}$ for each $\alpha \in E$, define a topology on X.

Proof

By definition, for any $\alpha \in E$, we have $\tilde{\tau}_\alpha = \left\{F(\alpha) \mid F_A \in \tilde{\tau}\right\}$. Now,

(1) $F_A, F_\varnothing \in \tilde{\tau}$ implies $F_A, F_\varnothing \in \tilde{\tau}_\alpha$.

(2) Let $\{F_i(\alpha) \mid i \in I\}$ be a collection of sets in $\tilde{\tau}_\alpha$. Since $F_{A_i} \in \tilde{\tau}$, for all $i \in I$

So that $\cup_{i \in I} F_i \in \tilde{\tau}$ thus $\cup_{i \in I} F_i(\alpha) \in \tilde{\tau}_\alpha$.

(3) Let $F(\alpha), G(\alpha) \in \tilde{\tau}_\alpha$ for some $F_A, F_B \in \tilde{\tau}$. Since $F_A \cap F_B \in \tilde{\tau}$

So $F(\alpha) \cap G(\alpha) \in \tilde{\tau}_\alpha$.

Thus $\tilde{\tau}_\alpha$ defines a topology on X for each $\alpha \in E$.

This shows that corresponding to each parameter $\alpha \in E$, we have a topology $\tilde{\tau}_\alpha$ on X. Thus a soft topology on X gives a parameterized family of topologies on X.

Example: 3.11

Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tilde{\tau} = \{F_A, F_\varnothing, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}\}$ where $F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}$ are soft sets over X, defined as follows

$F_{A_1}(e_1) = \{h_2\}$, $F_{A_1}(e_2) = \{h_1\}$,

$F_{A_2}(e_1) = \{h_2,h_3\}$, $F_{A_2}(e_2) = \{h_1,h_2\}$,

$F_{A_3}(e_1) = \{h_1,h_2\}$, $F_{A_3}(e_2) = X$,

$F_{A_4}(e_2) = \{h_1,h_2\}$, $F_{A_4}(e_2) = \{h_1,h_3\}$.

Then $\tilde{\tau}$ defines a soft topology on X and $(F_A, \tilde{\tau})$ is a soft topological space over X. It can be easily seen that, $\tilde{\tau}_{e_1} = \{F_A, F_\varnothing, \{h_2\}, \{h_2,h_3\}, \{h_1,h_2\}\}$ and $\tilde{\tau}_{e_2} = \{F_A, F_\varnothing, \{h_1\}, \{h_1,h_3\}, \{h_1,h_2\}\}$ are topologies on X.

Now we give an example to show that the converse of above proposition does not hold.

Example: 3.12

Let $X = \{h_1,h_2,h_3\}$, $E = \{e_1,e_2\}$ and $\tilde{\tau} = \{F_\varnothing, F_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}\}$ where $F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}$ are soft sets over X, defined as follows

$F_{A_1}(e_1) = \{h_2\}$, $F_{A_1}(e_2) = \{h_1\}$,

$F_{A_2}(e_1) = \{h_2,h_3\}$, $F_{A_2}(e_2) = \{h_1,h_2\}$,

$F_{A_3}(e_1) = \{h_1,h_2\}$, $F_{A_3}(e_2) = \{h_1,h_2\}$,

$F_{A_4}(e_2) = \{h_2\}$, $F_{A_4}(e_2) = \{h_1,h_3\}$.

Then $\tilde{\tau}$ defines a soft topology on X because $F_{A_2} \cup F_{A_3} = F_B$

Where $F_B(e_1) = X$, and $F_B(e_2) = \{h_1,h_2\}$ and so $F_B \notin \tilde{\tau}$

Also,

$\tilde{\tau}_{e_1} = \{F_\varnothing, F_A, \{h_2\}, \{h_2,h_3\}, \{h_1,h_2\}\}$ and

$\tilde{\tau}_{e_2} = \{F_{\varnothing}, F_A, \{h_1\}, \{h_1, h_3\}, \{h_1, h_2\}\}$ are topologies on X.

Hence example 3.11 shows that any collection of soft sets need not to be a soft topology on X, even if the collection corresponding to each parameter defines a topology on X.

Proposition: 3.13

Let $(F_A, \tilde{\tau}_1)$ and $(F_A, \tilde{\tau}_2)$ be two soft topological spaces over the same universe X, then $(F_A, \tilde{\tau}_1 \cap \tilde{\tau}_2)$ is a soft topological spaces over X.

Proof

(i) $F_{\varnothing}, F_A$ belong to $\tilde{\tau}_1 \cap \tilde{\tau}_2$.

(ii) Let $\{F_{A_i} / i \in I\}$ be a family of soft sets in $\tilde{\tau}_1 \cap \tilde{\tau}_2$. Then $F_{A_i} \in \tilde{\tau}_1$ and $F_{A_i} \in \tilde{\tau}_2$, for all $i \in I$, so $\cup_{i \in I} F_{A_i} \in \tilde{\tau}_1$ and $\cup_{i \in I} F_{A_i} \in \tilde{\tau}_2$. Thus $\cup_{i \in I} F_{A_i} \in \tilde{\tau}_1 \cap \tilde{\tau}_2$.

(iii) Let $F_A, F_B \in \tilde{\tau}_1 \cap \tilde{\tau}_2$. Then $F_A, F_B \in \tilde{\tau}_1$ and $F_A, F_B \in \tilde{\tau}_2$. Since $F_A \cap F_B \in \tilde{\tau}_1$ and $F_A \cap F_B \in \tilde{\tau}_2$, so $F_A \cap F_B \in \tilde{\tau}_1 \cap \tilde{\tau}_2$.

Thus $\tilde{\tau}_1 \cap \tilde{\tau}_2$ define a soft topology on X and $(F_A, \tilde{\tau}_1 \cap \tilde{\tau}_2)$ is a soft topological spaces over X.

Remark: 3.14

The union of two soft topologies on X may not be a soft topology on X.

Example: 3.15

Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tilde{\tau}_1 = \{F_{\varnothing}, F_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}\}$

$\tilde{\tau}_2 = \{ F_{\emptyset}, F_B, F_{B_1}, F_{B_2}, F_{B_3}, F_{B_4}\}$ be two soft topologies defined on X

where $F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}, F_{B_1}, F_{B_2}, F_{B_3}$ and F_{B_4} are soft sets over X, defined as follows

$F_{A_1}(e_1) = \{h_2\}$, $F_{A_1}(e_2) = \{h_1\}$

$F_{A_2}(e_1) = \{h_2, h_3\}$, $F_{A_2}(e_2) = \{h_1, h_2\}$,

$F_{A_3}(e_1) = \{h_1, h_2\}$, $F_{A_3}(e_2) = X$,

$F_{A_4}(e_1) = \{h_1, h_2\}$, $F_{A_4}(e_2) = \{h_1, h_3\}$,

and

$F_{B_1}(e_1) = \{h_2\}$ $F_{B_1}(e_1) = \{h_1\}$,

$F_{B_2}(e_1) = \{h_2, h_3\}$ $F_{B_2}(e_1) = \{h_1, h_2\}$,

$F_{B_3}(e_1) = \{h_1, h_2\}$ $F_{B_3}(e_1) = \{h_1, h_2\}$,

$F_{B_4}(e_1) = \{h_2\}$ $F_{B_4}(e_1) = \{h_1, h_3\}$,

Now we define

$$\tilde{\tau} = \tilde{\tau}_1 \cup \tilde{\tau}_2$$

$$= \{ F_{\emptyset}, F_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}, F_{B_3}, F_{B_4}\}$$

If we take

$$F_{A_2} \cup F_{B_3} = F_C$$

Then

$$F_C(e_1) = F_{A_2}(e_1) \cup F_{B_3}(e_1)$$

$$= \{h_2, h_3\} \cup \{h_1, h_2\}$$

$$= X$$

and

$$F_C(e_2) = F_{A_2}(e_2) \cup F_{B_3}(e_2)$$

$= \{h_1,h_2\} \cup \{h_1,h_2\}$

$= \{h_1,h_2\}$

But $F_C \notin \tilde{\tau}$. Thus $\tilde{\tau}$ is not a soft topology on X.

Definition: 3.16

Let $(F_A, \tilde{\tau})$ be a soft topological space. Then, every element of $\tilde{\tau}$ is called a **soft open set**. Clearly, $F_\varnothing$ and F_A are soft open sets.

Definition: 3.17

Let $(F_A, \tilde{\tau}_1)$ and $(F_A, \tilde{\tau}_2)$ be soft topological space. Then the following hold

If $\tilde{\tau}_2 \supseteq \tilde{\tau}_1$, then $\tilde{\tau}_2$ is soft finer than $\tilde{\tau}_1$.

If $\tilde{\tau}_2 \supseteq \tilde{\tau}_1$, then $\tilde{\tau}_2$ is soft strictly finer than $\tilde{\tau}_1$.

If either $\tilde{\tau}_2 \supseteq \tilde{\tau}_1$ or $\tilde{\tau}_2 \subseteq \tilde{\tau}_1$, then $\tilde{\tau}_1$ is comparable with $\tilde{\tau}_2$.

Example: 3.18

Let us consider the soft topologies on F_A that are given in example 3.4, Then

$\tilde{\tau}_2$ is soft finer than $\tilde{\tau}_1$ and $\tilde{\tau}_3$, and $\tilde{\tau}_3$ is soft finer than $\tilde{\tau}_1$. So $\tilde{\tau}_1$, $\tilde{\tau}_2$ and $\tilde{\tau}_3$ are comparable soft topologies.

Definition: 3.19

Let $(F_A, \tilde{\tau})$ be a soft topological space and $\tilde{B} \subseteq \tilde{\tau}$. If every element of $\tilde{\tau}$ can be written as the union of elements of $\tilde{B}$, then $\tilde{B}$ is called a **soft basis** for the soft

topology $\tilde{\tau}$. Each element of $\tilde{B}$ is called a **soft basis element.**

Example: 3.20

Let us consider example 3.2 and example 3.4

Then, $\tilde{B} = \left\{ F_{\emptyset}, F_{A_1}, F_{A_2}, F_{A_4}, F_{A_5} \right\}$ is a soft basis for the soft topology $\tilde{\tau}_2$.

Theorem: 3.21

Let $(F_A, \tilde{\tau})$ be a soft topological space and $\tilde{B}$ be a soft basis for $\tilde{\tau}$.

Then $\tilde{\tau}$ equals the collection of all soft unions of element of $\tilde{B}$.

Proof

This is clearly from Definition 3.19

Definition: 3.22

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then the collection

$\tilde{\tau}_{F_B} = \left\{ F_{A_i} \tilde{\cap} F_B : F_{A_i} \tilde{\in} \tau , i \in I \subseteq N \right\}$ is called a **soft subspace** topology on F_B.

Hence $(F_B, \tilde{\tau}_{F_B})$ is called a **soft topological subspace** of $(F_A, \tilde{\tau})$.

Theorem: 3.23

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then a soft subspace topological on F_B is a soft topology.

Proof

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$.

To Prove : A soft topology on F_B is a soft topology.

Indeed, it contains $F_\varnothing$ and F_B because $F_\varnothing \tilde{\cap} F_B = F_\varnothing$ and $F_A \tilde{\cap} F_B = F_B$.

Where $F_\varnothing$, $F_A \in \tilde{\tau}$. Since $\tilde{\tau} = \left\{F_{A_i} : F_{A_i} \tilde{\subseteq} F_A, i \in I\right\}$, it is closed under finite soft interections and arbitrary soft unions:

$$\tilde{\cap}_{i=1}^{n}(F_{A_i} \tilde{\cap} F_B) = (\tilde{\cap}_{i=1}^{n} F_{A_i}) \tilde{\cap} F_B \in \tilde{\tau}_{F_B}$$

$$\tilde{\cup}_{i=1}^{n}(F_{A_i} \tilde{\cap} F_B) = (\tilde{\cup}_{i=1}^{n} F_{A_i}) \tilde{\cap} F_B \in \tilde{\tau}_{F_B}$$

Therefore, a soft subspace topology on F_B is a soft topology.

Example: 3.24

Let us consider the soft topology $\tilde{\tau}_3 = \left\{ F_\varnothing, F_A, F_{A_2}, F_{A_{11}}, F_{A_{13}} \right\}$ on F_A given in example 3.4. If $F_B = F_{A_9}$ then $\tilde{\tau}_{F_B} = \left\{ F_\varnothing, F_{A_5}, F_{A_8}, F_{A_9} \right\}$ and so $(F_B, \tilde{\tau}_{F_B})$ is a soft topological subspace of $(F_A, \tilde{\tau}_3)$.

Theorem: 3.25

Let $(F_A, \tilde{\tau})$ and $(F_A, \tilde{\tau}')$ be soft topological spaces, and $\tilde{B}$ and $\tilde{B}'$ be soft bases for $\tilde{\tau}$ and $\tilde{\tau}'$, respectively. If $\tilde{B}' \subseteq \tilde{B}$, then $\tilde{\tau}$ is soft finer that $\tilde{\tau}'$.

Proof

Let $(F_A, \tilde{\tau})$ and $(F_A, \tilde{\tau}')$ be soft topological space, $\tilde{B}$ and $\tilde{B}'$ be soft base for $\tilde{\tau}$ and $\tilde{\tau}'$ respectively. Let $\tilde{B}' \subseteq \tilde{B}$.

Then for each $F_B \in \tilde{\tau}'$ and $F_C \in \tilde{B}'$.

To Prove : $\tilde{\tau}$ is soft finer than $\tilde{\tau}'$

$F_B = \cup_{F_c \in \tilde{B}'} F_C = \cup_{F_c \in \tilde{B}} F_C$.

$\Rightarrow F_B \in \tilde{\tau}$

Therefore $\tilde{\tau}' \subseteq \tilde{\tau}$.

Hence, $\tilde{\tau}$ is soft finer than $\tilde{\tau}'$.

Theorem: 3.26

Let $(F_A, \tilde{\tau})$ be a soft topological space. If $\tilde{B}$ is a soft basis for $\tilde{\tau}$, then the collection

$\tilde{B}_{F_B} = \left\{ F_{A_i} \tilde{\cap} F_B : F_{A_i} \in \tilde{B},\ i \in I \subseteq N \right\}$ is a soft basis for the soft subspace topology on F_B.

Proof :

Let $(F_A, \tilde{\tau})$ be a soft topological space and $\tilde{B}$ is a soft basis for $\tilde{\tau}$.

Given $\tilde{B}_{F_B} = \left\{ F_{A_i} \tilde{\cap} F_B : F_{A_i} \in \tilde{B},\ i \in I \subseteq N \right\}$.

To Prove:

$\tilde{B}_{F_B}$ is a soft basis for the soft subspace topology on F_B.

Take as given each $F_{A_i} \in \tilde{\tau}_{F_B}$.

From the definition of soft topology, $F_C = F_D \tilde{\cap} F_B$, where $F_D \tilde{\in} \tau$.

Because $F_D \tilde{\in} \tau$, $F_D \tilde{=} \cup_{F_{A_i} \tilde{\in} B} F_{A_i}$.

Therefore,

$F_C \tilde{=} (\cup_{F_{A_i} \tilde{\in} B} F_{A_i}) \tilde{\cap} F_B \tilde{=} (\cup_{F_{A_i} \tilde{\in} B} (F_{A_i} \tilde{\cap} F_B)$.

Hence, $\tilde{B}_{F_B}$ is a soft basis for the soft subspace topology $\tilde{\tau}_{F_B}$ on F_B.

Theorem: 3.27

Let $(F_A, \tilde{\tau})$ be a soft topological space, $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace, and $F_C \tilde{\subseteq} F_B$. If F_C is soft open in F_B, then F_C is soft open F_A.

Proof

This is clearly seen from Definition 3.22

Definition: 3.28

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then F_B is said to be **soft closed**. If the soft set $\tilde{F}_B^c$ is soft open.

Theorem: 3.29

Let $(F_A, \tilde{\tau})$ be a soft topological space. Then, the following conditions hold.

(i) The universal soft set F_E and $\tilde{F}_A^c$ are soft closed set.

(ii) Arbitrary soft intersections of the closed sets are soft closed.

(iii) Finite soft unions of the soft closed sets are soft closed.

Proof

Let $(F_A, \tilde{\tau})$ be a soft toplogical space.

i) To Prove : The universal soft set F_E and $\tilde{F}_A^c$ are soft closed sets.

Now, $\tilde{F}_A^c = F_\varnothing$ and $(\tilde{F}_A^c)^{\tilde{C}} = F_A$ are soft open sets.

Hence, F_E and $\tilde{F}_A^c$ are soft closed sets.

ii) Let $\left\{F_{A_i} : \tilde{F}_{A_i}^C \in \tilde{\tau}, i \in \subseteq N\right\}$ is a given collection of soft closed sets.

To prove: $\tilde{\cap}_{i \in I} F_{A_i}$ is a soft closed set.

Now, $(\tilde{\cap}_{i \in I} F_{A_i})^{\tilde{C}} = \tilde{\cup}_{i \in I} F_{A_i}^{\tilde{C}}$ is soft open.

Therefore, $\tilde{\cap}_{i \in I} F_{A_i}$ is a soft closed set.

iii) Similarly, let F_{A_i} is soft closed for i= 1,2,.......,n.

To prove : $\tilde{\cup}_{i \in I} F_{A_i}$ is a soft closed set.

Now, $(\tilde{\cup}_{i \in I} F_{A_i})^{\tilde{C}} = \tilde{\cap}_{i=1}^{n} \tilde{F}_{A_i}^C$ is soft open.

Hence, $\tilde{\cup}_{i \in I} F_{A_i}$ is a soft closed set.

CHAPTER 4

SOFT INTERIOR AND SOFT CLOSURE

4.1 Soft Interior

Definition: 4.1.1

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, the **soft interior** of F_B, denoted by F_B° is defined as the soft union of all soft open subsets of F_B.

Note that F_B° is the biggest soft open set that is contained by F_B.

Example: 4.1.2

Let us consider the soft topology $\tilde{\tau}_3 = \{ F_{\varnothing}, F_A, F_{A_2}, F_{A_{11}}, F_{A_{13}} \}$ on F_A given in example 3.4

If $F_B = F_{A_{12}} = \{(x_1,\{u_2\}),(x_2,\{u_2,u_3\})\}$,

then $F_B^{\circ} = F_{\varnothing} \tilde{\cup} F_{A_2} \tilde{\cup} F_{A_{11}} = F_{A_{11}}$.

Theorem: 4.1.3

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. F_B is a soft open set

if and only if $F_B = F_B^{\circ}$.

Proof

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$.

Suppose that F_B is an open soft set.

To prove : $F_B = F_B^{\circ}$

Since F_B is an open soft set, then the biggest soft open set that is contained by F_B is equal to F_B.

Therefore, $F_B = F_B^{\circ}$.

Conversely, suppose that $F_B = F_B^{\circ}$.

To prove : F_B is a soft open set.

We know that F_B°. Is a soft open set and if $F_B = F_B^{\circ}$ then F_B is a soft open set.

Therefore, F_B is a soft open set.

Theorem: 4.1.4

Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B, $F_C \tilde{\subseteq} F_A$. Then

i. $(F_B^{\circ})^{\circ} = F_B^{\circ}$

ii. $F_B \tilde{\subseteq} F_C \Rightarrow F_B^{\circ} \tilde{\subseteq} F_C^{\circ}$.

iii. $F_B^\circ \tilde{\cap} F_C^\circ = (F_B \tilde{\cap} F_C)^\circ$

iv. $F_B^\circ \tilde{\cup} F_C^\circ \tilde{\subseteq} (F_B \tilde{\cup} F_C)^\circ$

Proof

Let $(F_A, \tilde{\tau}\neg)$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$.

i) To prove : $(F_B^\circ)^\circ = F_B^\circ$.

Let $F_B^\circ = F_D$.

Then $F_D \tilde{\in} \tau\neg$ if and only if $F_D = F_D^\circ$.

Therefore, $(F_B^\circ)^\circ = F_B^\circ$.

ii) Let $F_B \tilde{\subseteq} F_C$.

To prove : $F_B^\circ \tilde{\subseteq} F_C^\circ$

From the definition of a soft interior , $F_B^\circ \tilde{\subseteq} F_B$ and $F_C^\circ \tilde{\subseteq} F_C$.

F_C° is the biggest soft open set that is contained by F_C.

Hence, $F_B \tilde{\subseteq} F_C \Rightarrow F_B^\circ \tilde{\subseteq} F_C^\circ$.

iii) To prove: $F_B^\circ \tilde{\cap} F_C^\circ = (F_B \tilde{\cap} F_C)^\circ$

By definition of a soft interior, $F_B^\circ \tilde{\subseteq} F_B$ and $F_C^\circ \tilde{\subseteq} F_C$.

Then , $F_B^\circ \tilde{\cap} F_C^\circ \tilde{\subseteq} F_B \tilde{\cap} F_C$.

$(F_B \tilde{\cap} F_C)^\circ$ is the biggest soft open set that is contained by $F_B \tilde{\cap} F_C$.

Hence, $F_B^\circ \tilde{\cap} F_C^\circ = (F_B \tilde{\cap} F_C)^\circ$.

Conversely, $\tilde{F}_B \cap \tilde{F}_C \subseteq \tilde{F}_B$ and $\tilde{F}_B \cap \tilde{F}_C \subseteq \tilde{F}_C$.

Then, $(\tilde{F}_B \cap F_C)^{\tilde{\circ}} \subseteq F_B^{\circ}$ and $(\tilde{F}_B \cap F_C)^{\tilde{\circ}} \subseteq F_C^{\circ}$

Hence, $(\tilde{F}_B \cap F_C)^{\tilde{\circ}} \subseteq \tilde{F}_B^{\circ} \cap F_C^{\circ}$

Therefore, $\tilde{F}_B^{\circ} \cap F_C^{\circ} = (\tilde{F}_B \cap F_C)^{\circ}$.

iv) To prove : $\tilde{F}_B^{\circ} \cup \tilde{F}_C^{\circ} \subseteq (\tilde{F}_B \cup F_C)^{\circ}$.

From the definition of a soft interior, $\tilde{F}_B^{\circ} \subseteq F_B$ and $\tilde{F}_C^{\circ} \subseteq F_C$.

Then $\tilde{F}_B^{\circ} \cup \tilde{F}_C^{\circ} \subseteq \tilde{F}_B \cup F_C$.

$(\tilde{F}_B \cup F_C)^{\circ}$ is the biggest soft open set that is contained by $\tilde{F}_B \cup F_C$.

Hence, $\tilde{F}_B^{\circ} \cup \tilde{F}_C^{\circ} \subseteq (\tilde{F}_B \cup F_C)^{\circ}$.

Definition: 4.1.5

Let $(\tilde{F}_A, \tau)$ be a soft topological space and $\alpha \in F_A$. If there is a soft open set F_B such that $\in F_B$, then F_B is called a **soft open neighbourhood** (or soft neighbourhood) of α. The set of all soft neighbourhoods of α, denoted by $\tilde{V}(\alpha)$, is called the family of soft neighbourhoods of α; that is,

$$\tilde{V}(\alpha) = \{ F_B : F_B \in \tilde{\tau},\ \alpha \in F_B \}$$

Example: 4.1.6

Let us consider the $(\tilde{F}_A, \tau_3)$ topological space in example 2.4 and $\alpha = (x_1, \{u_1, u_2\}) \in F_A$. Then, $\tilde{V}(\alpha) = \{F_A, F_{A_{13}}\}$.

Proposition: 4.1.7

Let $(F_A, \tilde{\tau})$ be a soft topological space over X, F_B be a soft set over X and

$x \in X$. If x is a soft interior point of F_B then x is an interior point of $G(\alpha)$ in

$(F_A, \tilde{\tau}_\alpha)$, for each $\alpha \in E$.

Proof

For any $\alpha \in E$, $G(\alpha) \subseteq X$. If $x \in X$ is a soft interior point of F_B

then there exists $F_A \tilde{\in} \tau$ such that $x \in F_A \tilde{\subset} F_B$. This means that, $x \in F(\alpha)$

$\subseteq G(\alpha)$.

As $F(\alpha) \in \tilde{\tau}_\alpha$, So $F(\alpha)$ is an open set in $\tilde{\tau}_\alpha$ and $x \in F(\alpha)$. This implies that x

is an interior point of $G(\alpha)$ in $\tilde{\tau}_\alpha$.

Proposition: 4.1.8

Let $(F_A, \tilde{\tau})$ be a soft topological space over X, then

(i) each $x \in X$ has a soft neighborhood;

(ii) if F_A and F_B are soft neighborhoods of some $x \in X$, then $F_A \cap F_B$ is also a soft neighborhood of x.

(iii) if F_A is a soft neighborhood of $x \in X$ and $F_A \tilde{\subset} F_B$, then F_B is also a soft neighborhood of $x \in X$.

Proof

(i) For any $x \in X$, $x \tilde{\in} X$ and since $\tilde{X} \tilde{\in} \tau$, so $x \in \tilde{X} \tilde{\subset} \tilde{X}$.

Thus $\tilde{X}$ is a soft neighborhood of x.

(ii) if F_A and F_B are soft neighborhoods of $x \in X$, then there exist F_{A_1}, $F_{A_2} \in \tilde{\tau}$ such that $x \in F_{A_1} \tilde{\subset} F_A$ and $x \in F_{A_2} \tilde{\subset} F_B$.

Now $x \in F_{A_1}$ and $x \in F_{A_2}$ implies that $x \in F_{A_1} \cap F_{A_2}$ and $F_{A_1} \cap F_{A_2} \in \tilde{\tau}$.

So we have $x \in F_{A_1} \cap F_{A_2} \tilde{\subset} F_A \cap F_B$.

Thus $F_A \cap F_B$ is a soft neighborhood of x.

(iii) F_A is a soft neighborhood of $x \in X$ and $F_A \tilde{\subset} F_B$. By definition there exist a soft open set F_A such that $x \in F_{A_1} \tilde{\subset} F_A \tilde{\subset} F_B$.

Thus $x \in F_{A_1} \tilde{\subset} F_B$.

Hence F_B is a soft neighborhood of x.

Proposition: 4.1.9

Let $(F_A, \tilde{\tau})$ be a soft topological space over X. For any soft open set F_A

over X, F_A is a soft neighborhood of each point of $\cap_{\alpha \in E} F(\alpha)$.

Proof

Let $F_A \in \tilde{\tau}$. For any $x \in \cap_{\alpha \in E} F(\alpha)$,

We have $x \in F(\alpha)$ for each $\alpha \in E$.

Thus $x \in F_A \tilde{\subset} F_A$ and so F_A is a soft neighborhood of x.

4.2 Soft Closure

Definition: 4.2.1

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\cup} F_A$. Then the **soft closure** of F_B, denoted by $\overline{F_B}$ is defined as the soft intersection of all soft closed supersets of F_B.

Note that $\overline{F_B}$ is the smallest soft closed set that containing F_B.

Example: 4.2.2

Let us consider the soft topology $\tilde{\tau}_3 = \{F_\varnothing, F_A, F_{A_2}, F_{A_{11}}, F_{A_{13}}\}$ on F_A given in example 3.4

If $F_B = F_{A_9} = \{(x_1, \{u_1\}), (x_2, \{u_2, u_3\})\}$,

then $\tilde{F}^C_{A_2} = \{(x_1, \{u_1, u_3\}), (x_2, U, (x_3, U))\}$ and $\tilde{F}^C_\varnothing = \tilde{F}_E$ are soft closed supersets of F_B.

Hence, $\overline{F_B} = \tilde{F}^C_{A_2} \tilde{\cap} \tilde{F}_E = \tilde{F}^C_{A_2}$.

Theorem: 4.2.3

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. F_B is a closed soft set

if and only if $F_B = \overline{F_B}$.

Proof

The proof is trivial.

Theorem: 4.2.4

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then $F_B^\circ \tilde{\subseteq} F_B \tilde{\subseteq}$

$\overline{F_B}$.

Proof

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$.

To prove : $F_B^{\circ} \tilde{\subseteq} F_B \tilde{\subseteq} \overline{F_B}$

Indeed, $F_B^{\circ} \tilde{=} \cup \left\{F_{B_i} : F_{B_i} \tilde{\in} \tau,\ \ F_{B_i} \tilde{\subseteq} F_B,\ i \in I \tilde{\subseteq} N\right\}$.

Then $f_{B_i}(x) \tilde{\subseteq} f_B(x)$ and $\cup_{i \in I} f_{B_i}(x) \tilde{\subseteq} f_B(x)$ for all $x \in E$.

So, $F_B^{\circ} \tilde{\subseteq} F_B$.

Now, $\overline{F_B} \tilde{=} \cap \left\{F_{A_i} : \tilde{F}_{A_i}^{C} \tilde{\in} \tau,\ \ F_B \tilde{\subseteq} F_{A_i},\ i \in j \tilde{\subseteq} N\right\}$.

Then, $f_B(x) \tilde{\subseteq} f_{A_i}(x)$ and $f_B(x) \tilde{\subseteq} \cap_{i \in j} f_{A_i}(x)$ for all $x \in E$.

So $F_B \tilde{\subseteq} \overline{F_B}$.

Hence $F_B^{\circ} \tilde{\subseteq} F_B \tilde{\subseteq} \overline{F_B}$.

Theorem: 4.2.5

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then

i) $\overline{\overline{(F_B)}} = \overline{F_B}$.

ii) $(\tilde{F}_B)^{\tilde{C}} = (\tilde{F}_B^{C})^{\circ}$

iii) $F_C \tilde{\subseteq} \overline{F_B} \Rightarrow \overline{F_C} \tilde{\subseteq} \overline{F_B}$.

iv) $\overline{F_B} \tilde{\cap} \overline{F_C} \tilde{\subseteq} \overline{(F_B \tilde{\cap} F_C)}$.

v) $\overline{F_B}\ \tilde{\cup}\ \overline{F_C} = \overline{(F_B \tilde{\cup} F_C)}$

Proof

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$.

i) To prove : $(\overline{\overline{F_B}}) = \overline{F_B}$.

Let $\overline{F_B} = F_D$.

Then F_D is a soft closed set.

Therefore, F_C and $\overline{F_D}$ are equal.

Hence, $(\overline{\overline{F_B}}) = \overline{F_B}$.

ii) To prove : $\widetilde{(F_B)}^{\tilde{c}} = (F_B^{\tilde{c}})^{\circ}$

If we consider the definitions of a soft closure and a soft interior,

we obtain $\widetilde{(F_B)}^{\tilde{c}} = \left(\tilde{\cap}_{F_B \tilde{\subseteq} F_{A_i},\ F_{A_i}^{\tilde{c}} \tilde{\in} \tilde{\tau}} F_{A_i}\right)^{\tilde{c}}$.

$$= \tilde{\cup}\ F_{A_i}^{\tilde{c}}$$

$$= (F_B^{\tilde{c}})^{\circ}$$

Hence $\widetilde{(F_B)}^{\tilde{c}} = (F_B^{\tilde{c}})^{\circ}$.

iii) To prove : $F_C \tilde{\subseteq} F_B \Rightarrow \overline{F_C} \tilde{\subseteq} \overline{F_B}$.

Let $F_C \tilde{\subseteq} F_B$

By the definition of a soft closure, $F_B \tilde{\subseteq} \overline{F_B}$ and $F_C \tilde{\subseteq} \overline{F_C}$.

$\overline{F_C}$ is the smallest soft closed set that containing F_C.

Then $\overline{F_C} \tilde{\subseteq} \overline{F_B}$.

iv) To prove: $\overline{F_B} \tilde{\cap} \overline{F_C} \tilde{\subseteq} \overline{(F_B \tilde{\cap} F_C)}$.

Now, $\overline{F_B}$ and $\overline{F_C}$ are soft closed sets.

So, $\overline{F_B} \tilde{\cap} \overline{F_C}$ is a soft closed set.

Since $F_B \tilde{\cap} F_C \tilde{\subseteq} \overline{(F_B \tilde{\cap} F_C)}$ and $\overline{(F_B \tilde{\cap} F_C)}$ is the smallest soft closed set that

containing $F_B \tilde{\cap} F_C$.

Therefore, $\overline{F_B} \tilde{\cap} \overline{F_C} \tilde{\subseteq} \overline{(F_B \tilde{\cap} F_C)}$.

v) To prove : $\overline{F_B} \tilde{\cup} \overline{F_C} = \overline{(F_B \tilde{\cup} F_C)}$

By the definition of a soft closure, $F_B \tilde{\subseteq} \overline{F_B}$ and $F_B \tilde{\subseteq} \overline{F_B}$.

Then $F_B \tilde{\cup} F_C \tilde{\subseteq} \overline{F_B} \tilde{\cup} \overline{F_C}$.

Since $\overline{(F_B \tilde{\cup} F_C)}$ is the smallest soft closed set that containing $F_B \tilde{\cup} F_C$.

Therefore $\overline{(F_B \tilde{\cup} F_C)} \tilde{\subseteq} \overline{F_B} \tilde{\cup} \overline{F_C}$.

Conversely, $F_C \tilde{\subseteq} F_C \tilde{\subseteq} \overline{(F_B \tilde{\cup} F_C)}$ and $F_B \tilde{\subseteq} \overline{F_B} \tilde{\subseteq} \overline{(F_B \tilde{\cup} F_C)}$.

Therefore, $\overline{F_B} \tilde{\cup} \overline{F_C} \tilde{\subseteq} \overline{(F_B \tilde{\cup} F_C)}$.

Hence $\overline{F_B} \tilde{\cup} \overline{F_C} = \overline{(F_B \tilde{\cup} F_C)}$.

Theorem: 4.2.6

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$.

Then the following hold.

(i) $\alpha \in \overline{F_B}$ if and only if every soft open F_C containing α soft intersects F_B.

(ii) Supposing the soft topology of F_A is given by a soft basis, then $\alpha \in \overline{F_B}$ if and only if every soft basis element F_D containing α soft intersects F_B.

Proof

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$.

(i) The hypothesis is equivalent to $\alpha \notin \overline{F_B}$ if and only if there exists a soft open set F_C. Containing α that does not soft intersect F_B.

If $\notin \overline{F_B}$, the soft set $F_C = \left(\overline{F_B}\right)^c$ is a soft open set containing α that does not soft intersect F_B as required.

Conversely, if there exits soft open set F_C containing α which does not soft intersect F_B.

By the definition of the soft closure $\overline{F_B}$, the soft set $\overline{F_C}^{c}$ must contain $\overline{F_B}$.

Therefore α cannot be in $\overline{F_B}$.

(ii) If $\alpha \in \overline{F_B}$, then every soft open set F_C containing α soft intersects F_B, and so every soft intersects F_B.

Conversely, if every soft element F_D containing α soft intersects F_B, then every soft open set F_C containing α α soft intersects F_C containing α soft intersects F_B.

Hence $\alpha \in \overline{F_B}$.

CHAPTER 5

SOFT LIMIT POINT AND SOFT HAUSDORFF SPACE

5.1 Soft Limit point

Definition 5.1.1

Let $(F_A, \tilde{\tau})$ be a soft topological space and $\alpha \in F_A$. If there is a soft open set F_B such that $\in F_B$, then F_B is called a soft open neighbourhood (or soft neighbourhood) of α. The set of all soft neighbourhoods of α, denoted by $\tilde{V}(\alpha)$, is called the **family of soft neighbourhoods of α**; that is,

$$\tilde{V}(\alpha) = \{ F_B : F_B \in \tilde{\tau}, \alpha \in F_B\}$$

Example: 5.1.2

Let us consider the $(F_A, \tilde{\tau}_3)$ topological space in example 2.4 and $\alpha = (x_1, \{u_1, u_2\}) \in F_A$. Then, $\tilde{V}(\alpha) = \{F_A, F_{A_{13}}\}$.

Definition: 5.1.3

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_A \tilde{\subseteq} F_B$, and $\alpha \in F_A$. If every neighbourhood of α soft intersects F_B in some points other than α itself, then α is called a **soft limit point** of F_B. The set of all limit points of F_B is denoted by $F_B^{'}$.

In other words, if $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$ and

$\alpha \in F_A$.

Then $\alpha \in F_B' \Leftrightarrow F_C \tilde{\cap} (F_B \tilde{\setminus} \{\alpha\}) \neq F_\varnothing$ for all $F_C \tilde{\in} V(\alpha)$.

Example: 5.1.4

Let us consider example 5.1.2

If $F_B = F_{A_{13}}$ and $\alpha = (x_1, \{u_1, u_2\}) \in F_A$,

Then $\alpha \in F_B'$.

Since $F_C \tilde{\cap} (F_B \tilde{\setminus} \{\alpha\}) \neq F_\varnothing$ and $F_{A_{13}} \tilde{\cap} (F_B \tilde{\setminus} \{\alpha\}) \neq F_\varnothing$.

Theorem: 5.1.5

Let $(\tilde{F_A}, \tau)$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then $F_B \tilde{\cup} F_B' = \overline{F_B}$.

Proof

If $\alpha \in F_B \tilde{\cup} F_B'$, then $\alpha \in F_B$ or $\alpha \in F_B'$.

To Prove: $\alpha \in \overline{F_B}$

In this $\alpha \in F_B$, then $\alpha \in \overline{F_B}$

If $\alpha \in \overline{F_B}$, then $F_C \tilde{\cap} (F_B \tilde{\setminus} \{\alpha\}) \neq F_\varnothing$ for all $F_C \tilde{\in} V(\alpha)$, and so

$F_C \tilde{\cap} F_B \neq F_\varnothing$ for all $F_C \tilde{\in} V(\alpha)$;

Hence $\alpha \in \overline{F_B}$.

Conversely, if $\alpha \in \overline{F_B}$, then $\alpha \in F_B$ or $\alpha \in \overline{F_B}$.

To Prove: $\alpha \in F_B \tilde{\cup} F_B'$.

In this case, if $\alpha \in F_B$, it is trivial that $\alpha \in F_B \tilde{\cup} F_B'$.

If $\alpha \notin F_B$, then $F_C \tilde{\cap} (F_B \tilde{\setminus} \{\alpha\}) \neq F_{\emptyset}$ for all $F_C \tilde{\in} V(\alpha)$.

Therefore, $\alpha \in F_B'$, so $\alpha \in F_B \tilde{\cup} F_B'$.

Hence $F_B \tilde{\cup} F_B' = \overline{F_B}$.

Theorem: 5.1.6

Let $(\tilde{F}_A, \tau)$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then F_B is soft closed if and only if $F_B' \tilde{\subseteq} F_B$.

Proof

Let $(\tilde{F}_A, \tau)$ be a soft topological space and $F_B \tilde{\subseteq} F_A$.

To Prove:

F_B is soft closed if and only if $F_B' \tilde{\subseteq} F_B$.

Now, F_B is soft closed $\Leftrightarrow \overline{F_B} = F_B$

$$\Leftrightarrow F_B \tilde{\cup} F_B' = F_B$$

$$\Leftrightarrow F_B' \tilde{\subseteq} F_B.$$

Theorem: 5.1.7

Let $(\tilde{F}_A, \tau)$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then

(i) $F_B' \tilde{\subseteq} \overline{F_B}$

(ii) $F_B \tilde{\subseteq} F_C \Rightarrow F_B' \tilde{\subseteq} F_C'$

(iii) $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_B' \tilde{\cap} F_C'$

(iv) $(F_B \tilde{\cup} F_C)' = F_B' \tilde{\cup} F_C'$

(v) F_B is a soft closed set $\Leftrightarrow F_B' \tilde{\subseteq} F_B$.

Proof

Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$.X

(i) To Prove: $F_B' \tilde{\subseteq} \overline{F_B}$

From the definition of a soft closure, the proof is trivial.

(ii) To Prove: $F_B \tilde{\subseteq} F_C \Rightarrow F_B' \tilde{\subseteq} F_C'$

Let $F_B \tilde{\subseteq} F_C$.

Since $F_B \tilde{\setminus} \{\alpha\} \tilde{\subseteq} F_C \tilde{\setminus} \{\alpha\}$, then $\overline{F_B \tilde{\setminus} \{\alpha\}} \tilde{\subseteq} \overline{F_C \tilde{\setminus} \{\alpha\}}$.

And we obtain $F_B' \tilde{\subseteq} F_C'$

Hence $F_B \tilde{\subseteq} F_C \Rightarrow F_B' \tilde{\subseteq} F_C'$.

(iii) To Prove: $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_B' \tilde{\cap} F_C'$

WKT, $F_B \tilde{\cap} F_C \tilde{\subseteq} F_B$ and $F_B \tilde{\cap} F_C \tilde{\subseteq} F_C$.

Then $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_B'$ and $(F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_C'$

$\Rightarrow (F_B \tilde{\cap} F_C)' \tilde{\subseteq} F_B' \tilde{\cap} F_C'$.

(iv) To Prove : $(F_B \tilde{\cup} F_C)' = F_B' \tilde{\cup} F_C'$

For all $\alpha \in (F_B \tilde{\cup} F_C)' \Leftrightarrow \alpha \in \overline{(F_B \tilde{\cup} F_C) \setminus \{\alpha\}}$, therefore

$$\overline{(F_B \tilde{\cup} F_C) \setminus \{\alpha\}} = \overline{(F_B \tilde{\cup} F_C) \tilde{\cap} \{\alpha\}^{\tilde{c}}}$$

$$= \overline{(F_B \tilde{\cap} \{\alpha\}^{\tilde{c}}) \tilde{\cup} (F_C \tilde{\cap} \{\alpha\}^{\tilde{c}})}$$

$$= \overline{(F_B \tilde{\cap} \{\alpha\}^{\tilde{c}})} \tilde{\cup} \overline{(F_C \tilde{\cap} \{\alpha\}^{\tilde{c}}}$$

$$= \overline{(F_C) \setminus \{\alpha\})} \tilde{\cup} \overline{(F_C) \setminus \{\alpha\})}$$

$\Leftrightarrow \alpha \in F_B' \tilde{\cup} F_C'$.

Hence $(F_B \tilde{\cup} F_C)' = F_B' \tilde{\cup} F_C'$

(v) To Prove:

F_B is a soft closed set $\Leftrightarrow F_B' \tilde{\subseteq} F_{B.}$

Now F_B is a soft closed set $\Leftrightarrow \overline{F_B} = F_{B.}$

$\Leftrightarrow F_{B.} \tilde{\cup} F_B' = F_{B.}$

$\Leftrightarrow F_B' \tilde{\subseteq} F_{B.}$

Hence, $F_{B.}$ is a soft closed set $\Leftrightarrow F_B' \tilde{\subseteq} F_{B.}$

5.2 Hausdorff space

Definition: 5.2.1

Let $(F_A, \tilde{\tau})$ be a soft topological space. If $\forall\ \alpha_1, \alpha_2 \in F_A$

$(\alpha_1, \neq \alpha_2)$, there exist $F_{B_1} \tilde{\in} v(\alpha_1)$ and $F_{B_2} \tilde{\in} v(\alpha_2)$ such that $F_{B_1} \tilde{\cap} F_{B_2} = F_{\emptyset}$.

Then $(F_A, \tilde{\tau})$ is called a **soft Hausdorff space.**

Example: 5.2.2

Let $U = \{u_1, u_2, u_3\}$ and $E = \{x_1, x_2, x_3\}$.

Clearly, $F_E = \{(x_1, \{u_1, u_2\}), (x_2\{u_2, u_3\}), (x_3, \{u_1, u_3\})\} \in S(U)$.

If $F_{E_1} = \{(x_1, \{u_1, u_2\}),, (x_3, \{u_1,\})\}$ and $F_{E_2} = \{(x_2\{u_2, u_3\}), (x_3, \{u_3\})\}$

Then $\tilde{\tau} = \{F_{\emptyset}, F_E, F_{E_1}, F_{E_2}\}$ is a soft topology on F_E.

Hence, $(F_A, \tilde{\tau})$ is a soft Hausdorff space.

Theorem: 5.2.3

Every finite point soft set in a soft Hausdorff space is a soft closed set.

Proof

To prove: Every finite point soft set in a Hausdorff space is a soft closed set.

Let $(F_A, \tilde{\tau})$ is a soft Hausdorff space.

It suffices to show that every point $\{\alpha_1\}$ is soft closed.

If α_2 is a point of F_A different from α_1, then α_1 and α_2 have disjoint soft neighbourhoods F_{B_1} and F_{B_2}, respectively.

Since F_{B_1} does not soft intersect $\{\alpha_2\}$, point α_1 cannot belong to the soft closure of the set $\{\alpha_2\}$.

As a result the soft closure of the set $\{\alpha_1\}$ is $\{\alpha_1\}$ itself, so it is soft closed.

Definition: 5.2.4

Let (F_A, τ) be a soft topological space, and $F_B \tilde{\subseteq} F_A$. Then the soft boundary

of F_B, denoted by F_B^b is defined by $F_B^b = \overline{F_B} \tilde{\cap} \overline{F_B^c}$.

Example: 5.2.5

Let us consider example 3.6. For F_B, $\overline{F_B} = F_{A_2}^c$ and $\overline{F_B^c} = \tilde{F}_E$.

Then $F_B^b = \overline{F_B} \tilde{\cap} \overline{F_B^c} = F_{A_2}^c$.

Theorem: 5.2.6

Let (F_A, τ) be a soft topological space, and $F_B, F_C \tilde{\subseteq} F_A$. Then

(i) $F_B^b \tilde{\subseteq} \overline{F_B}$

(ii) $F_B^b = (F_B^c)^b$

(iii) $F_B^c = \overline{F_B} \tilde{\setminus} F_B^\circ$

Proof

Let (F_A, τ) be a soft topological space, and $F_B, F_C \tilde{\subseteq} F_A$.

i) To prove : $F_B^b \tilde{\subseteq} \overline{F_B}$

From the definition of a soft boundary, the proof is trivial.

ii) **To prove**: $F_B^b = (\tilde{F}_B^c)^b$

Take as given $\alpha \in F_B^b \Leftrightarrow, \tilde{F}_C \cap F_B \neq F_\emptyset$ and $\tilde{F}_C \cap \tilde{F}_B^c \neq F_\emptyset$ for all $, F_C \tilde{\in} v(\alpha)$.

$\Leftrightarrow \tilde{F}_C \cap \tilde{F}_B^c \neq F_\emptyset$ and $\tilde{F}_C \cap (\tilde{F}_B^c)^{\tilde{c}} \neq F_\emptyset$ for all $, F_C \tilde{\in} v(\alpha)$.

Hence $F_B^b = (\tilde{F}_B^c)^b$.

iii) **To prove**: $\tilde{F}_B^c = .\ \bar{F}_B \tilde{\setminus} F_B^\circ$

Now, $\bar{F}_B \tilde{\setminus} F_B^\circ = \bar{F}_B \tilde{\cap} (F_B^\circ)^{\tilde{c}}$

$= \bar{F}_B \tilde{\cap} (\tilde{\cup}_{F_{B_i} \tilde{\subseteq} F_B,\, F_{B_i} \tilde{\in} \tau} F_{B_i})^{\tilde{c}}$

$\bar{F}_B \tilde{\setminus} F_B^\circ = \bar{F}_B \tilde{\cap} (\tilde{\cap} \tilde{F}_{B_i}^c)$

$= \bar{F}_B \tilde{\cap} \overline{\tilde{F}_B^c}$

$= \tilde{F}_B^c$.

Hence $\tilde{F}_B^c = .\ \bar{F}_B \tilde{\setminus} F_B^\circ$

CHAPTER 6

SOFT FUNCTION

6.1 Open base of a soft topology

Definition: 6.1.1

Let $(\tilde{X}, \tilde{\tau}, A)$ be a soft topological space. Then a subcollection B of $\tilde{\tau}$, containing $(\tilde{\emptyset}, A)$, is said to be an **open base** τ iff $\forall \tilde{x} \tilde{\in} (\tilde{X}, A)$ and for any soft open set (F, A) containing the soft element $\tilde{x}$, there exists (G, A) $\in$ B such that

$\tilde{x} \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$.

Examples: 6.1.2

Let X = {x, y, z, t}, A = {α, β} and

$\tilde{\tau} = \{(\tilde{\Phi}, A), (\tilde{X}, A), (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A), (F_7, A)\}$ where

$F_1(\alpha) = \{x\}$, $F_1(\beta) = \{t\}$;

$F_2(\alpha) = \{y\}$, $F_2(\beta) = \{z\}$;

$F_3(\alpha) = \{t\}$, $F_3(\beta) = \{x\}$;

$F_4(\alpha) = \{x, y\}$, $F_4(\beta) = \{z, t\}$;X

$F_5(\alpha) = \{y, t\}$, $F_5(\beta) = \{z, x\}$;

$F_6(\alpha) = \{x, t\}$, $F_6(\beta) = \{x, t\}$;

$F_7(\alpha) = \{x, y, t\}$, $F_7(\beta) = \{x, z, t\}$.

Then τ is a soft topology on $(\tilde{X}, A)$.

Let B = {($\tilde{\Phi}$, A), ($\tilde{X}$, A), (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A)}.

Then B forms an open base for $\tilde{\tau}$.

Proposition: 6.1.3

Let ($\tilde{X}$, $\tilde{\tau}$, A) be a soft topological space and B is an open base for $\tilde{\tau}$. Then every member of $\tilde{\tau}$ can be expressed as the elementary union of some members of B.

Remark: 6.1.4

Converse of Proposition 6.1.3 is not true. Consider the soft topological space ($\tilde{X}$, $\tilde{\tau}$, A).

Let B = {($\tilde{\Phi}$, A), ($\tilde{X}$, A), (F_1, A), (F_2, A), (F_3, A)}.

Then B satisfies the condition of Proposition 6.1.3 but B is not an open base for $\tilde{\tau}$ as the soft element $\bar{x} \tilde{\in}$ (F_6, A) but there is no soft set in B containing $\bar{x}$ and contained in (F_6, A).

Proposition: 6.1.5

If a collection B of soft sets of $\tilde{S}(X)$ forms an open base of a soft topological space ($\tilde{X}$, $\tilde{\tau}$, A), then the following conditions are satisfied:

(i) ($\tilde{\Phi}$, A) ∈ B.

(ii) $(\tilde{X}, A)$ is elementary union of some members of B.

(iii) If $(F_1, A), (F_2, A) \in B$ and $\tilde{x} \tilde{\in} (F_1, A) \cap (F_2, A)$, then there exists

$(F_3, A) \in B$ such that $\tilde{x} \tilde{\in} (F_3, A) \tilde{\subseteq} (F_1, A) \cap (F_2, A)$.

Remark: 6.1.6

Converse of Proposition 6.1.5 is not true.

Consider the soft topological space $(\tilde{X}, \tilde{\tau}, A)$ of Example 6.1.2 and let B as in Remark 61..4. Then B satisfies all the condition of Proposition 6.1.5 but B is not base for the soft topology $\tilde{\tau}$.

6.2 Soft function and soft continuous function

Definition of soft mapping has been given using 'soft point' concept, we introduce here a definition of soft function using the concept of 'soft element'.

Definition: 6.2.1

Let X and Y be two non- empty sets and $\{f_\lambda : X \to Y, \lambda \in A\}$ be a collection of functions. Then a function $f : SE(\tilde{x}) \to SE(\tilde{y})$ defined by $[f(\tilde{x})](\lambda) = f_\lambda(\tilde{x}(\lambda)), \forall \lambda \in A$ is called a **soft function**.

Definition: 6.2.2

Let $f : SE(\tilde{x}) \to SE(\tilde{y})$ be a soft function. Then

(i) image of a soft set (F, A) over X under the soft function f, denoted by $f[(F, A)]$, is defined by $f[(F, A)] = SS\{f(SE(F, A))\}$

i.e. $f[(F, A)](\lambda) = f_\lambda(F(\lambda)), \forall \lambda \in A$.

(ii) inverse image of a soft set (G, A) over Y under the soft function f, denoted by $f^{-1}[(G, A)]$, is defined by $f^{-1}[(G, A)] = SS\{f^{-1}(SE(G, A))\}$ i.e. $f^{-1}[(G, A)](\lambda) = f^{-1}(G(\lambda)), \forall \lambda \in A$.

Definition 6.2.3

Let $f : SE(\tilde{x}) \to SE(\tilde{y})$ be a soft function associated with the family of functions $\{f_\lambda : X \to Y, \lambda \in A\}$. Then f is said to be

(i) injective if $\tilde{x} \neq \tilde{y}$ implies $f(\tilde{x}) \neq f(\tilde{y})$.

(ii) surjective if $f(\tilde{x}, A) = (\tilde{y}, A)$.

(iii) bijective if both injective and surjective.

Proposition: 6.2.4

Let X and Y be two non- empty sets and A be the parameter set.

Also let $f : SE(\tilde{x}) \to SE(\tilde{y})$ be a soft function associated with the family of functions $\{f_\lambda : X \to Y, \lambda \in A\}$. If $(F, A) \in S(\tilde{x})$ then

(i) $ff^{-1}(F, A) \tilde{\subseteq} (F, A)$.

(ii) $(F, A) \tilde{\subseteq} f^{-1}f(F, A)$.

Proposition: 6.2.5

Let X and Y be two non- empty sets and A be the parameter set.

Also let $f : SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft function associated with the family of functions $\{f_\lambda : X \to Y, \lambda \in A\}$. If $(F_1, A), (F_2, A) \in S(\tilde{X})$ then

(i) $(F_1, A) \tilde{\subseteq} (F_2, A) \Rightarrow f[(F_1, A)] \tilde{\subseteq} f[(F_2, A)]$.

(ii) $f[(F_1, A) \cup (F_2, A)] = f[(F_1, A)] \cup f[(F_2, A)]$.

(iii) $f[(F_1, A) \cap (F_2, A)] \tilde{\subseteq} f[(F_1, A)] \cap f[(F_2, A)]$.

(iv) $f[(F_1, A) \cap (F_2, A)] = f[(F_1, A)] \cap f[(F_2, A)]$, if f is one- one.

Proposition: 6.2.6

Let X and Y be two non- empty sets and A be the parameter set.

Also let $f : SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft function associated with the family of functions $\{f_\lambda : X \to Y, \lambda \in A\}$. If $(F_1, A), (F_2, A) \in S(\tilde{X})$ then

(i) $(F_1, A) \tilde{\subseteq} (F_2, A) \Rightarrow f^{-1}[(F_1, A)] \tilde{\subseteq} f^{-1}[(F_2, A)]$.

(ii) $f^{-1}[(F_1, A) \cap (F_2, A)] = f^{-1}[(F_1, A)] \cap f^{-1}[(F_2, A)]$.

(iii) $f^{-1}[(F_1, A) \cup (F_2, A)] = f^{-1}[(F_1, A)] \cup f^{-1}[(F_2, A)]$.

Definition: 6.2.7

Let $(\tilde{X}, \tilde{\tau}, A)$ and $(\tilde{Y}, \tilde{\gamma}, A)$ be two soft topological spaces and

$f : SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft function associated with the family of

functions

$\{f_\lambda : X \to Y, \lambda \in A\}$.

Then we denote this soft function as $f : (\tilde{X}, \tilde{\tau}, A) \to (\tilde{Y}, \tilde{\gamma}, A)$.

Now $f : (\tilde{X}, \tilde{\tau}, A) \to (\tilde{Y}, \tilde{\gamma}, A)$ is said to be **soft continuous** at $\tilde{x}_0 \tilde{\in}$

$(\tilde{X}, A)$, if for every $(V, A) \in \tilde{\gamma}$ such that $\tilde{f}\ x_0 \tilde{\in} (V, A)$, there exists $(U, A) \in \tilde{\tau}$ such that $\tilde{x}_0 \tilde{\in} (U, A)$ and $\mathbf{f}\ (U, A) \tilde{\subseteq} (V, A)$.

f is said to be **soft continuous** on $(\tilde{X}, \tilde{\tau}, A)$ if it is soft continuous at each soft element $\tilde{x}_0 \tilde{\in} (\tilde{X}, A)$.

Proposition 6.2.8

Let $(\tilde{X}, \tilde{\tau}, A)$ and $(\tilde{Y}, \tilde{\gamma}, A)$ be two soft topological spaces and

$\mathbf{f} : (\tilde{X}, \tilde{\tau}, A) \to (\tilde{Y}, \tilde{\gamma}, A)$ be a soft function.

Then the followings are related as follows:

(i) $\Leftrightarrow$ (ii), (ii) $\Leftrightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

(ii) f is soft continuous.

(iii) For all $(V, A) \in v$, $\mathbf{f}^{-1}(V, A) \in \tilde{\tau}$.

(iv) There exists a sub base $\wp$ for v such that $f^{-1}(V, A) \in \tilde{\tau}$ for all $(V, A) \in \wp$.

For any closed soft set $(F, A) \in S(\tilde{Y})$ in $(\tilde{Y}, \tilde{\gamma}, A)$,

$\mathbf{f}^{-1}(F, A)$ is soft closed in $(\tilde{X}, \tilde{\tau}, A)$.

Remark: 6.2.9

In Proposition 6.2.8 (iv) $\Rightarrow$ (i) does not hold.

Let $X = \{x, y, z\}$ and $A = \{\alpha, \beta\}$.

Let $\tilde{\tau}_1 = \{(\tilde{\Phi}, A), (\tilde{X}, A), (F, A)\}$ and

$\tilde{\tau}_2 = \{(\tilde{\Phi}, A), (\tilde{X}, A)\}$,

where $F(\alpha) = \{x, y, z\}$, $F(\beta) = \{x, z\}$.

Then $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are soft topologies on $(\tilde{X}, A)$.

Consider the soft function $i : (\tilde{X}, \tilde{\tau}_2, A) \to (\tilde{X}, \tilde{\tau}_1, A)$ corresponding to the identity function $i : X \to X$.

Then $i^{-1} : (\tilde{X}, \tilde{\tau}_1, A) \to (\tilde{X}, \tilde{\tau}_2, A)$ maps all soft closed sets of $\tilde{\tau}_1$ to soft closed sets of $\tilde{\tau}_2$.

But $i^{-1}(F, A) \not\in \tilde{\tau}_2$.

Therefore, $i : (\tilde{X}, \tilde{\tau}_2, A) \to (\tilde{X}, \tilde{\tau}_1, A)$ is not soft continuous function.

Definition: 6.2.10

A soft function $\mathbf{f} : (\tilde{X}, \tilde{\tau}, A) \to (\tilde{Y}, \tilde{\gamma}, A)$ is said to be

(i) soft open if f maps soft open sets of $\tilde{\tau}$ to soft open sets of $\tilde{\gamma}$.

(ii) soft closed if f maps soft closed sets of $\tilde{\tau}$ to soft closed sets of $\tilde{\gamma}$.

Definition: 6.2.11

A soft function $\mathbf{f} : (\tilde{X}, \tilde{\tau}, A) \to (\tilde{Y}, \tilde{\gamma}, A)$ is said to be soft homeomorphism if

(i) f is bijective

(ii) f is soft continuous

(iii) f^{-1} is soft continuous.

Proposition: 6.2.12

Let $f : (\tilde{X}, \tilde{\tau}, A) \to (\tilde{Y}, \tilde{\gamma}, A)$ be a soft function.

Then the followings are equivalent:

(i) f is a soft homeomorphism.

(ii) f is bijective, f , f^{-1} are soft continuous.

(iii) f is bijective, soft open and soft continuous.

(iv) f^{-1} is a soft homeomorphism.

A STUDY ON MODERN CRYPTOGRAPHY

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

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Submitted by

M. MUGESH PRIYA

Reg.No: 19SPMT19

Under the guidance of

Ms. J. JENIT AJITHA M.Sc., M.Phil.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON MODERN CRYPTOGRAPHY** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by

M. MUGESH PRIYA (Reg. No: 19SPMT19)

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON MODERN CRYPTOGRAPHY"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. J. JENIT AJITHA M.Sc., M.Phil.,** Assistant Professor, Department of Mathematics (SSC),**St. Mary's College (Autonomous), Thoothukudi.**

M. Mugesh Priya

Station: Thoothukudi **Signature of the Student**

Date: 10 . 04 . 2021

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CONTENT

Chapter 1

CHAPTER 1

INTRODUCTION

Cryptography is defined as the study of mathematical techniques related to some aspects of information security. Cryptography is used in applications present in technologically advanced societies; examples include the security of ATM cards, computer passwords and electronic commerce. The goals of cryptography are Confidentiality, Data integrity, Authentication and Non-repudiation. Confidentiality is a service used to keep the content of information secret from all expect the authorized users. Data integrity is a service which addresses the unauthorized alteration (addition, deletion, modification) of data. Authentication is a service related to identification. This function is applied to both entities and information itself. In entity authentication, the claimant must identify himself or herself to the verifier. This can be done with one of the three kinds of witness: something known, something possessed or something in him or her. A secret known only by the claimant for example a password, a PIN comes under something known. Some identification card, a passport or a driving license possessed by the claimant belong to the second kind. Some inherent characteristics like conventional signature, finger print, voice, facial characteristics, retinal pattern belong to the third kind.

Knowledge of the password is assumed to guarantee that the user is authentic. The weakness in this system for transactions is that passwords can often be stolen, accidentally revealed, or forgotten. For this reason, Internet business and many other transactions required a more stringent authentication process. The use of digital certificates issued and verify by a Certificate Authority (CA) as part of a public key infrastructure is likely to become the standard way to perform authentication on the internet. Non-repudiation is a service which prevents an entity from denying previous commitments or actions. When disputes arise due to an entity denying that certain actions were taken, a mean to resolve the situation is necessary. A procedure involving a trusted third party is needed to resolve the dispute.

Assume a sender Alice (as in commonly used) wants to send a message m to a receiver referred to as Bob. She uses an insecure communication channel. For example, the channel could be a computer network or a telephone line. There is a problem if the message contains confidential information. The message could be intercepted and read by a eavesdropper or even worse, the adversary might be able to modify the message during transmission in such a way that the legitimate receipt Bob does not detect the manipulation.

Providing confidentiality is not the only objective of cryptography. Cryptography is also used to provide solutions for other problems like Data Integrity: the receiver of a message should be able to check whether the message was modified during transmission, either accidentally or deliberately. No one should be able to substitute a false message for the original message, or parts of it. Authentication: the receiver of a message should be able to verify its origin. No one should be able to send a message to Bob and pretend to be Alice (data origin authentication). When initiating a communication, Alice and Bob should be able to identify each other (entity authentication). Non-repudiation: the sender should not be able to later deny that she or he didn't send a message.

Many of us realize that mathematics has more application than we know; growing up we were constantly reminded that math can apply to almost everything. One of those things is keeping our sensitive data secure. One of the earliest uses of cryptography dates back two-thousand years, developed when Julius Caesar realized the need to encode military messages to and from his commanders. It was a simple substitution cipher where every letter would be replaced by a different letter. For Caesar's Cipher the shift was 3 letters; so every "A" in your original message would be replaced by a "D", every "B" by a "E" and so on. To decrypt the message simply shift 3 letters in the opposite direction. Frequency analysis provides a method to easily break substitution ciphers like Caesar's Cipher, especially with the aid technology, so cryptographers had to invent new ways to encode information. Symmetric ciphers like the Data Encryption Standard (DES) and the Advanced Encryption Standard (AES) encode blocks of information at a time using a private key that all parties trying to communicate must know. Asymmetric ciphers like RSA and Diffie-Helman key exchange use computationally hard mathematical problems to allow one key to be public, so anyone can send an encrypted message, but also incorporate a private key that only selected individuals know. The private key allows one to decrypt the message.

Sometimes we are required to have sensitive data stored on a database, like a password or a pin number to a debit card. These are not messages that you want someone to decode using a key, you just want the data stored to verify it is you every time you log into a website or make a withdrawal from an ATM. For this, hash functions have proven useful. Recent applications of hash functions have exposed the world of cryptography to a much larger audience. For example, the invention Blockchain technology, and as a consequence Bitcoin, is shaping the further: from being able to securely send or receive money from someone you may not trust, to establishing a smart contract between two parties that everyone in the network can bear witness to. As we will see, Cryptography is a revolutionary application of mathematics.

Chapter 2

CHAPTER 2

DATA ENCRYPTION STANDARD

2.1 Motivation

In 1977, the National Security Agency (NSA) announced that there was a need for a secure standardized cipher for commercial use. IBM created the Data Encryption Standard (DES), a block cipher that encodes blocks of bits at a time, as opposed to a stream cipher, which encodes only one bit at a time. Although stream ciphers are impossible to break using a cryptographic secure random number generator, they are too impractical for general applications since keys cannot be reused. Block ciphers are more practical because keys are of reasonable length, at most 256 bits, and can be used securely for a whole session of communication.

2.2 Security

In the 1970's, block ciphers were in high demand because of how efficient they are in encoding information. For example, the DES can encrypt 64 bits at a time. The challenge was creating a block cipher both secure from attacks that plague the block cipher of the past and, as later discovered, differential cryptanalysis attacks. The inventor of information theory, Claude Shannon, stated that a secure block cipher must have two main attributes: confusion, which is a basis substitution table, and diffusion, which means a single bit should affect the entire message. Although differential cryptanalysis was not public knowledge until 18 years after the publication of DES, the NSA and IBM research teams knew of the attack and created their substitution tables to be resilient to this attack.

Today, DES is no longer secure after being under the microscope of so many cryptographers for over 20 years. But 3DES, which encrypts using DES three times in a row, is still a commonly used block cipher.

2.3 Understanding the Data Encryption Standard

DES is not too complex from an abstract perspective; encryption relies on confusion and diffusion functions. First, DES takes 64 bits of data at a time. The 64 bits will go through 16 rounds that all have the same functionality. First, the 64 bits are split into two groups, each consisting of 32 bits. Let us denote the first set of 32 bits L_0, the last 32 bits by R_0. In any round of DES, only the first 32 bits are encrypted. For example, round 1 would encrypt L_0 and set the encrypted 32 bits to be R_1. The set R_0 is kept the same and set to L_1. For round 2 of encryption, L_1 is encrypted and set to be R_2, while R_1 is unchanged and set to L_2. This process continues for 16 rounds.

2.4 Encryption Method

Now, with a general understanding of how DES encrypts, we can look at the inner workings of each round and how the encryption actually works. As stated before, 64 bits are encrypted over 16 rounds. All rounds have the same operations, so we need only understand one round. For simplicity, let us look at the first round of DES for 64 bits of data. The 64 bits are divided in half, giving us $L_0 and\ R_0$ as described above. The block of 32 bits, R_0, has two roles in the first round: R_0 is set to be L_1 ready to be encrypted in the next round, and R_0 along with the round key K_0, are input into a particular function f, which we described below. We set $R_1 = L_0 \oplus f(R_0, K_0)$, where $\oplus$ denotes the XOR operation.

"Exclusive or" (XOR) is a logical operation that outputs true only when inputs differ. For example,

$$
\begin{aligned}
& 10001110_2 \\
\oplus\ & 10011000_2 \\
=\ & 00010110_2
\end{aligned}
$$

The function f is where the confusion and diffusion take place in the form of

Table 2.1: Expansion Function bit Mapping

Input	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Output	31	0	1	2	3	4	3	4	5	6	7	8	7	8	9	10
	11	12	11	12	13	14	15	16	15	16	17	18	19	20	19	20
	21	22	23	24	23	24	25	26	27	28	27	28	29	30	31	0

Table 2.2: S_1 Table With Decimal Numbers

S_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	14	4	13	1	2	15	11	8	3	10	6	12	5	9	0	7
1	0	15	7	4	14	2	13	1	10	6	12	11	9	5	3	8
2	4	1	14	8	13	6	2	11	15	12	9	7	3	10	5	0
3	15	12	8	2	4	9	1	7	5	11	3	14	10	0	6	13

four operations. First, the 32 bits pass through an expansion function E, making the 32 bits into 48 bits. Second, the 48 bits are XOR'd with the 48-bit round sub-key. Third, the 48 bits are separated into eight groups of 6 bits each and are changed via corresponding substitution table S: S_1 for the first group, S_2 for the second group, and so on. Each substitution table inputs 6 bits and outputs 4 bits, leaving us with 32 bits after the substitution. Finally the 32 bits are permuted.

The expansion function E applies diffusion, something essential for block ciphers E takes 32 bits and outputs 48 bits. This is done by sending half of the 32 bits to two location. The other half of the 32 bits are mapped to just one bit in the output of E. See Table:2.1 for a complete mapping of bits performed by E.

After the expansion function, the 48 bits are then XOR'd with the 48 bit round key. We then still have 48 bits.

The 48 bits, in order from left to right, are separated into 8 groups, each containing 6 bits of data. Each group of 6 bits have a particular address in a substitution box. There are 8 different substitution boxes, one for each group. Intuitively, the first group would use box S_1, the second group box S_2, and so on. We will illustrate this process with an example. For simplicity we will look at only the first 6 bits of data and calculate the output using S_1.

Reading the S_1 table may seem somewhat counterintuitive at first, but the process is fairly straightforward after seeing it done. For example, if the first 6 bits of data are 011001_2, we would convert the middle four bits 1100_2 to decimal numbers giving us 12. This directs us to look at column 12. We then take the outside two bits 01_2 and convert it to the corresponding decimal number 1. This tells us to look at row 1. Finally, we take the entry in column 12, row 1, and convert back to base 2. Thus, $S_1(011001_2) = 9 = 1001_2$. Another computation, $S_1(110101_2) = 3 = 0011_2$. So, 6 bits in gives us 4 bits out. After all 8 groups have passed through their corresponding S-Box, we are left with 32 bits. This S-box is the element of confusion that DES uses.

Finally, the bits go through a permutation which are then output to encrypt L_i via $R_{i+1} = L_i \oplus f(R_i, K_i)$.

2.5 Key Generation

DES is a block cipher that encrypts data over 16 rounds. For each of those rounds an encryption key is used. DES starts with a 64-bit key that undergoes multiple permutations to generate a unique key for each round. The key generation starts by permutation choice 1 (PC1), Table 2.3. The number in Table 2.3 represents the new location for each bit. For example, the fifty-seventh bit in the original 64-bit key would be the first bit after the permutation, the forty-ninth bit would become the second bit, and so on. Permutation choice 1 is only performed once, on the original 64-bit key. We also note that only 56 bits of the original 64 bits remain after PC1. For this reason DES has a cryptographic strength of a 56-bit cipher, not the strength of a 64-cipher. Next, the 56 bits are split into two groups, each group being 28 bits long. The next permutation is a left side of bits in the two groups. For rounds 1,2,9 and 16, the shift is one spot to the left. For all the remaining rounds, the shift is two spots to the left. We observe that the left shift is one spot to the left for four rounds and two spots for the remaining twelve rounds; thus the total number of left shifts is 28 spots. This coincides with the fact that the two groups are each 28 bits long. The final permutation is called permutation choice 2 (PC2) and works in the same way as PC1. See Table 2.4 for the permutation that PC2 performs. The 48 bits that remain after PC2 are used as the encryption key. In general, key generation would go as follows. The original 64-bit key undergoes the first permutation PC1, leaving 56 bits. The 56 bits would be split into two groups each 28 bits long.

Table 2.3: Permutation Choice 1

57	49	41	33	25	17	9	1	58	50	42	34	26	18
10	2	59	51	43	35	27	19	11	3	60	52	44	36
63	55	47	39	31	23	15	7	62	54	46	38	30	22
14	6	61	53	45	37	29	21	13	5	28	20	12	4

Table 2.4: Permutation Choice 2

14	17	11	24	1	5	3	28	15	6	21	10
23	19	12	4	26	8	16	7	27	20	13	2
41	52	31	37	47	55	30	40	51	45	33	48
44	49	39	56	34	53	46	42	50	36	29	32

Let us denote the first 28 bits c_0 and the second 28 bits d_0. Both c_0 and d_0 are permuted via a left shift of one spot. We will call the first group c_1 and the second group d_1 after the left shift. Next, the 56 bits that make up c_1 and d_1 undergo the second permutation choice PC2.The 48 bits that remain are used as the first key for encryption. For the second round of key generation, c_1 and d_1 undergo a left shift of one spot; after the left shift c_2 and d_2 are permuted via PC2 where the resulting 48 bits are used for the second round of encryption. For the third round of encryption c_2 and d_2 are shifted two spots to the left. The resulting groups c_3 and d_3 are permuted via PC2 where the third key for encryption is made. This process continues une til 16 keys are generated for each round of encryption.

2.6 Decryption

After all 16 rounds of encryption, the sender sends the final results L_{16} and R_{16} to the intended receiver. Thus, the receiver will have L_{16} and R_{16} directly from the sender. For symmetric ciphers, it is necessary for all parties to have the encryption key, so it is assumed that the receiver has access to the key and the ability to compute all round sub-keys. Once the receiver has all round keys, they need only do the steps in reverse order.

Starting from the last round, L_{16}=R_{15}, so the first round of decryption needs only the calculation of L_{15}. We know from encryption that R_{16}=$L_{15}\oplus f(R_{15}, K_{15})$, thus $R_{16}\oplus f(R_{15}, K_{15}) = L_{15}$. This process continues until L_1 and R_1, the original message , are retrieved.

Chapter 3

CHAPTER 3

RSA CRYPTOSYSTEM

3.1 Motivation

The advanced encryption standard is a secure and efficient symmetric cipher. The main disadvantage to AES is the need to establish a key that all parties wishing to communicate must know. If the parties cannot physically meet, they must send the key over a channel, which runs the risk of their key being intercepted and, in turn, all their encrypted message may be easily broken. A solution to this problem is to have one key that encrypts a message, a public key, and one key that decrypts the message, a private key.

3.2 Security

The simplicity of the RSA cryptosystem's security adds to its beauty. It places its bet on the simple fact that it is considered a hard problem to factor numbers. More specifically, numbers composed of two primes. Factoring numbers will always involve some trial and error. A problem like prime factoring 91 can easily be done by hand. Factoring 4168723211, which is the product of two 5-digit primes, is much more difficult to do without the assistance of a computer, even knowing the length of the two primes. However, when the prime factors multiplied together are larger than 2^{2048}, even modern day super computers take countless years of computing time to factor the composite number. On the other hand, multiplication is considered an easy problem because there is a direct way to compute products. It is not difficult to compute the product 59359×70229

without the help of a computer. This is the core principal of an asymmetric cipher. It is easy to compute one way, multiplication, but very difficult to take the inverse, factoring.

3.3 Method

To set up an RSA system, there are six variables that a user, Alice, would first have to determine. Since RSA is an asymmetric algorithm, a sender, Bob, does not have to compute a key; most of the work is done by Alice. First, Alice would select two primes, normally denoted as p and q. Then, Alice computes $n = p * q$. Next, Alice computes $\Phi(n)$, which normally would not be easy. However, since Euler's Phi Function Φ is multiplicative, along with the fact that $\Phi(u) = u - 1$ if u is prime we have,

$$\begin{aligned}\Phi(n) &= \Phi(p * q)\\ &= \Phi(p) * \Phi(q)\\ &= (p-1) * (q-1).\end{aligned}$$

So, $\Phi(n) = (p-1) * (q-1)$, easy enough for Alice to compute. Alice would then select a value e that is relatively prime to both $(p-1)$ and $(q-1)$, making it relatively prime to $\Phi(n)$.

Finally, Alice calculates d such that $e * d \equiv 1 \; mod \; \Phi(n)$. Proof of $d's$ existence is a consequence of Bezout's Identity.

Theorem 1 (Bezout's Identity)

Let a and b be integers with greatest common divisor d. Then, there exist integers x and y such that $ax + by = d$. More generally, the integers of the form $ax + by$ are exactly the multiples of d.

For our application, take $a = e, b = \Phi(n)$, and $d = GCD(e, \Phi(n)) = 1$. Note that the d from the theorem is not the same d that we are trying to show exists. Now, from the theorem we have,

$$de + k\Phi(n) = 1$$

Thus

$$de + k\Phi(n) \ mod \ \Phi(n) = 1 \ mod \ \Phi(n),$$

from which it follows that

$$de \ mod \ \Phi(n) = 1 \ mod \ \Phi(n).$$

In practice, Alice would calculate d via the Extended Euclidean Algorithm. Alice publishes (e, n) for anyone to encrypt with and keeps (d, n) as a personal decryption key. The method, in general, works as follows. Bob. who wants to send plain-text message m, would first compute $c \equiv m^e \ mod \ n$. Bob sends c to Alice. Alice computes $m \equiv c^d \ mod \ n$, thus getting the plain text back. An eavesdropper, Eve, knows e and n, as they are publicly known, as well as c, which is the message Bob sent. Eve is missing the decryption key d, as it is known only by Alice. Why does this work? After all, e and d

were inverse with respect to $\mathbb{Z}_{\Phi(n)}$ and the actual cipher text is computed in $\mathbb{Z}_{\mathrm{n}}$. The answer is a consequence of Euler's Theorem.

Theorem 2 (Euler's Theorem)

Let a and n be two coprime positive integers, then $a^{\Phi(n)} \equiv 1 \; mod \; n$.

Manipulation of Euler's Theorem gives us the following result:

$$a^{\Phi(n)} \equiv 1 \; mod \; n$$

$$\Longrightarrow \quad a^{\mathrm{k} \cdot \Phi(n)} \equiv 1 \; mod \; n$$

$$\Longrightarrow \quad a * a^{\mathrm{k} \cdot \Phi(n)} \equiv a * 1 \; mod \; n$$

$$\Longrightarrow \quad a^{\mathrm{k} \cdot \Phi(n)+1} \equiv a \;\; mod \; n.$$

Now, since $ed = 1 \; mod \; \Phi(n)$, there is some $k \in \mathbb{Z}$ such that $ed = k * \Phi(n) + 1$. So Alice's

Decryption is computed as follows:

$$\begin{aligned} c^d \; mod \; n &\equiv (m^e)^d \; mod \; n \\ &\equiv m^{ed} \; mod \; n \\ &\equiv m^{k\Phi(n)+1} \; mod \; n \\ &\equiv m \; mod \; n. \end{aligned}$$

Ergo, the plain text message is retrieved.

3.4 A Worked Example

Let Alice be the user, Bob the sender, and Eve an eavesdropper that intercepts everything sent between Alice and Bob. Alice randomly selects $p = 5$ and $q = 11$ and computes $n = 55$. Alice also computes $\Phi(55) = 4 * 10 = 40$. She has free choice for e and select 3 since it is relatively prime to both 4 and 10. Using the Extended Euclidean Algorithm, Alice computes $d = 27$. Alice then publishes (3, 55) for anyone to use. Bob wants to send Alice the secret message $m = 18$. Bob computes

$$18^3 \; mod \; 55 \equiv 5832 \; mod \; 55$$

$$\equiv 2 \; mod \; 55,$$

and sends Alice $c = 2$. Now Alice, having the private key $d = 27$, computes

$$2^{27} \; mod \; 55 \equiv 134217728 \; mod \; 55$$

$$\equiv 18 \; mod \; 55,$$

Table 3.1 A brute-force attack

m	$\frac{m^3-2}{55}$
1	$-\frac{1}{55}$
2	$\frac{6}{55}$
3	$\frac{7}{55}$
$\vdots$	$\vdots$
18	$\frac{5830}{55} = 106$
$\vdots$	$\vdots$
73	7073
$\vdots$	$\vdots$
128	38130
$\vdots$	$\vdots$

retrieving the plain-text message that Bob sent. Eve has cipher text 2, public modulus 55, and public key 3. Eve knows Bob's encryption scheme:

$m^3 \, mod \, 55 \equiv 2$. Therefore, $m^3 = 55k + 2$ for some $k \in \mathbb{N}$. Thus, $\frac{m^3-2}{55} = k \in \mathbb{N}$. In this case the code may be somewhat easily broken via trial and error and using technology. Since $\frac{18^3-2}{55} = 106 \in \mathbb{N}$, we see that $m = 18$ is a possible message. However, there are more possible messages as shown in Table 3.1, so Eve would need to see if any make sense. For the purposes of illustration we had Alice choose very low primes for p and q and a very simple public key. In practice, the primes would be much larger and the private key less trivial, thereby rendering an encryption that would, with todays technology, be nearly impossible to crack via brute force.

One may immediately think, $\Phi(n)$ is very close to n since $n = p * q$ and $\Phi(n) = (p-1)(q-1)$, and that a descending approach from n to find $\Phi(n)$ could be a possible attack. Of course, if $\Phi(n)$ was discovered by an attacker they need not worry about factoring n, as they can just compute d via the Extended Euclidean Algorithm. The problem with this strategy is that in practice, $p \geq 2^{1024}$ and $q \geq 2^{1024}$, making $n \geq 2^{2048}$, and $\Phi(n) \geq 2^{2048}$ which, unfortunately for any attacker, yields 2^{2047} integers with bit length 2048. Let us put this in perspective by converting 2^{2047} bits to something more tangible, decimal digits:

$$2^{2047} = 10^u$$

$$\Rightarrow \qquad log\, 2^{2047} = \log 10^u$$

$$\Rightarrow \qquad 616 \approx u.$$

Thus, 2047 bits approximately 616 decimal digits. The most powerful super computers in the world (as of this writing) can compute 20 decimal digits per second. Thus, we find it would take approximately 10^{596} seconds or greater than $3\times\ 10^{589}$ years to find $\Phi(n)$ with a backward approach starting at n. Needless to say, users following the RSA protocol are safe from the naïve attacker.

3.5 Application

In practice, RSA encryption involves extremely large numbers and it is not always practical to encrypt a whole message using a RSA protocol. A potential application of RSA would be the establishment of a session key, a key used for a particular session of sending encrypted information. This is called a key exchange protocol. It could work many ways, one way would be to encrypt a message using AES. We know AES is an extremely efficient encryption algorithm; the main downside is that the two parties communicating need to know the key. One could encrypt the AES key with RSA and send the encrypted key and the encrypted message over an insecure network. The receiver would be able to calculate the AES key using their RSA private key. Then using the decrypted AES key, decrypt the message.

Chapter 4

CHAPTER 4

HASH FUNCTIONS

4.1 Motivation

Hash functions are a different type of cryptographic tool. The purpose of a hash function is not to encode information, rather it serves as a one-way function to verify the integrity of a message. An example is Alice and Bob playing a game of rock-paper-scissors over the Internet. How might we prevent one person from changing their answer in order to win? This is where a hash function comes in. Like any function, the same input yields the same output. For example, suppose Alice selects scissors and Bob selects rock. They each will hash their choice and sent the result to one another. Bob reveals to Alice that he selected rock, if Alice were to respond with a lie and claim she selected paper, Bob can hash "paper" and compare the result with the hash originally sent by Alice. Bob will realize that the hash does not match and know Alice was lying. Another typical use of a hash function is storing pin numbers in a database. When you first create a pin number it is hashed and the result is stored in a database. The purpose of this is to ensure that even if an attacker were to access the database, all they would be able to retrieve are the hash results, not the actual pin. When you use your debit card and enter your pin, the pin is hashed and compared to the hash recorded on the database, if the two hashes match, it is assumed that the entered pin was correct and the ATM will give you the requested money.

A hash function h is considered secure if it has the following attributes. First, a hash function must be able to take an arbitrary number of characters and output a fixed number of characters. If "Hello" is hashed through Sha-256, a very popular hashing algorithm, the result is the 64 character output

$$66a045b452102c59d840ec097d59d9467e13a3f34f6494e539ffd32c1bb35f18.$$

If "Hello my name is Samuel, I am a math major at California State University San Bernardino" is hashed through Sha-256, the result is

$$b813f6fe19598833d21e21b14eda378f90ace49dfe84da3301a9b7ea7e073$$

Which has 64 characters, the same number of characters as the previous hash.

The second property of a secure hash function is that it must be second pre-image resistant. A $collision$ occurs when two different messages have the same hash. Collision is unavoidable as a consequence of the first property since there are fewer possible outputs than inputs. Second pre-image resistance is described as follows. Given a message x_1 and hash $y_1 = h(x_1)$, it should be infeasible to come up with an input $x_2 \neq x_1$ such that $y_1 = h(x_2)$.

A third property is collision resistance. If should be infeasible to find two distinct messages x_1 and x_2 such that $h(x_1) = h(x_2)$.

A fourth property ties in with the idea of collision resistance. Given a hash y_1, it should be impossible to create a plain-text message x_1 where $h(x_1) = y_1$.

A fifth property is similar to what we have seen in previous chapters, diffusion. A small change in plain-text results in a completely different hash. Compare the hash of "Hello" given earlier as

66a045b452102c59d840ec097d59d9467e13a3f34f6494e539f f d32c1bb35f18.

to the hash of "hello" which is

5891b5b522d5df086d0 f f0b110fbd9d21bb4fc7163af34d08286a2e846f6be03.

Notice the apparent complete lack of similarity between the two hashes.

The final property is that hash functions should be one-way functions. Given a hash, there should not be a way to invert the hash to retrieve the plain-text. A hash is secure only if it has all the attributes described above.

4.2 Understanding Properties of Secure Hash Functions

We will now look at the purpose of the various properties for secure hash algorithms by considering potential attacks. The first property is primarily for efficiency. In practice it would be extremely inefficient to manually hash blocks of a message and then for the receiver to again hash blocks of the message and check every single block to verify that the message was not changed. By allowing any length messages, a hash must only be computed and checked once.

The rest of the properties are for security purposes. Second pre-image resistance aims to thwart the following attack. Imagine Alice sending \$10 to Eve, an active attacker who wants to change the dollar amount Alice is sending. If Eve were to change the \$10 to \$20, the hash would not match and the bank would not release the funds. It is extremely unlikely that the hashes would match for any random amount that Eve might choose. Eve must find a dollar amount x_1 such that $h(x_1) = h(10)$. If the hash function is not secure, Eve will be able to calculate such a value for x_1 and replace \$10 with the new amount.

Since the hash results would match, the bank would release the funds to Eve. This attack becomes easier to defend against by having a sufficiently large pool of hash results. For example, SHA-256 has a 256-bit hash output resulting in 2^{256} possible results. Along with the other properties, the only way to find a particular hash is by brute force. Every dollar amount hashed has a probability of $\frac{1}{2^{256}}$ to have the same hash as \$10.

Collision attacks are statistically harder to defend against. Imagine Eve finds two dollar amounts that have the same hash, $h(x_1) = h(x_2)$. Eve could list an item for sale at dollar amount x_1, then when someone is buying the item, switch x_1 with x_2, which would execute because the hashes match. This attack is very powerful because of the cascading effect that occurs while checking hash results. It goes as follows. Assume Eve is trying to find a collision in SHA-256 and can choose any dollar amount x_1 to start with. Then Eve can choose any $x_2 \neq x_1$ and check if $h(x_2) = h(x_1)$. The probability of a match is $\frac{1}{2^{256}}$, which implies the probability of the results not colliding is $1 - \frac{1}{2^{256}}$. If x_1 and x_2 do not collide, Eve chooses x_3 and computes $h(x_3)$. The major danger is that now Eve can check to see if $h(x_3) = h(x_1)$ or $h(x_1) = h(x_2)$. The probability of a collision is $\frac{2}{2^{256}}$ which implies the probability of not having a collision is $1 - \frac{1}{2^{256}}$. The probability of there not being a collision after 3 dollar amounts are hashed and checked is $\left(1 - \frac{1}{2^{256}}\right)\left(1 - \frac{2}{2^{256}}\right)$. If a collision still has not occurred, Eve can then select x_4 and compute $h(x_4)$ to check if $h(x_4) = h(x_1)$ or $h(x_4) = h(x_2)$ or $h(x_4) = h(x_3)$. The probability of a collision on this step is $\frac{3}{2^{256}}$, thus the probability of no collisions after four dollar amounts are hashed and checked is $(1 - \frac{1}{2^{256}})(1 - \frac{2}{2^{256}})(1 - \frac{3}{2^{256}})$. The probability that atleast t checks are required before achieving a collision is $\prod_{i=1}^{t-1}(1-\frac{i}{2^{256}})$. The formula may be used to estimate t, hence $t \approx 2^{\frac{n+1}{2}}\sqrt{\ln\frac{1}{1-\lambda}}$ where n is the number of bits and λ is the probability for atleast one collision. As mentioned before SHA-256 has 2^{256} different possible hash results or 256 bits. How many checks would be required to provide a 50% chance of collision? Substituting n = 256 and λ = .5 into the formula we would have

$$t \approx 2^{\frac{256+1}{2}}\sqrt{\ln(\frac{1}{1-.5})}$$

$$= \sqrt{2^{257}.\ \ln(2)}$$

$$< \sqrt{2^{257}.2}$$

$$= 2^{128}$$

We can see that the number of steps required for achieving a collision attack with 50% probability is about the square root of n. This indicates that SHA-256 is still secure from this type of attack because of its colossal pool of possible hash results. It is still infeasible to execute 2^{128} steps with today's technology, which is why SHA-256 is still used.

The remaining properties for hash functions are to ensure that the only viable attacks are brute force attacks.

BLOCKCHAIN TECHNOLOGY

4.3 "Satoshi Nakamoto's" White Paper

In 2008, a new type of cryptosystem protocol was published in the form of a white paper under the pseudonym "Satoshi Nakamoto". It is unknown who "Satoshi" is, but since the publication of his white paper, thousands of crypto assets have been developed. The technology went unseen by the majority of the population for many years. More recently, the boom in value towards various crypto assets caught the attention of mainstream media and investors hoping to ride Bitcoin or other altcoins to a Lamborghini. The term altcoin is used to refer to all crypto assets that are not Bitcoin.

4.4 Security of Blockchain

The core cryptographic property used by blockchains are hash functions; the Bitcoin blockchain uses SHA-256. The white paper starts by claming there is a problem with double spending money. The current way to avoid the double-spending problem is by including a mutually trusted third party, the bank for example. Satoshi proposed an alternate solution: "we propose a solution to the double-spending problem using a peer-to-peer network. The network timestamps transactions by hashing them into an ongoing chain of hash-based proof-of-work, forming a record that cannot be changed without redoing the proof-of-work" (Nakamoto 1). There are two main principles being introduced here. The first is that all transactions to be timestamped enter a block. That block is hashed along with the hash of the previous block. By chaining the blocks, using the hash of the previous block in the current block, an ongoing chain of timestamps is created in which each validates the timestamps of the previous block. Proof-of-work is the key security feature that makes the blockchain secure. The "work" for Bitcoin is to find a solution to a computationally hard mathematical problem. The problem is to find a value x_1 such that $h(x_1)$ starts with a given number zero bits. As we know from hash functions, the best way to find a particular hash is by brute force. The idea is that all transactions and items enter a block; if there is no current block then a block is created.

For the block to be completed a solution for the problem described above must be found. All CPUs on the network work together to find a solution for the block. Once the solution is found, it is published for other CUPs to verify. It is much easier to verify a solution than to find a solution as verification requires only hashing the proposed solution and checking to see if it is indeed a true solution. If the solution is correct, the block is verified and the CUPs work on creating a the next block. The hash of the previous block is included in the current block, forming a chain of blocks. Visit [4] to see the Bitcoin blockchain in action.

The second principle introduced in the quote is that the record of transactions cannot be changed without redoing the proof-of-work. A major part of the security is that for a typical attacker, the amount of CUP power and money required to attack the network is not profitable. If someone want to attack the network, for example reverse a previous transaction, the attacker must resolve the proof-of-work problem for the block where the transaction is stored, as well as all blocks that were chained after. In order for the attack to be successful, the attacker must force the network to accept the chain that contains the attacked block. The network accepts the longest chain as the true chain, so the attacker must chain more blocks on the attacked block than the original chain has. This type of attack is called the 51% attack because if an attacker possessed at least 51% of all computing power on the network, they can solve blocks faster than all of the rest of the network combined. Thus, if the attackers change a transaction they can chain blocks faster than the rest of the network and force the whole network to accept the chain with manipulated blocks.

4.4 Bitcoin

The first recorded use of Bitcoin as a currency was its use in an order of two pizzas from Papa John's. Reportedly the two parties agreed to trade 10,000 Bitcoin for the pizzas. On January 9, 2018, the value of a single Bitcoin was listed at $147,700,00 USD.

Chapter 5

CHAPTER 5

ADVANCED ENCRYPTION STANDARD

5.1 Motivation

After attacks on DES became too efficient, there was a call by the NSA for a new encryption standard. Unlike DES, which was developed in secret and then eventually published a few years later, the need for a new encryption standard was announced as a kind of competition. Many prominent cryptographers of the time formed teams and created ciphers. In addition to the requirement to be very secure, there were other criteria for submissions. Submissions must be 128-bit block ciphers, supporting 128-bit, 196-bit, and 256-bit key lengths. Submissions must also be computationally efficient, enough so for the commercial use the cipher was intended for. After a long evaluation process, Rijndael, the submission by Vincent Rijmen and Joan Daemen, was chosen and became the Advanced Encryption Standard (AES).

5.2 Security

Like DES, the security of AES comes from confusion and diffusion. The major improvements to security in AES were the increased block length to 128 bits and the key length. AES supports three different key lengths, 128, 198, and 256 bits. The different key lengths require a different number of rounds to be considered secure. A 128-bit key needs 10 rounds, a 196-bit key requires 12 rounds, and a 256-bit key calls for 14 rounds. The number of rounds for the various key lengths was determined by the team that developed the cipher.

Table 5.1: S-box for AES in Hexadecimal

S-Box	00	01	02	03	04	05	06	07	08	09	0A	0B	0C	0D	0E	0F
00	63	7C	77	7B	F2	6B	6F	C5	30	01	67	2B	FE	D7	AB	76
10	CA	82	C9	7D	FA	59	47	F0	AD	D4	A2	AF	9C	A4	72	A0
20	B7	FD	93	26	36	3F	F7	CC	34	A5	E5	F1	71	D8	31	15
30	04	C7	23	C3	18	96	05	9A	07	12	80	E2	EB	27	B2	75
40	09	83	2C	1A	1B	6E	5A	A0	52	3B	D6	B3	29	E3	2F	84
50	53	D1	00	ED	20	FC	B1	5B	6A	CB	BE	39	4A	4C	58	CF
60	D0	EF	AA	FB	43	4D	33	52	45	F9	02	7F	50	3C	94	A8
70	51	A3	40	8F	92	9D	38	F5	BC	B6	DA	21	10	FF	F3	D2
80	CD	OC	13	EF	5F	97	44	17	CF	A7	7E	3D	64	5D	19	73
90	60	B1	4F	DC	22	2A	90	88	46	EE	B8	14	DE	5E	0B	DB
A0	E0	32	3A	0A	49	06	24	5C	C2	D3	AC	62	91	95	E4	79
B0	E7	C8	37	6D	8D	D5	4E	A9	6C	56	F4	EA	65	7A	AE	08
C0	BA	78	25	2E	1C	A6	B4	C6	E8	DD	74	1F	4B	BD	8B	8A
D0	70	3E	B5	66	48	03	F6	0E	61	35	57	B9	86	C1	1D	9E
E0	E1	F8	98	11	69	D9	8E	94	9B	1E	87	E9	CE	55	28	DF
F0	8C	A1	89	0D	BF	E6	42	68	41	99	2D	0F	B0	54	BB	16

5.3 Understanding the Advanced Encryption Standard

The AES is a byte cipher in the sense that the operations that apply confusion and diffusion work with bytes; bytes are simply larger bits, 8 bits makes a single byte. The AES is also a round cipher, confusion and diffusion take place in multiple rounds. Before the first round, the original key K_0 is XOR'd byte by byte to the original 16 bytes of data. In all the rounds except the last, there are four operations. The first operation is the use of a substitution box. As we know from DES, this is an element of confusion. Unlike DES, there is only one substitution box in AES. After substitution, there is a shift row operation followed by a mix column operation, together applying diffusion among the bytes. Finally, there is a key, which is XOR'd byte for byte. The last round skips the mix column operation. The generation of the S-box for AES requires some general understanding of Galois fields, in particular GF(2^8). This is significant because the S-box has a mathematical generation that can be understood, as opposed to the S-boxes of DES that were made in secret and seemingly random.

3.4 Encryption Method

The key length does not actually change what operations take place in the encryption, although longer keys have more rounds. For simplicity we will look at encryption using a 128-bit key length which calls for 10 rounds. AES will encrypt 128 bits at a time, regardless of key length. AES works with bytes, so the 128 bits are broken up into bytes producing $\frac{128}{8} = 16$ bytes. In this paper, the 16 bytes will be converted to hexadecimal. The 16 bytes are XOR'd with the original key. Then the 16 bytes pass through the substitution box AES employs. The S-box for AES is more intuitive than the

S-box of DES. For example, if the byte in question was "C3", we would use "C" to determine which row to use, and "3" to determine which column. Referring to Table 5.1, we see "C3" would have an output of "2E". A byte of "F1" would use row F and column 1, giving output "A1". This is the confusion element of AES. The remaining operations for the round are easier to understand via a matrix. We name the 16 bytes of data after the original key: $A_0, A_1, A_2, \dots A_{15}$. Suppose after the S-box we have $S(A_0)=B_0$, $S(A_1) = B_1, \dots, S(A_{15}) = B_{15}$, Still leaving us with 16 bytes. Entering the bytes column-wise into a 4×4 matrix produces,

$$M_O = \begin{bmatrix} B_0 & B_4 & B_8 & B_{12} \\ B_1 & B_5 & B_9 & B_{13} \\ B_2 & B_6 & B_{10} & B_{14} \\ B_3 & B_7 & B_{11} & B_{15} \end{bmatrix}$$

A shift row operation will shift the first row zero spots to the left, the second row one spot to the left, the third row two spots to the left, and the fourth row three spots to the left. Hence, $M_O$is mapped to,

$$M_1 = \begin{bmatrix} B_0 & B_4 & B_8 & B_{12} \\ B_5 & B_9 & B_{13} & B_1 \\ B_{10} & B_{14} & B_2 & B_6 \\ B_{15} & B_3 & B_7 & B_{11} \end{bmatrix}$$

A mix column operation is the next step and it involves multiplying M_1 by the constant matrix

$$C = \begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix}$$

and "APPLE SAUCE OLI!" is

4150504C45205341554345204F4C4921.

This is the key used before the first round of encryption. Let us compute the key expansion with the given 128-bit starting key:

$$K_0 = \begin{bmatrix} 41 & 45 & 55 & 4F \\ 50 & 20 & 43 & 4C \\ 50 & 53 & 45 & 49 \\ 4C & 41 & 20 & 21 \end{bmatrix}$$

We have $w_0 = \{41,50,50,4C\}$,

$w_1 = \{45,20,53,41\}$,

$w_2 = \{55,43,45,20\}$, and

$w_3 = \{4F, 4C, 49,21\}$.

We must now compute $g(w_3)$, obtaining $\{4C, 49,21,4F\}$ after the left shift of the bytes. Next, using the S-box, Table 5.1, we get $\{29,3B, FD, 84\}$. Last, we XOR the word with the first round constant, $\{01,00,00,00\}$. Consequently we have, $29=00101001_2$,

$$\begin{aligned} 29_{16} \oplus 01_2 &= 00101001_2 \oplus 01_2 \\ &= 00101001_2 \\ &= 28_{16}. \end{aligned}$$

The remaining entries of w_3 are unchanged. This yields $g(w_3) = \{28,3B, FD, 84\}$. We compute w_4 as described above:

at a time, for each of the ten rounds of encryption. To compute the next four words we would first compute $g(w_3)$ and set $w_4 = w_0 \oplus g(w_3), w_5 = w_4 \oplus w_1, w_6 = w_5 \oplus w_2$, and $w_7 = w_6 \oplus w_3$. In general, to calculate w_i, w_{i+1}, w_{i+2}, and w_{i+3}, where i is divisible by 4, we would first compute $w_i = w_{i-4} \oplus g(w_{i-1})$. We then use w_i to calculate

$$w_{i+1} = w_i \oplus w_{i-3},$$

$$w_{i+2} = w_{i+1} \oplus w_{i-2}, \text{ and}$$

$$w_{i+3} = w_{i+2} \oplus w_{i-1}.$$

Now, the function g operates in three steps. First it will shift the bytes to the left. So, for example, w_3 would be $\{k_{13}, k_{14}, k_{15}, k_{12}\}$. The bytes would then be substituted via the S-box as described in the encryption section. Finally, a round constant, c_i is to be XOR'd. The round constant is generated recursively. To start, $c_1 = 01_2$. Then, the remaining round constants may be calculated by $c_i = 02 \times c_{i-1}$. It is important to note that x is polynomial multiplication over GF (2^8) reduced by the AES modulo $x^8 + x^4 + x^3 + x + 1$. We note that 02 over GF $(2^8)$is multiplication with the corresponding polynomial c_{i-1} by x.

5.6 Example 0f Encryption a 128-bit Message

Alice would like to send the message "THE BOSS IS HERE" to Bob. Alice and Bob have already agreed to use the key "APPLE SAUCE OLI!" in case of an emergency. Throughout the paper all text to her conversion is done via the American Standard Code for Information Interchange (ASCII). Now, "THE BOSS IS HERE" in hexadecimal is

54484520424F5353204953204845524 5

Where the matrix multiplication done over GF (2^8). The resulting matrix would then be XOR'd entry-wise with K_1, thus completing round 1.

5.5 Key Expansion

It would be unwise to use the same key for each round of AES encryption. So, an algorithm was developed to take a 128-bit key, and using the original 128 bits, generate unique keys for each of the 10 rounds. This process is called key expansion and goes as follows

Given an original 128 bit key, 16 bytes, we enter the bytes column-wise into a 4×4 matrix. Let the first byte of the key be k_0, the second byte be k_1, and so on. Then we have the key matrix

$$K_0 = \begin{bmatrix} k_0 & k_4 & k_8 & k_{12} \\ k_1 & k_5 & k_9 & k_{13} \\ k_2 & k_6 & k_{10} & k_{14} \\ k_3 & k_7 & k_{11} & k_{15} \end{bmatrix}$$

The columns are then stored in what are called words,

$w_0 = \{k_0, k_1, k_2, k_3\}$,

$w_1 = \{k_4, k_5, k_6, k_7\}$,

$w_2 = \{k_8, k_9, k_{10}, k_{11}\}$, and

$w_3 = \{k_{12}, k_{13}, k_{14}, k_{15}\}$.

We need a total of forty-four words for a 128-bit key; four words, the original four, are used before the plain-text enters the first round. The remaining forty words are used four at a time, for each of the ten rounds of encryption. To compute the next four words we would first compute $g(w_3)$ and set $w_4 = w_0 \oplus g(w_3), w_5 = w_4 \oplus w_1, w_6 = w_5 \oplus w_2,$ and $w_7 = w_6 \oplus w_3$. In general, to calculate w_i, w_{i+1}, w_{i+2}, and w_{i+3}, where i is divisible by 4, we would first compute $w_i = w_{i-4} \oplus g(w_{i-1})$. We then use w_i to calculate

$$w_{i+1} = w_i \oplus w_{i-3},$$

$$w_{i+2} = w_{i+1} \oplus w_{i-2}, \text{ and}$$

$$w_{i+3} = w_{i+2} \oplus w_{i-1}.$$

Now, the function g operates in three steps. First it will shift the bytes to the left. So, for example, w_3 would be $\{k_{13}, k_{14}, k_{15}, k_{12}\}$. The bytes would then be substituted via the S-box as described in the encryption section. Finally, a round constant, c_i is to be XOR'd. The round constant is generated recursively. To start, $c_1 = 01_2$. Then, the remaining round constants may be calculated by $c_i = 02 \times c_{i-1}$. It is important to note that x is polynomial multiplication over GF (2^8) reduced by the AES modulo $x^8 + x^4 + x^3 + x + 1$. We note that 02 over GF $(2^8)$is multiplication with the corresponding polynomial c_{i-1} by x.

5.6 Example Of Encryption a 128-bit Message

Alice would like to send the message “THE BOSS IS HERE” to Bob. Alice and Bob have already agreed to use the key “APPLE SAUCE OLI!” in case of an emergency. Throughout the paper all text to her conversion is done via the American Standard Code for Information Interchange (ASCII). Now, “THE BOSS IS HERE” in hexadecimal is

54484520424F5353204953204845524 5

and "APPLE SAUCE OLI!" is

4150504C45205341554345204F4C4921.

This is the key used before the first round of encryption. Let us compute the key expansion with the given 128-bit starting key:

$$K_0 = \begin{bmatrix} 41 & 45 & 55 & 4F \\ 50 & 20 & 43 & 4C \\ 50 & 53 & 45 & 49 \\ 4C & 41 & 20 & 21 \end{bmatrix}$$

We have $w_0 = \{41,50,50,4C\}$,

$w_1 = \{45,20,53,41\}$,

$w_2 = \{55,43,45,20\}$, and

$w_3 = \{4F,4C,49,21\}$.

We must now compute $g(w_3)$, obtaining $\{4C,49,21,4F\}$ after the left shift of the bytes. Next, using the S-box, Table 5.1, we get $\{29,3B,FD,84\}$. Last, we XOR the word with the first round constant, $\{01,00,00,00\}$. Consequently we have, $29=00101001_2$,

$$29_{16} \oplus 01_2 = 00101001_2 \oplus 01_2$$

$$= 00101001_2$$

$$= 28_{16}.$$

The remaining entries of w_3 are unchanged. This yields $g(w_3) = \{28,3B,FD,84\}$. We compute w_4 as described above:

$$w_4 = w_0 \oplus g(w_3)$$

$$= \{41 \oplus 28, 50 \oplus 3B, 50 \oplus FD, 4C \oplus 84\}$$

$$= \{D9, 6B, AD, C8\}$$

Performing the remaining XOR operations in the same way gives

$w_5 = \{9C, 4B, FE, 89\}$, $w_6 = \{C9, 08, BB, A9\}$ and $w_7 = \{86, 44, F2, 88\}$, giving us our first round key,

$$K_1 = \{D9, 6B, AD, C8, 9C, 4B, FE, 89, C9, 08, BB, A9, 86, 44, F2, 88\}$$

Now, as before, to compute the next four words we start by computing $g(w_7)$:

$$\{86, 44, F2, 88\} \rightarrow \{44, F2, 88, 86\}$$

$$\rightarrow \{1B, 89, C4, 44\}$$

$$\rightarrow \{19, 89, C4, 44\}.$$

Multiplication by 2 in binary shifts the number in question to the left one place value and inserts a zero into the one's place, in much the same way as multiplying by 10 with decimal numbers. Thus, computing the round constant is trivial until the 9th round, where the previous round constant was 10000000_2 and multiplication by 02 would push the '1' out of the 8 bits that make the byte. To fix this, the polynomial representation of 10000000×02= $x^7.x = x^8$, must be reduced back into the field GF (2^8) via the polynomial $x^8 + x^4 + x^3 + x + 1$. So, the polynomial representation of our 9th round constant will be

$x^8 \bmod (x^8 + x^4 + x^3 + x + 1) \equiv x^4 + x^3 + x + 1 \bmod (x^8 + x^4 + x^3 + x + 1)$,

which translates to our 9^{th} round constant $c_9 = \{00011011,00,00,00\}$. We compute the 10^{th} round constant by multiplying the 9^{th} round constant by 02, which yields $c_{10} = \{00110110,00,00,00\}$. Following the key expansion scheme as described, we obtain all 10 round keys:

$$K_0 = \{41,50,50,4C,45,20,53,41,55,43,45,20,4F,4C,49,21\}$$

$$K_1 = \{D9,6B,AD,C8,9C,4B,FE,89,C9,08,BB,A9,86,44,F2,88\}$$

$$K_2 = \{C0,E2,69,8C,5C,A9,97,05,95,A1,2C,AD,13,E5,DE,24\}$$

$$K_3 = \{1D,FF,5F,F1,41,56,C8,F4,D4,F7,E4,59,C7,12,3A,7D\}$$

$$K_4 = \{DC,7F,CD,37,9D,29,05,C3,49,DE,E1,9A,8E,CC,DB,E7\}$$

$$K_5 = \{87,C5,59,2E,1A,EC,5C,ED,53,32,BD,77,DD,FE,66,90\}$$

$$K_6 = \{1C,F6,39,EF,06,1A,65,02,55,28,D8,75,88,D6,BE,E5\}$$

$$K_7 = \{AA,58,E0,2B,AC,42,85,29,F9,6A,5D,5C,91,BC,E3,B9\}$$

$$K_8 = \{4F,59,39,EF,E3,0B,BC,C6,1A,61,E1,9A,8B,DD,02,23\}$$

$$K_9 = \{95,2E,1F,D2,76,25,A3,14,6C,44,42,8E,E7,99,40,AD\}$$

$$K_{10} = \{4D,27,8A,46,3B,02,29,52,57,46,6B,DC,B0,DF,2B,71\}$$

Now, we can start encryption process. Entering our plain-text message into a matrix as described in Section 5.4 we have

$$M_O = \begin{bmatrix} 54 & 42 & 20 & 48 \\ 48 & 4F & 49 & 45 \\ 45 & 53 & 53 & 52 \\ 20 & 53 & 20 & 45 \end{bmatrix}$$

We then XOR M_O entry-wise with the original key K_O, obtaining

$$\begin{bmatrix} 54 & 42 & 20 & 48 \\ 48 & 4F & 49 & 45 \\ 45 & 53 & 53 & 52 \\ 20 & 53 & 20 & 45 \end{bmatrix} \oplus \begin{bmatrix} 41 & 45 & 55 & 4F \\ 50 & 20 & 43 & 4C \\ 50 & 53 & 45 & 49 \\ 4C & 41 & 20 & 21 \end{bmatrix} = \begin{bmatrix} 15 & 07 & 75 & 07 \\ 18 & 6F & 0A & 09 \\ 15 & 00 & 16 & 1B \\ 6C & 12 & 00 & 64 \end{bmatrix}$$

Next, we substitute each entry of our current matrix via the S-box from Table 5.1, which gives us

$$\begin{bmatrix} 59 & C5 & 9D & C5 \\ AD & A8 & 67 & 01 \\ 59 & 63 & 47 & AF \\ 50 & C9 & 63 & 43 \end{bmatrix}$$

We then apply the shift row operation to get

$$\begin{bmatrix} 59 & C5 & 9D & C5 \\ A8 & 67 & 01 & AD \\ 47 & AF & 59 & 63 \\ 43 & 50 & C9 & 63 \end{bmatrix}$$

The next step in Round 1 involves multiplying by the constant matrix

$$C = \begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix}$$

obtain the matrix

$$\begin{bmatrix} 55 & C7 & B2 & 7D \\ 98 & B1 & BD & 42 \\ BA & 17 & 6E & 0B \\ 82 & 3C & 6D & 5C \end{bmatrix}$$

Finally, the above matrix is XOR'd entry-wise with K_1, producing

$$M_1 = \begin{bmatrix} 8C & 5B & 7B & FB \\ F3 & FA & B5 & 06 \\ 17 & E9 & D5 & F9 \\ 4A & B5 & C4 & D4 \end{bmatrix}$$

Round 1 is complete. This process repeats for 9 more rounds. The only exception is Round 10 omits the shift column operation. Even after one round of AES, the 16 bytes

8CF3174A5BF AE9B57BB5D5C4F B06F9D4

have few recognizable characters, none of which were in the original plain-text message. Continuing after Round 2 we have

$$M_2 = \begin{bmatrix} 34 & 05 & 43 & 18 \\ 91 & 47 & 80 & CE \\ FE & 33 & E3 & C4 \\ 9E & B5 & E0 & 1E \end{bmatrix}$$

After Round 3 the matrix is

$$M_3 = \begin{bmatrix} 60 & CC & 08 & 3C \\ 92 & 93 & E3 & 19 \\ E9 & 4B & 7C & B3 \\ 36 & 8E & F6 & 0A \end{bmatrix}$$

Let us look at a new example. We will encrypt "THE BOSS IS HERE" and see how the change of a single character, a capital I to lowercase i, affects the matrix after a single round of AES. The key will be the same, so the key expansion is the same. We start as before, with an XOR operation. Our initial matrix this time is

$$M_0^* = \begin{bmatrix} 54 & 42 & 20 & 48 \\ 48 & 4F & 69 & 45 \\ 45 & 53 & 53 & 52 \\ 20 & 53 & 20 & 45 \end{bmatrix}$$

We compute

$$\begin{bmatrix} 54 & 42 & 20 & 48 \\ 48 & 4F & 69 & 45 \\ 45 & 53 & 53 & 52 \\ 20 & 53 & 20 & 45 \end{bmatrix} \oplus \begin{bmatrix} 41 & 45 & 55 & 4F \\ 50 & 20 & 43 & 4C \\ 50 & 53 & 45 & 49 \\ 4C & 41 & 20 & 21 \end{bmatrix} = \begin{bmatrix} 15 & 07 & 75 & 07 \\ 18 & 6F & 2A & 09 \\ 15 & 00 & 16 & 1B \\ 6C & 12 & 00 & 64 \end{bmatrix}$$

Again, we substitute via the S-box from Table 5.1 to get,

$$\begin{bmatrix} 59 & C5 & 9D & C5 \\ AD & A8 & E5 & 01 \\ 59 & 63 & 47 & AF \\ 50 & C9 & 63 & 43 \end{bmatrix}$$

Applying the shift row operation yields the matrix

$$\begin{bmatrix} 59 & C5 & 9D & C5 \\ A8 & E5 & 01 & AD \\ 47 & AF & 59 & 63 \\ 43 & 50 & C9 & 63 \end{bmatrix}$$

So, unsurprisingly our matrix still has only one byte different from the original example; we have only applied confusion. The next step, mix column, applies the necessary diffusion.

$$\begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix} \otimes \begin{bmatrix} 59 & C5 & 9D & C5 \\ A8 & E5 & 01 & AD \\ 47 & AF & 59 & 63 \\ 43 & 50 & C9 & 63 \end{bmatrix} = \begin{bmatrix} 55 & 5A & B2 & 7D \\ 98 & AE & BD & 42 \\ DA & 95 & 6E & 0B \\ 82 & BE & 6D & 5C \end{bmatrix}$$

Finally, our current matrix is XOR'd with K_1, which yields

$$M_1^* = \begin{bmatrix} 8C & C6 & 7B & FB \\ F3 & E5 & B5 & 06 \\ 17 & 6B & D5 & F9 \\ 4A & 37 & C4 & D4 \end{bmatrix}$$

Thus, after one round of AES with a single character changed, we have M_1 differing from M_1^* by an entire column. Moreover, in the second round, the shift row step will shift one of these four differing elements into each column. After the mix column operation, the one differing element in each column will affect the entire matrix. Hence, a single byte change has diffused into the entire message after the second round. The matrix after Round 2 becomes

$$M_2^* = \begin{bmatrix} 33 & 04 & 0C & 0A \\ 62 & CA & CF & 6D \\ 0A & BE & 2C & 06 \\ 6A & 39 & 7E & 7F \end{bmatrix}$$

which is a completely different matrix than our original example. This helps illustrate how effectively AES diffuses a single bit flip throughout the entire message.

5.7 Decryption

Decryption is required to retrieve the original message. The steps must be reversed and operations inverted. The XOR operation is its own inverse, $X \oplus X = 0$ for all $X \in \mathbb{R}$, so the first operation for decryption would be to XOR the last round key. The first round of decryption would skip inverting the mix column operation as it was not performed in the last round of encryption. For all other rounds, we multiply by the inverse of the constant matrix

$$C^{-1} = \begin{bmatrix} 0E & 0B & 0D & 09 \\ 09 & 0E & 0B & 0D \\ 0D & 09 & 0E & 0B \\ 0B & 0D & 09 & 0E \end{bmatrix}$$

To invert the shift row operation, we must keep the first row the same, shift the second row one spot to the right, the third row to the right twice, and the fourth row three times to the right. Finally, to undo the S-box, one would find their entry inside the table and use the corresponding row and column to find the entry's pre-image. For example, if we wished to know what "6D" was before the S-box, we would locate "6D" in Table 5.1 and notice that it is in row "B0" and column "03". Hence $S^{-1}(6D) = B3$.

Chapter 6

CHAPTER 6

DISCRETE LOGARITHM PROBLEM

6.1 Motivation

The demand for an asymmetric cipher was still high even after RSA. There were already very secure and computationally efficient symmetric algorithms in DES and later AES. Their main weakness was that the establishment of keys could be difficult if the two parties could not physically meet before communication began. The Diffie-Hellman key exchange was invented to satisfy the demand.

6.2 Diffie-Hellman Key Exchange

The Diffie-Hellman key exchange uses cyclic groups, a class of groups with generators. Let us say that $\mathbb{G}$ is a cyclic group and that $a \in \mathbb{G}$ is a generator of $\mathbb{G}$. So, for all $b \in \mathbb{G}$ there exists some $\alpha \in \mathbb{N}$ such that $a^{\alpha} = b \bmod \mathbb{G}$. .Now, say Alice and Bob would like to exchange a key but do not have a secure chain of communication. They could publicly select a cyclic group, $\mathbb{G}$, and generator, a. They then each select a group element and keep that element secret. Suppose Alice picks r and Bob selects s. Alice would compute $A = a^r \bmod \mathbb{G}$ and send the result to Bob. Bob computes $A^s \bmod \mathbb{G}$. Bob now has the private key he and Alice will use for encryption. The process is repeated for Alice to have the private key: Bob computes $B = a^s \bmod \mathbb{G}$ and sends the result to Alice. Alice then computes $B^r \bmod \mathbb{G}$ and they now have the same key. An eavesdropper, Eve, knows the group $\mathbb{G}$, the generator a, and the public messages between Alice and Bob, A and B. Eve can formulate the equation $a^r = A \bmod \mathbb{G}$. The only missing variable for Eve is r. If Eve could solve for r, then the system has been broken. ... mod $\mathbb{G}$ is easy to compute. The way to solve for an exponent is by using logarithms. Unfortunately for Eve, the typical logarithm approach learned in algebra class is useless ... The approach to solve for the missing exponential in the Diffie-Hellman key exchange leads to the discrete logarithm problem; the security of this key exchange relies on the difficulty of the problem.

6.3 Discrete Logarithm Problem

The discrete logarithm problem is formulated as follows: Given prime number p, an element $\beta \in \mathbb{Z}_p^*$, and primitive element α, find x such that $\alpha^x \equiv \beta \bmod p$. If a group has integer elements, as does $\mathbb{Z}_p^*$, there are powerful algorithms that can solve this problem if p is not too large. To have the same security as AES with a 128-bit key, p must be 15360 bits long. With numbers this large, trivial operations run slow. What we will see is a cyclic group whose elements are points on an elliptic curve. Using this type of group, a level of security comparative to AES with a 256-bit key requires p to be 512 bits. This gives a huge boost in efficiency when compared to the former discrete logarithm problem.

6.4 Small-step Giant -step Algorithm

A very powerful algorithm to solve the discrete algorithm problem, and in turn a powerful attack against the Diffie-Hellman key exchange, is the small-step giant-step algorithm. Consider the discrete algorithm problem. The small step is to select some integer k, Compute $a^1, a^2, a^3, \dots, a^{k-1}$, and also compute $ba^{-k}, ba^{-2k}, ba^{-3k}, \dots ba^{-rk}$

CONCLUSION

Mathematics has played a crucial role in cryptography throughout the centuries, particularly in the last century. As computers continue to get faster and, perhaps more troublesome, as algorithms become more efficient, encryption based on computationally hard mathematical problems may become less useful. Problems like factoring and the discrete logarithm problem, where security relies in having a pool too large for computers to attack encryption via brute force, will be easily broken with quantum computers and quantum computing algorithms. When quantum computing becomes a reality, the cryptography world will have to move to post quantum cryptography. But for now, computationally hard mathematical problem are the key to effective encryption methods.

REFERENCES

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A STUDY ON DUAL GRAPHS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

G. PADMAPRIYA

Reg. No: 19SPMT20

Under the guidance of

Ms. P. SUGANYA M.Sc., M.Phil., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April - 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON DUAL GRAPHS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmanium Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **G. PADMAPRIYA (Reg. No. 19SPMT20)**

P. Sr—

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON DUAL GRAPHS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. P. Suganya M.Sc., M.Phil., SET.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

Station: Thoothukudi

G. Padma Priya
Signature of the Student

Date: 10. 04. 2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

1. PRELIMINARIES

Definition: 1.1

A Graph $G = (V, E)$ consists of a pair of V and E. The elements of V are called **vertices** and the elements of E are called **edges**. Each edge has a set of one or two vertices associated to it, which are called its end points.

Definition: 1.2

Let E be an unordered set of two elements subsets of V. If we considered ordered pair of elements of V then the graph is called $G = (V, E)$ a **directed graph or digraph**.

Definition: 1.3

A **walk** consists of an alternating sequence of vertices and edges consecutive elements of which are incident, that begins and ends with a vertex. A walk is **closed** if it begins and ends at the same vertex.

Definition: 1.4

A **cycle** is a closed walk in which all the vertices are distinct except u=v, that is the initial and terminal points of the walk coincide.

Definition: 1.5

A graph G is called **acyclic or forest** if, it has no cycles.

Definition: 1.6

A **bipartite graph** is one whose vertex can be partitioned into two subsets X and Y so that each edge has one end in X and one end in Y such a partition (X, Y) is called a Bipartition of the graph.

Definition: 1.7

A **tree** is an acyclic connected graph.

Definition: 1.8

For subsets S and S' of V denote by [S, S'] the set of edges with one end in S and the other end in S'. An **edge cut** of G is a E of the form [S, S'] where S is a non-empty proper subset of V and S'=V\S.

Definition: 1.9

A minimal non-empty edge cut of G is called a **Bond or cut-set**.

Definition: 1.10

A graph G is said to be **connected** if between every pair of vertices x and y in G, there always exists a path in G. Otherwise, G is called disconnected.

Definition: 1.11

A vertex v of a graph G is a **cut-vertex** if the edges set E can be partitioned into two non-empty subsets E_1 and E_2 such that G(E_1) and G(E_2) have just the vertex v in common.

Definition: 1.12

An edge set E of a graph G is a **cut edge** of G if W(G-e)>W(G). In particular, the removal of a cut edge from a connected graph makes the graph disconnected.

Definition: 1.13

An edge with identical ends is called a **loop.**

Definition: 1.14

A connected graph that has no cut vertices is called a **Block**.

Definition: 1.15

A graph G is **planar** if it can be drawn in the plane in such a way that no two edges meet except at a vertex with which they both are incident. Any such drawing is a plane drawing of G.

A graph G is **non-planar** if no plane drawing of G exists.

Definition: 1.16

A planar graph is an **Outer Planar graph** if it has an embedding on the plane such that every vertex of the graph is a vertex belonging to the same (usually exterior) region.

Definition: 1.17

A **Tour** of G is a closed walk of G which includes every edge of G at least once.

Definition: 1.18

An **Euler Tour** of G is a tour which includes each edge of G exactly once.

Definition: 1.19

A graph G is called **Eulerian** or Euler if it has an Euler Tour.

Definition: 1.20

A plane graph G partitions the rest of the plane into a number of arc-wise connected open sets. The sets are called the **faces** of G.

2. DUAL GRAPHS

2.1 INTRODUCTION

A map on the plane or the sphere can be viewed as a plane graph in which the faces are the territories, the vertices are places where boundaries meet and the edges are the parties of the boundaries that join two vertices from any plane graph we can form a related plane graph called its “Dual”.

2.2 DUAL GRAPHS

Definition: 2.2.1

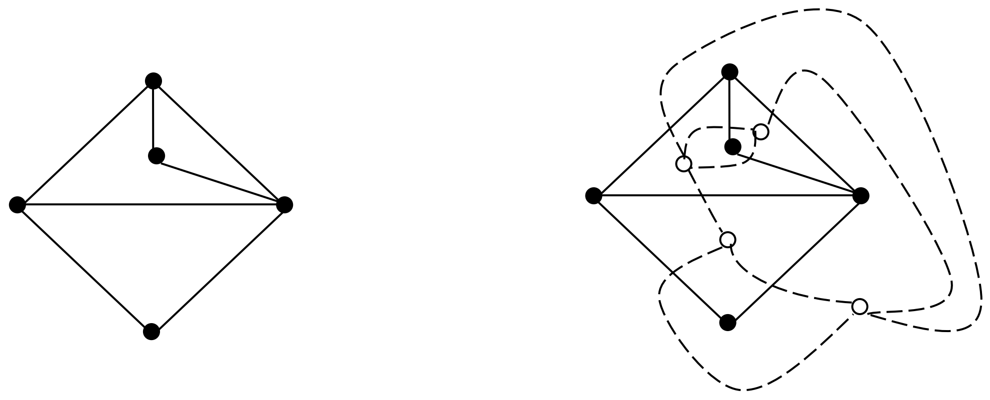
Let G be a connected planar graph. Then a **dual graph** G^* is constructed from a plane drawing of G, as follows

Draw one vertex in each face of the plane drawing, these are vertices of G^*. For each edge e of a plane drawing, draw a line joining the vertices of G^* in faces on either side of e, these lines are the edges of G^*.

Example: 2.2.2

(1) Isomorphic dual graphs:

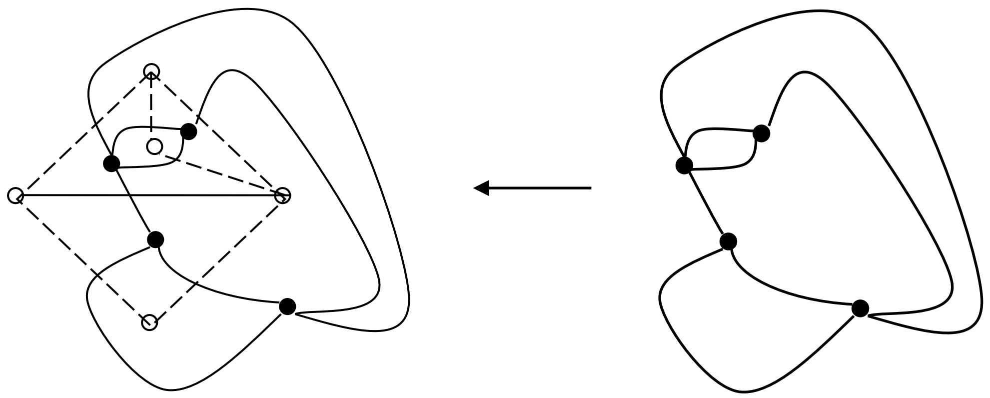
We always assume that we have been presented with a plane drawing of G. The procedure is illustrated below.



G **G***

Figure: 2.1

Also if G is a plane drawing of a connected planar graph, then so its dual G^*, and we can thus construct $(G^*)^*$, the dual of G^*.



$(\mathbf{G}^*)^*$ $\mathbf{G}^*$

Figure: 2.2

The above diagrams demonstrated that the construction that gives rise to G^* from G can be reversed to give G from G^*. It follows that $(G^*)^*$ is isomorphic to G.

(2) Non-isomorphic dual graphs:

Dual graphs are not unique, in the sense that the same graph can have non-isomorphic dual graphs because the dual graph depends on a particular plane embedding. In Figure 2.3, the blue graphs are isomorphic but their duals red graphs are not. The upper red dual has a vertex with degree 6 (corresponding to the outer face of the blue graph) while in the lower red graph all degrees are less than 6.

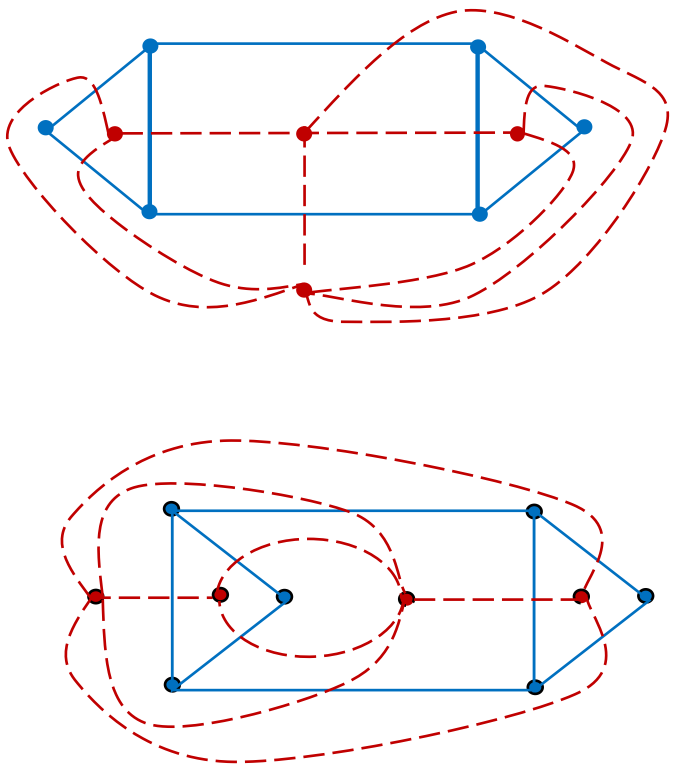


Figure: 2.3

(3) Uniqueness of dual graphs:

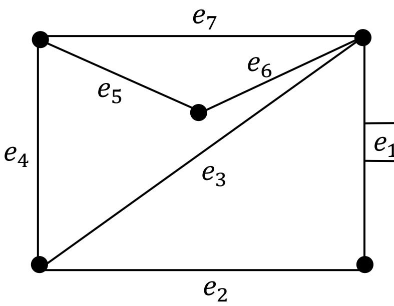
(1) Consider the graph $G_1$and its dual ${G_1}^*$. Also consider the graph G_2 and its dual ${G_2}^*$ (see Figure: 2.4)

(2) Observe that graph $G_1$and G_2 are two different planar representations of a same graph.

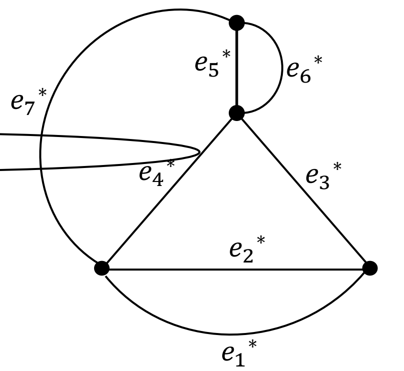
(3) The graph ${G_2}^*$ contains a vertex of degree 5, and the graph ${G_1}^*$ contains no

vertex of degree 5. Therefore, ${G_1}^*$ and ${G_2}^*$ are non-isomorphic. So, we have that $G_1 \cong G_2$ but ${G_1}^* \ncong {G_2}^*$.

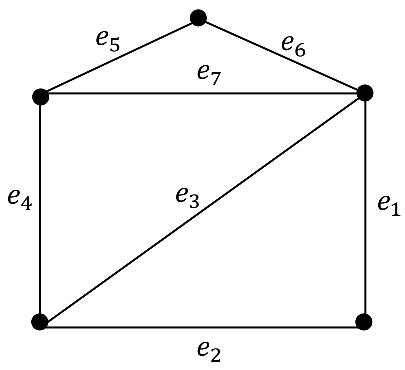
From (3), we may conclude that two isomorphic planar graphs may have distinct non-isomorphic duals.



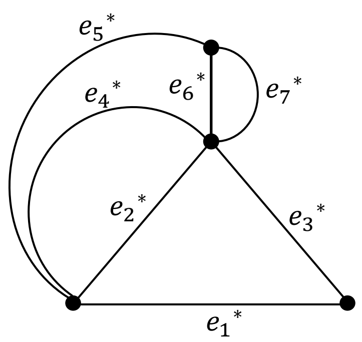
$\mathbf{G_1}$



$\mathbf{G_1}^*$



$\mathbf{G_2}$



$\mathbf{G_2}^*$

Figure: 2.4

There are many forms of duality in graph theory.

Result: 2.2.3

(1) The dual of a plane graph is planar multigraph – a graph that may have loops and multiple edges.

(2) If G is a connected graph and if G^* is a dual of G then G is a dual of G^*.

Definition: 2.2.4

Let $m(G)$ be the cycle rank of a graph G, $m^*(G)$ be the co-cycle rank, and the relative complement $G - H$ of a subgraph H of G be defined as that subgraph obtained by deleting the lines of H. Then a graph G^* is a **combinatorial dual** of G if there is one-to-one correspondence subsets of lines,

$$m^*(G - Y) = m^*(G) - m(Y^*)$$

where $<Y^*>$ is the subgraph of G^* with the line set Y^*.

Whitney showed that the geometric dual graph and combinatorial dual graph are equivalent, and so may be called "the" dual graph.

Result: 2.2.5

A graph is plane if and only if it has a combinatorial dual.

Definition: 2.2.6

The **weak dual** of an embedded planar graph is the sub graph of the dual graph whose vertices correspond to the bounded faces of the primal graph.

Result: 2.2.7

A planar graph is outer planar if and only if its weak dual is a forest.

A planar graph is a Halin graph if and only if its weak dual is biconnected and outer planar.

2.3 PLANE DUALITY

Proposition: 2.3.1

The dual of any plane graph is connected.

Proof:

Let G be a plane graph and G^* a plane dual of G. Consider any two vertices of G^*. There is a curve in the plane connecting them which avoids all vertices of G. The sequence of faces and edges of G traversed by this curve corresponds in G^* to a walk connecting the two vertices.

Definition: 2.3.2

A simple connected plane graph in which all faces have degree three is called a **plane triangulation** or, for a short triangulation.

Proposition: 2.3.3

A simple connected plane graph is a triangulation if and only if its dual is cubic.

Deletion-contraction duality:

Let G be a planar graph and $\tilde{G}$ be a plane embedding of G. For any edge e of G, a plane embedding of G\e can be obtained by simply deleting the line e from $\tilde{G}$. Thus deletion of an edge from a planar graph results in a planar graph. Although less obvious, the contraction of an edge of a planar graph also results in a planar graph. Indeed, given any edge e of a planar graph G and a planar embedding $\tilde{G}$ of G, the line e of $\tilde{G}$ can be contracted to a single point (and the lines incident to its ends redrawn). So, that the resulting plane graph is a planar embedding of G\e.

The following two propositions show that the operations of contracting and deleting edges in plane graph are related in a natural way under duality.

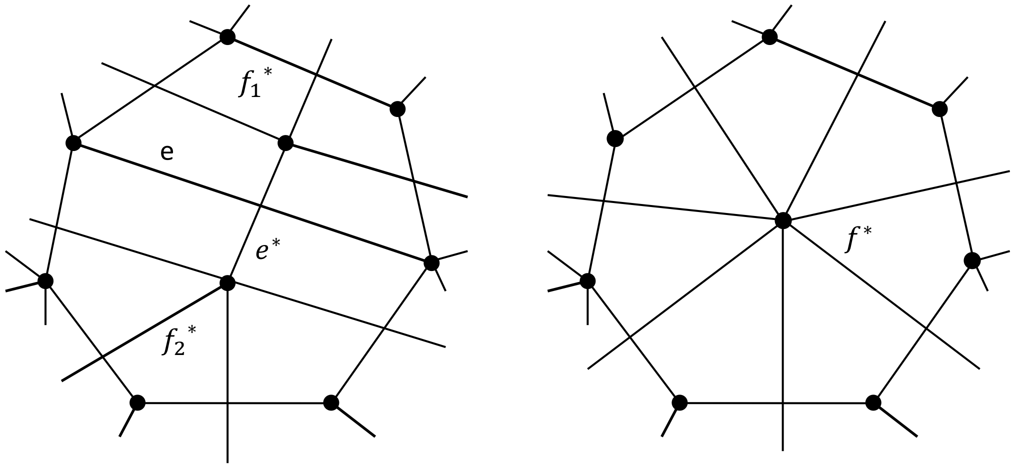
Proposition: 2.3.4

Let G be a connected plane graph, and let e be an edge of G that is not a cut edge. Then $(G\backslash e)^* \cong G^*/e^*$.

Proof:

Because e is not a cut edge, the two faces of G incident with e are distinct; denote them by f_1 and f_2. Deleting e from G results in a amalgamation of f_1 and f_2 into a single face f (see Figure: 2.5). Any face of G that is adjacent to f_1 and f_2 is adjacent in G\e to f; all other faces and adjacencies between them are unaffected by the deletion of e.

Correspondingly, in the dual, the two vertices ${f_1}^*$ and ${f_2}^*$ of G^* which correspond to the faces f_1 and f_2 of G are now replaced by a single vertex of $(G\backslash e)^*$, which we may be denote by f^*, and all other vertices of G^* are vertices of $(G\backslash e)^*$. Furthermore, any vertex of G^* that is adjacent to ${f_1}^*$ and ${f_2}^*$ is adjacent in $(G\backslash e)^*$ to f^*, and adjacencies between vertices of $(G\backslash e)^*$ other than v are the same as in G^*. The assertion follows from these observations.



(a) **(b)**

G and G^* **$G\backslash e$ and G^*/e^***

Figure: 2.5

Proposition: 2.3.5

Let G be a connected plane graph and let e be a link of G. Then $(G/e)^* \cong G^*\backslash e^*$.

Proof:

Because, G is connected $G^{**} \cong G$. Also because is not a loop of G, the edge e^* is not a cut edge of G^*, so $G^*\backslash e^*$ is connected by proposition: 2.3.4,

$$(G^*\backslash e^*)^* \cong G^{**}/e^{**} \cong G/e.$$

The proposition follows on taking duals.

We now apply propositions 2.3.1 and 2.3.3 to show that non separable plane graphs have non separable duals. This fact turns out to be very useful.

Theorem: 2.3.6

The dual of a non separable plane graph is non separable.

Proof:

By induction on the number of edges, let G be a non separable plane graph. The theorem is clearly true if G has at most one edge, so we may assume that G has atleast two edges, hence no loops or cut edges. Let e be an edge of G. Then either G\e or G/e is non separable. If G\e is non separable so is $(G\backslash e)^* \cong G^*/e^*$, by the induction hypothesis and proposition 2.3.4. And we deduce that G* is non separable. The case where G/e is non separable can be established by an analogous argument.

2.4 COMBINATORIAL DUAL

Proposition: 2.4.1

Let G be a 2-connected plane multi graph, and let H be its geometric dual. Then H is a combinatorial dual of G. Moreover, G is a geometric dual graph (and hence a combinatorial dual) of H.

Proof:

Since the minimal cuts of G are the minimal separating sets of G, We now have:

(A) If $E \subseteq E(G)$ is the edge set of a cycle in G, then E^* is cut in H.

(B) If E is the edge set of a forest in G, then $H - E^*$ is connected.

Imply that H is a combinatorial dual of G. In particular, H is 2-connected contains at least three vertices (Otherwise, G is a cycle and the claims are easy to verify). To prove that G is a geometric dual of H, it suffices to prove that, for each facial cycle C^* in H, has only one vertex in the face F of H bounded by C^*, (clearly, G has no edge inside F). But, if G has two or more vertices in F, then some two vertices of C^* can be joined by a simple arc inside F having only its ends in common with $G \cup H$. But, this is impossible by the definition of H.

Whitney [wh33a] proved that combinatorial duals are geometric duals. This gives rise to another characterization of planar graphs.

Theorem: 2.4.2

Let G be a 2-connected multigraph. Then G is a planar if and only if it has a combinatorial dual. If G^* is a combinatorial dual of G, then G has an embedding in the plane such that G^* is isomorphic to the geometric dual of G. In particular, also G^* is planar, and G is a combinatorial dual of G^*.

Proof:

By proposition 2.4.1, it suffices to prove the second part of the theorem. The proof will be done by induction on the number of edges of *G*.

If G is a cycle, then any two edges of G^* are in a 2-cycle and hence G^* has only two vertices. Clearly, G and G^* can be represented as a geometric dual pair.

If G is not a cycle, then G is the union of a 2-connected subgraph $G^{'}$ and a path P such that $G' \cap P$ consists of the two end vertices of P. By the induction hypothesis and by the proposition, " If G^* is a combinatorial dual of G and $E \subseteq E(G)$ is a set of edges of G such that $G - E$ has only one component containing edges, then G^*/e^* is a combinatorial dual of $G - e$ (minus isolated vertices)", $H = G^*/E(P^*)$ is a combinatorial dual of G'. By the induction hypothesis, G' and H can be represented as a geometric dual pair, and G' is also a combinatorial dual of H.

If e_1 , e_2 are two edges of P, then ${e_1}^*, {e_2}^*$,are two edges of G^* which belong to a cycle C^* of G^*. If C^* has length at least 3, then it is easy to find a minimal cut in G^* containing e, but not e_2 . But, this is impossible since any cycle in G containing e_1 also contains e_2 . Hence, all edges of $E(P)^*$ are parallel in G^* and join two vertices z_1, z_2 say, in G^*.

Let z_0 be the vertex in H which corresponds to z_1, z_2. The edges in H incident with z_0 form a minimal cut in H. Let C be the corresponding cycle in G'. As $E(C)^*$ separates z_0 from $H - z_0$ in H, C is a simple closed curve separating z_0 from $H - z_0$. In particular, C is facial in G'.

Let C_1, C_2 be the two cycles in $C \cup P$ containing P such that $E(C_i)^*$is the minimal cut consisting of the edges incident with z_i , for $i = 1,2$. Now we draw P inside the face F of G' bounded by C and represent z_i inside C_i for $i = 1,2$. This way we obtain a representation of G^* as a geometric dual of G.

Proposition: 2.4.3

Let G be a 2-connected multigraph and let G^* be its combinatoinal dual. Then G^* is 3-connected if and only if G is 3-connected.

Proof:

By Theorem 2.4.2, it sufficies to prove that G is 3-connected whenever G^* is 3-connected. Suppose that this is not a case if G has a vertex of degree 2, then G^* has parallel edges, a contradiction. So, G has minimum degree at least 3. Then we can write $G = G_1 \cup G_2$ where $G = G_1 \cap G_2$ consists of two vertices, $E(G_1) \cap E(G_2) = \emptyset$, and each of $G_1,\ G_2$ contains atleast three vertices.

By Theorem 2.4.2, G is planar. Then G has a facial cycle C such that $C \cap G_i$ is path P_i for $i = 1, 2$. Clearly, $G/E(C)$ has two edges which are not in the same block.

By proposition, "If, G^* is a combinatorial dual of G and $E \subseteq E(G)$ is a set of edges of G such that $G - E$ has only one component containing edges, then G^*/E^* is a combinatonal dual of $G - E$ (minus isolated vertices)", and Theorem 2.4.2, $G^* - E(C)^*$ has two edges which are not in the same block. As $E(C)^*$ is the set of edges incident with a vertex of G^*, G^* is not 3-connected.

Theorem: 2.4.4

A necessary and sufficient condition for two planar graphs G_1 and G_2 to be duals of each other is as follows. There is a one to one correspondence between the edges in G_1 and the edges in G_2 such that a set of edges in G_1 forms a circuit if and only if the corresponding set in G_2 forms a cut-set.

Proof:

Let us consider a plane representation of a planar graph G. Let us also draw (geometrically) a dual G^* of G. Then consider an arbitrary circuit Γ in G. Clearly, Γ will form some closed simple curve in the plane representation of G- dividing the plane into two areas (Jordan curve theorem). Thus the vertices of G^* are partitioned into non-empty, mutually exclusive subsets- one Γ and other outside.

In other words, the set of edges Γ* in G^* corresponding to the set Γ in G is a cut-set in G^*. (No proper subset of Γ* will be a cut-set in G^*). Likewise it is apparent that corresponding to a cut-set S^* in G^* there is a unique circuit consisting of the corresponding edge-cut S is a circuit. This proves the necessity of the theorem.

To prove the sufficiency, let G be a planar graph and let $G^{'}$ be the graph for which there is a one-to-one correspondence between the cut-sets of G and circuits of $G^{'}$, and vice-versa. Let G^* be a dual graph of G. There is a one-to-one correspondence between the circuits of $G^{'}$ and cut-sets of G, and also between the cut-sets of G and circuits of G^*. There is one-to-one correspondence between the circuits of $G^{'}$ and G^*, implying that $G^{'}$ and G^* are 2-isomorphic.

By a theorem, "All duals of a planar graph G are 2-isomorphic; and every graph 2-isomorphic to a dual of G is also a dual of G", $G^{'}$ must be a dual of G.

Theorem: 2.4.5

Edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.

Proof:

Consider $D \subseteq E(G)$. If D contains no cycle in G, then D encloses no region. It remains possible to reach the unbounded face of G from every face without crossing D. Hence, $G^* - D^*$ connected, and D^* contains no edge cut.

If D is the edge set of a cycle in G, then the corresponding edge set $D^* \subseteq E(G^*)$ contains all dual edges joining faces inside D to faces outside D. Thus D^* contains an edge cut.

If D contains a cycle and more, then D^* contains an edge cut and more.

Thus D^* is a minimal edge cut if and only if D is a cycle.

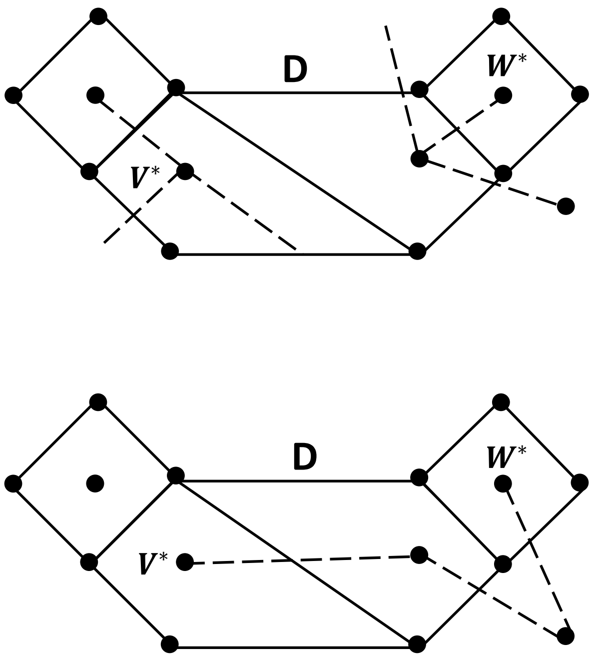


Figure: 2.6

Theorem: 2.4.6

The following are equivalent for a plane graph G.

(A) G is bipartite.

(B) Every face of G has even length.

(C) The dual graph G^* is Eulerian.

Proof:

A⇒B. A face boundary consists of closed walks. Every odd closed walk contains an odd cycle. Therefore, in a bipartite plane graph the contributions to the length of faces are all even.

B⇒A. Let C be a cycle in G. Since G has no crossings, C is laid out as a simple closed curve; let F be the region enclosed by C. Every region of G wholly within F or wholly outside F. If we sum the face lengths for the regions inside F, we obtain an even number. Since each face length is even. This sum counts each edge of C once. It also counts each edge inside F twice, since each such edge belongs twice to faces in F. Hence the parity of the length of C is the same as the parity of the full sum, which is even.

B⇔C. The dual graph G^* is connected and its vertex degrees are the face lengths of G.

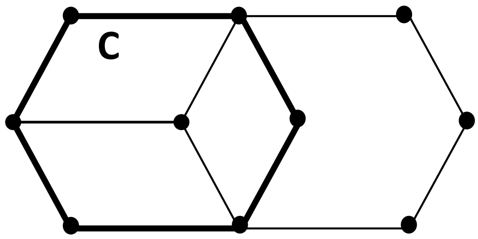


Figure: 2.7

Theorem: 2.4.7

A graph has a dual if and only if it is planar.

Proof:

We need to prove just the “only if” part. That is, we have only to prove that a non-planar graph does not have a dual. Let G be a non-planar graph. Then G contains K_5 or $K_{3,3}$ or a graph homeomorphic to either of these. We have already seen that a graph G can have a dual only if every subgraph g of G and every homeomorphic to g has a dual. Thus if we can show that neither K_5 nor $K_{3,3}$ has a dual, we have proved the theorem. This we shall prove by contradiction as follows:

(a) Suppose that $K_{3,3}$ has dual D. Observe that the cut-sets in $K_{3,3}$ correspond to circuits in D and viceversa, since $K_{3,3}$ has no cut-set consisting of two edges, D has no circuit consisting of two edges. D contains no pair of parallel edges. Since every circuit in $K_{3,3}$ is of length four or six, D has no cut-set with les than four edges. Therefore, the degree of every vertex in D is at least four. As D has no parallel edges and the degree of every vertex is at least four, D must have $(5 \times 4)/2 = 10$ edges. This is a contradiction, because $K_{3,3}$ has nine edges and so must its dual. Thus $K_{3,3}$ cannot have a dual. Likewise,

(b) Suppose that the graph K_5 has a dual H. Note that K_5 has (1) 10 edges, (2) no pair of parallel edges, (3) no cut-set with two edges, and (4) cut-sets with only four or six edges. Consequently, graph H must have (1) 10 edges, (2) no vertex with degree less than three, (3) no pair of parallel edges, and (4) circuits of length four and six only. Now graph H contains a hexagon (a circuit of length six), and no more than three edges can be added to a hexagon without creating a circuit of length three or a pair of parallel edges. Since both of these are forbidden in H and H has 10 edges, there must be at least seven vertices in at least three. The degree of each of these vertices is at least three. This leads to H having at least 11 edges which is a contradiction.

3. SELF DUAL GRAPHS

3.1 INTRODUCTION

Self-dual graph was developed by Brigitte Servatius and Herman Servatius. The three forms of self-duality that can be exhibited by a planar graph G, map self-duality, graph self-duality, matroid self-duality. They shown how these concepts are related with each other and with the connectivity of G. We use the geometry of self-dual polyhedral together with the structure of the cycle matroid to construct all self-dual graphs.

3.2 FORMS OF SELF-DUALITY

Definition: 3.2.1

A planar graph is isomorphic to its own dual is called a **self-dual graphs**.

Example: 3.2.2

K_4 is a self-dual graph.

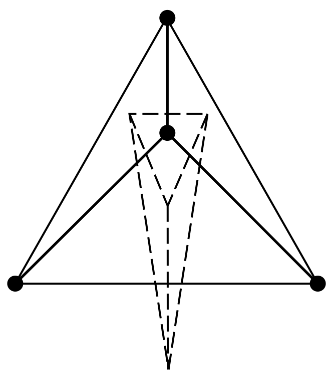


Figure: 3.1

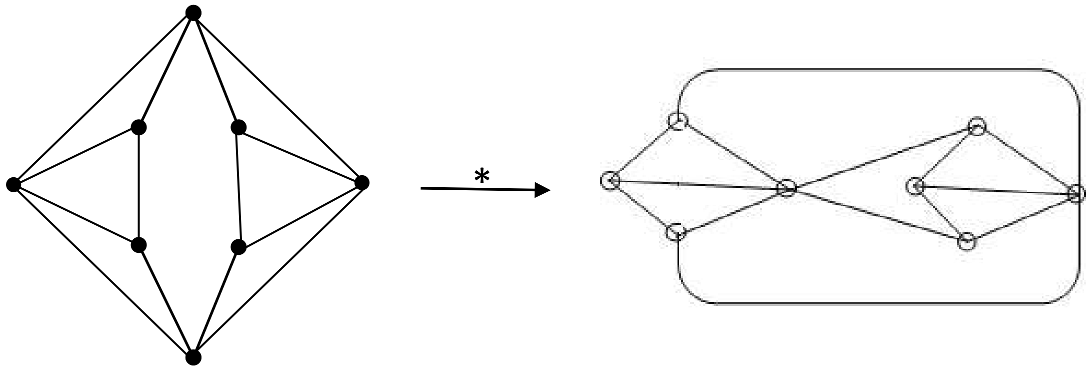
Definition: 3.2.3

Given a planar graph $G = (V, E)$ any regular embedding of the topological realization of G into a sphere partitions the sphere into regions called the faces of the embedding, and we write the embedded graph, called a **map**, as $M = (V, E, F)$. G may have loops and parallel edges.

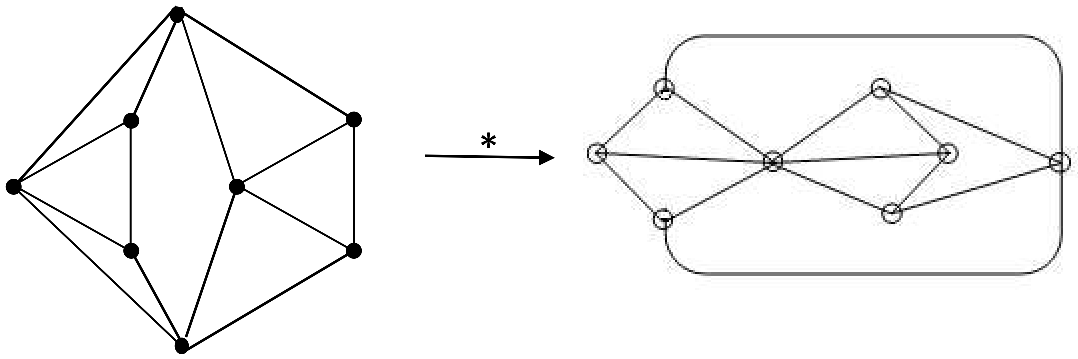
Definition: 3.2.4

Given a map M, we form the **dual map**, M^* by placing a vertex f_1^* in the centre of each face f, and for each edge e of M bounding two faces f_1 and f_2, we draw a dual edge e^* connecting the vertices f_1^* and f_2^* and crossing e once transversely. Each vertex v of M will then correspond to a face v^* of M^* and we write $M^* = (F^*, E^*, V^*)$.

If the graph G has distinguishable embeddings, then G may have more than one dual graph, see figure 3.2. In this example a portion of the map (V, E, F) is flipped over on a separating sets of two vertices to form (V, E, F').



(V, E, F) (F^*, E^*, V^*)



$(V, E, F^{'})$ (F'^{*}, E^{*}, V^{*})

Figure: 3.2

Such a move is called Whitney flip, and the duals of (V, E, F) and $(V, E, F^{'})$ are said to differ by a Whitney twist. If the graph (V, E) is 3-connected, then there is a unique embedding in the plane and so the dual is determined by the graph alone.

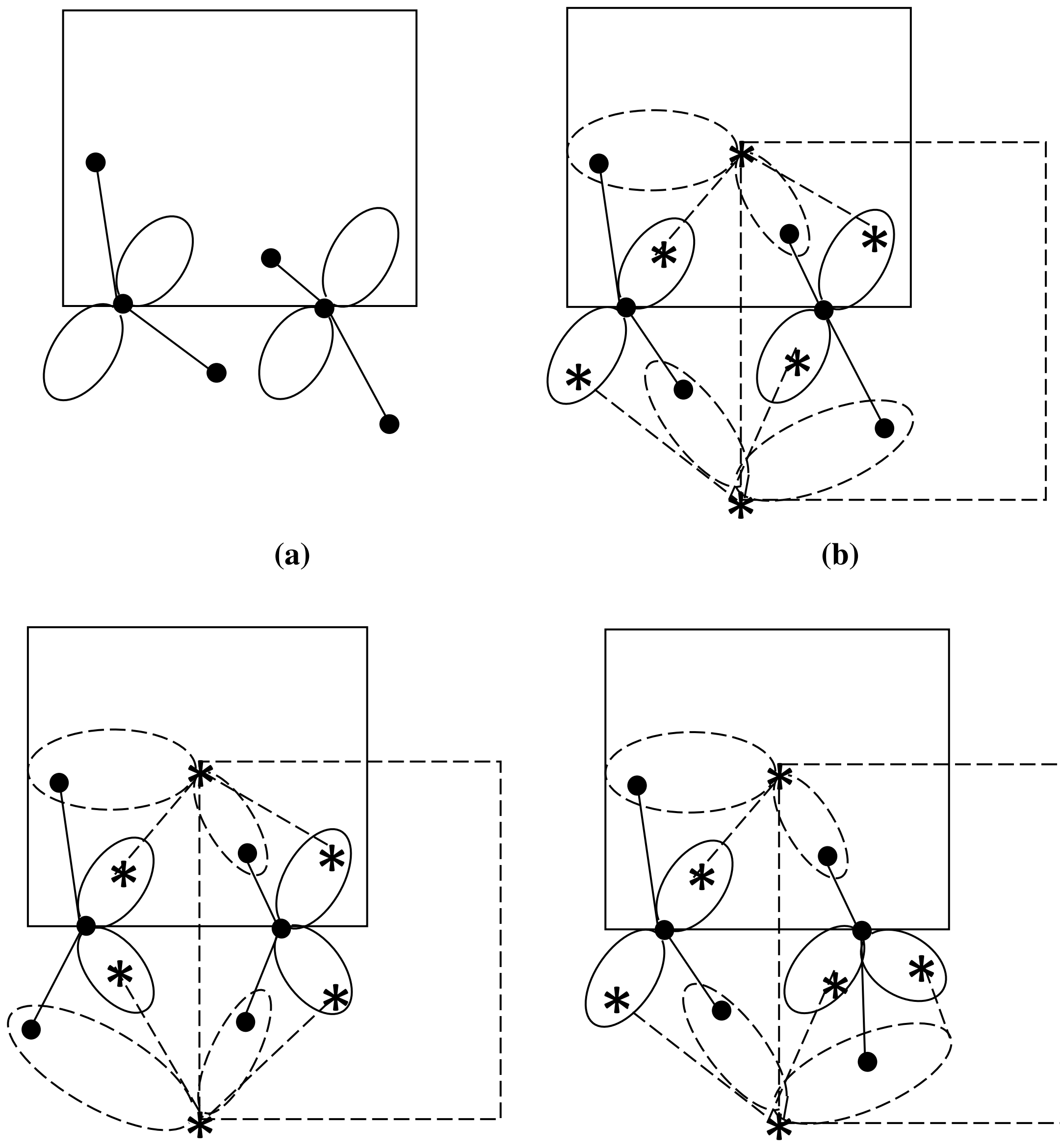
Given a map $X = (V, E, F)$ and its dual $X^{*} = (F^{*}, E^{*}, V^{*})$, there are three notions of self-duality. The strongest, map self-duality, requires that X and X^{*} are isomorphic as maps, that is, there is an isomorphism $\delta: (V, E, F) \to (F^{*}, E^{*}, V^{*})$ preserving incidences. A weaker notion requires only a graph isomorphism $\delta: (V, E) \to (F^{*}, E^{*})$, in which case we say that the map (V, E, F) is graph self-dual, and we say that $G = (V, E)$ is a self-dual graph.

Definition: 3.2.5

A **geometric duality** is a bijection $g: E(G) \to E(G^{*})$ such that $e \in E$ is the edge dual to $g(e) \in E(G^{*})$. If M is 2-cell, then M is connected; so if M is a 2-cell embedding, then $(M^{*})^{*} \cong M$ (we use * to indicate the geometric dual operation).

Definition: 3.2.6

An **algebraic duality** is a bijection $g: E(G) \rightarrow E(\hat{G})$ such that P is a circuit of G if and only if $g(p)$ is a minimal edge cut of $\hat{G}$. Given a graph $G = (V, E)$, an algebraic dual of G is a graph $\hat{G}$ for which there exist an algebraic duality $g: E(G) \rightarrow E(\hat{G})$.



(a) **(b)**

(c) **(d)**

Figure: 3.3

The geometric duals are shown in dotted lines. Embedding b) is map self-dual, c) is graphically self-dual and d) is algebraically self-dual.

We now define several forms of self-duality. Let $G = (V, E)$ be a graph and let $M = (V, E, F)$ be a fixed map of G, with geometric dual $M^* = (F^*, E^*, V^*)$.

Definition: 3.2.7

1. M is **map self-dual** if $M \cong M^*$.
2. M is **graphically self-dual** if $(V, E) \cong (F^*, E^*)$.
3. G is **algebraically self-dual** if $G \cong G^*$, where Ĝ is some algebraic dual of G.

Remark: 3.2.8

In the literature, the term matroidal or abstract is sometimes used where we use algebraic.

We will use the geometric duality operation and, unless specified, we will describe a graph as self-dual if it is graphically self-dual. Since, the dual of a graph is always connected, we know that a self-dual graph is connected.

The following are a few known results about self-dual graphs.

Corollary: 3.2.9

Let $M = (V, E, F)$ be a 2-cell embedding on an orientable surface. If M is self-dual, then $|E|$ is even.

Proof:

Since M is self-dual, By theorem (Euler), “Let $M = (V, E, F)$ be a 2-cell embedding of a graph in the orientable surface of genus k. Then,

$|V| - |E| + |F| = 2 - 2k$”.

$\Rightarrow \quad |E| = 2 - 2k - |V| - |F|$

$= 2(1 - k - |V|)$.

Theorem: 3.2.10

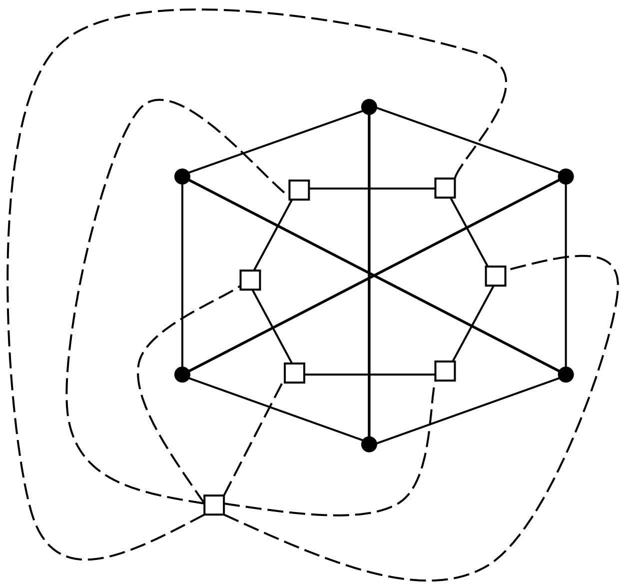
The complete graph K_n has a self-dual embedding on an orientable surface, if and only if $n \equiv 0 \; or \; 1 \; (mod \; 4)$.

Theorem: 3.2.11

For $w \geq 1$, there exists a self-dual embedding of some graph G of order n on $S_{n(w-1)+1}$ if and only if $n \geq 4w + 1$.

Note that a self-dual graph need not be self-dual on the surface of its genus. A single loop is planar; however it has a (non 2-cell) self-dual embedding on the torus.

Also note that there are infinitely many self-dual graphs. One such infinite family for the plane is the wheels. A wheel W_n consists of cycle of length n and a single vertex adjacent to each vertex on the cycle by means of a single edge called a Spoke. The complete graph on four vertices is also W_3 see Figure 3.4 for W_6.



The 6-wheel and its dual

Figure: 3.4

3.3 MATROIDS

Matroids may be considered a natural generalization of graphs. Thus when discussing a family of graphs, we should also consider the matroidal implications.

Definition: 3.1.1

Let S be a finite set, the ground set, and let I be a set of subsets of S, the independent sets. Then $M = (S, I)$ is a **matroid** if

1. $\emptyset \in I$;

2. If $J' \subseteq J \in I$, then $J' \in I$; and

3. For all $A \subseteq S$, all maximal independent subsets of A have the same cardinality.

An isomorphism between two matroids $M_1 = (S_1, I_1)$ and $M_2 = (S_2, I_2)$ is a bijection $\chi : S_1 \rightarrow S_2$ such that $I \in I_1$ if and only if $\chi(I) \in I_2$. If such a χ exists, then M_1 and M_2 are isomorphic denoted $M_1 \cong M_2$.

Given a graph $G = (V, E)$, the cycle matroid $M(G)$ of G is the matroid with ground set E, and $F \subseteq E$ is independent if and only if F is a forest. A matroid M is graphic if there exists a graph G such that $M = M(G)$.

For a matroid $M = (S, I)$ the dual matroid $M^* = (S, I^*)$ has ground set S and $I \subseteq S$ in I^* if there is a maximal independent set B in M such that $I \subseteq S \backslash B$. A matroid M is co-graphic if M^* is graphic. It is easily shown that if G is a connected planar graph, then $M^*(G) = M(G^*)$.

It is well known that G is algebraically self-dual if and only if cycle matroid of G and G^* are isomorphic.

4. A COMPARISON OF FORMS OF SELF-DUALITY

It is clear that for a map (V, E, F) we have,

Map self-duality $\Rightarrow$ Graph self-duality $\Rightarrow$ Matroid self-duality. However, In general, these implications cannot be reversed. But, we are concerned to what extent these implications can be reversed. The next two results assert that, in the most general case they cannot.

Result: 4.1

There exist a map (V, E, F) such that $(V, E) \cong (E^*, V^*)$, but $(V, E, F) \ncong (F^*, E^*, V^*)$.

Result: 4.2

There exist a map (V, E, F) such that $M(E) \cong M(E^*)^*$, but $(V, E) \ncong (F^*, E^*)$.

4.1 SELF-DUAL MAPS AND SELF-DUAL GRAPHS

A planar 3-connected simple graph has a unique embedding on the sphere, in the sense that if p and q are embeddings, then there is a homeomorphism h of the sphere so that $p = hq$. Any isomorphism between the cycle matroids of a 3-connected graph is carried by a graph isomorphism. Thus for a 3-connected graph

Map self-duality $\Leftarrow$ Graph self-duality $\Leftarrow$ Matroid self-duality.

So self-dual 3-connected graphs, as well as self-dual 3-connected graphic matroids, reduce to the case of self-dual maps. Since, the examples in figure 3.3 are only 1-connected, we must consider the 2-connected case. In figure 4.1 we see an example of a graphically self-dual map whose graph is 2-connected which is not map self-dual.

Theorem: 4.1.1

There exists a 2-connected map (V, E, F) which is graphically self-dual, so that $(V, E) \cong (F^*, V^*)$, but for which every map (V', E', F') such that $M(E) \cong M(E')$ is not map self-dual.

Proof:

Consider the map in figure 4.1, which is drawn on an unfolded cube. The graph is obtained by gluing two 3-connected self-dual maps together along an edge (a,b) and erasing the common edge. One map has only two reflections as self-dualities, both fixing the glued edge; the other has only two rotations of order four as dualities, again fixing the glued edge. The graph self-duality is therefore a combination of both, an order 4 rotation followed by a whitney twist of the reflective hemisphere. It is easy to see that all the embeddings of this graph, as well as the graph obtained after the whitney flip have the same property.

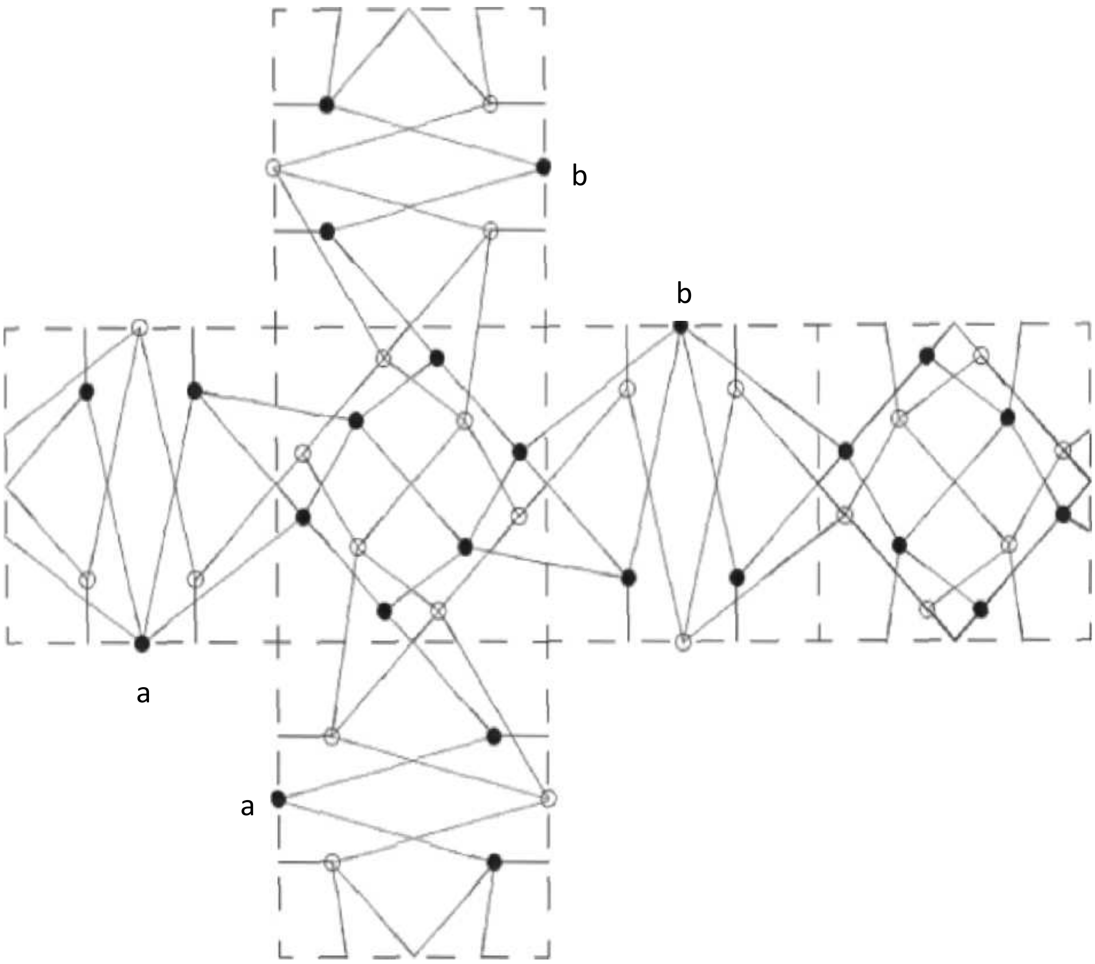


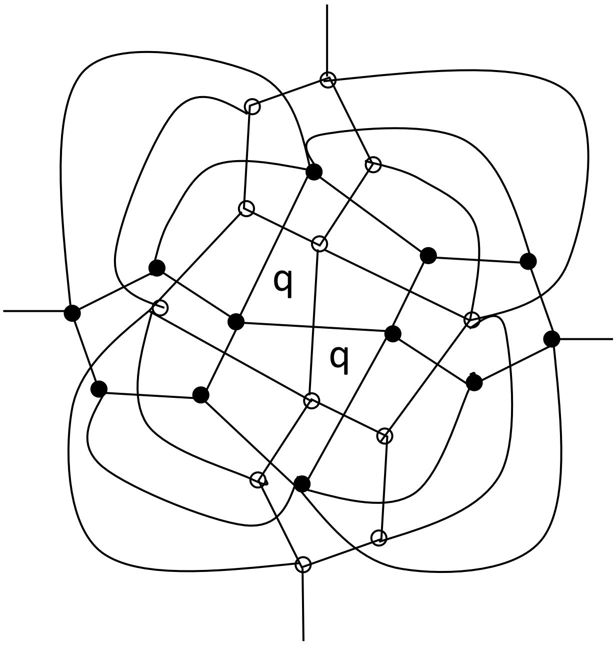
Figure: 4.1

Theorem: 4.1.2

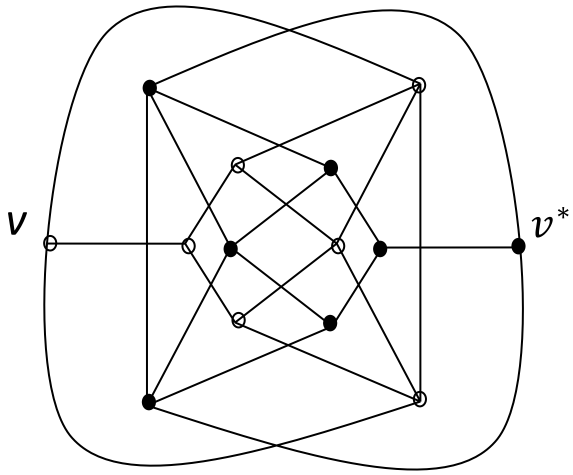
There is a graphically self-dual map (V, E, F) with (V, E) 1-connected and having only 3-connected blocks, but for which every map (V', E', F') such that $M(E) \cong M(E')$ is not map self-dual.

Proof:

Consider the 3-connected self-dual maps in Figure 4.2. X_1 has only self-dualities of order 4, two rotations and two flip rotations, while X_2 has only a left-right reflection and a 180° rotation as a self-duality. Form a new map X by gluing two copies of X_2 to X_1 in the quadrilateral marked with q's, with the gluing at the vertices marked v and v^*. X is graphically self-dual, as can easily be checked, but no gluing of two copies of X_2 can give map self-duality since every quadrilateral in X_1 has order 4 under any self-duality.



$\boldsymbol{X_1}$



$\boldsymbol{X_2}$

Figure: 4.2

In particular, self-dual graphs of connectivity less than 3 cannot in general be re-embedded as self-dual maps.

4.2 SELF-DUAL GRAPHS AND MATROIDS

If G is 1-connected, then its cycle matroid has a unique decomposition as the direct sum of connected graphic matroids, $M(G) = M_1 \oplus M_2 \oplus \dots \dots \dots \oplus M_k$, and if G^* is a planar dual of G, then $M(G^*) = M(G)^* = {M_1}^* \oplus {M_2}^* \oplus \dots \dots \dots \oplus {M_k}^*$. If G is a graph self-dual, then there is a bijection $\delta : M(G) \to M(G^*)$ sending cycles to cycles, and so there is a partition π of {1, 2, ...k} such that $\delta : M_i \to M_{\pi(i)}$, and we that $M(G)$ is the direct sum of self-dual connected matroids, together with some pairs of terms consisting of a connected matroid and its dual.

In the next theorem we see that not every self-dual matroid arises from a self-dual graph.

Theorem: 4.2.1

There exists a self-dual graphic matroid M such that for any graph $G = (V, E)$ with $M(G) = M$, and any embedding (V, E, F) of G, $(V, E) \ncong (F^*, E^*)$.

Proof:

Consider M_1 and M_2, the cycle matroids of two distinct 3-connected self-dual maps X_1 and X_2 whose only self-dualities are the antipodal map.

The matroid $M_1 \oplus M_2$ is self-dual, but its only map realizations are as the

1-vertex union of X_1 and X_2, which cannot be self-dual, since the cut vertex cannot simultaneously be sent to both "antipodal" faces.

So for 1-connected graphs, the three notions of self-duality are all distinct. For 2-connected graphs, however we have the following.

Theorem: 4.2.2

If $G = (V, E)$ is a planar 2-connected graph such that $M(E) \cong M(E)^*$, then G has an embedding (V, E, F) such that $(V, E) \cong (F^*, E^*)$.

Proof:

Let (V, E, F) be any embedding of G. Then G is 2-isomorphic, in the sense of whitney [15] to (F^*, E^*), and thus there is a sequence of whitney flips which transform (F^*, E^*, V^*) into an isomorphic copy of G and act as re-embeddings of G. thus the result is a new embedding (V, E, F') of G such that $(V, E, F') \cong (F'^*, E^*, V^*)$.

Thus, to describe 2-connected self-dual graphs it is enough up to embedding, to describe self-dual 2-connected graphic matroid.

4.3 SELF-DUAL MATROIDS

Definition: 4.3.1

A polyhedron P is said to be self-dual if there is an isomorphism $\delta : P \rightarrow P^*$, where P^* denotes the dual of P. We may regard δ as a permutation of the elements of P which sends vertices to faces and vice versa, preserving incidence.

As noted earlier 3-connected self-dual graphic matroids are classified via self-dual polyhedral.

On the other hand, 1-connected self-dual matroids are easily understood via the direct sum. Also we show how a 2-connected self-dual matroid M with self-duality δ arises via 3-connected graphic matroids by recursively constructing its 3-block tree $T(M)$ by adding orbits of pendent nodes.

The following theorem shows that this construction is sufficient to obtain all 2-connected self-dual matroids.

Theorem: 4.3.2

Let M be a self-dual connected matroid with 3-block tree T. Let T' be the tree obtained from T by deleting all the pendent nodes, and let M' be the 2-connected matroid determined by T'. Then M' is also self-dual.

Proof:

Let M be a self-dual connected matroid on a set E, so there is a matroid isomorphism $\Delta : M \to M^*$, so δ is a permutation of E sending cycles to co-cycles. The 3-block tree of M^* is obtained from that of M by replacing every label with the dual label, so Δ corresponds to a bijection $(\delta, \{\delta_\alpha\})$ of T onto itself, such that for each node α of T, $\delta_\alpha : M_\alpha \to M_{f(\alpha)}$ send cycles of M_α to co-cycles of $M_{f(\alpha)}$.

The restriction of $(\delta, \{\delta_\alpha\})$ to T' has the same property and so corresponds to a self-dual permutation of M'.

Theorem: 4.3.3

Suppose M is a self-dual 2-connected matroid with self-dual permutation δ and let $e_1 \in M$. Let $\{e_1, e_2, \dots \dots, e_k\}$ be the orbit of e_1 under δ. Suppose one of the following:

(1) k is even and M_0 is a 3-connected matroid or a cycle and δ_0 is a matroid automorphism of M_0 fixing an edge e_0.

(2) k is odd and M_0 is a 3-connected self-dual matroid with self-dual permutation δ_0 fixing an edge e_0.

For $i = 1,2, \dots \dots, k$ set $M_{2i+1} = M_0$ and $M_{2i} = {M_0}^*$. Let M' be the matroid obtained from M by 2-sums with the matroids M_i, amalgamating e_0 or ${e_0}^*$ in M_i with e_i.

Let δ' be defined by $\delta'(e)$ for $e \in M - \{e_1, e_2, \dots \dots, e_k\}$, $\delta' \colon M_k \to M_1$. Then M' is a 2-connected self-dual matroid with self-dual permutation δ'. Moreover, every 2-connected self-dual matroid and its self-duality is obtained in this manner.

Proof:

The fact that this construction gives a 2-connected self-dual matroid follows at once, since to check if δ' is a self-duality, it suffices to check that $(\delta')_{\alpha}$ send cycles to co-cycles on each 3-block.

The fact that M_0 must be self-dual if K is odd follows by considering that δ^{1k} is a self-duality and maps $M_0 = M_1$ onto itself.

To see that all self-dualities arise this way, let $\delta': M' \to M'$ be a self-duality, let α be a pendant node of T, and set $M_0 = M_\alpha$.

Let M be the self-dual matroid that results from removing from $T(M')$ the K nodes corresponding to the orbit of the node α. δ' induces $\delta: M \to M$. Then the desired δ_0 is $(\delta^k)_\alpha$.

4.4 THE STRUCTURES OF SELF-DUAL GRAPHS

Given the results of the previous section, we may construct all 2-connected self-dual graphs; start with any self-dual 2-connected graphic matroid M and chose any realization of M as a cycle matroid of a graph G. Theorem: 4.2.2, asserts that G has an embedding as a self-dual graph.

Alternatively, we may carry out a recursive construction in the spirit of Theorem: 4.4.1 at the graph level, paying careful attention to the orientations in the 2-sums.

The following theorem gives a more geometric construction.

Theorem: 4.4.1

Every 2-connected self-dual graph is 2-isomorphic to a graph which may be decomposed via 2-sums into self-dual maps such that the 2-sum on any two of the self-dual maps is along two edges, one of which is the pole of a rotation of order 4 and the other edge fixed by a reflection.

Proof:

Case: 1

In case 1 of Theorem: 4.3.3, we can always choose δ_0 to be the identity, and simply glue in the copies of the maps corresponding to M_0 and ${M_0}^*$ compatibly to make a self-dual map.

Case: 2

In case 2 we must have that M_0 is a self-dual 3-block containing a self-duality fixing e_0, hence it corresponds to a self-dual map and δ_0 must be a reflection or an order 4 rotation fixing e_0, and likewise the 3-block to which it is attached must be such an edge. If both are of the same kind, then the 3-blocks may be 2-summed into a self-dual map. This leaves only the mismatched pair.

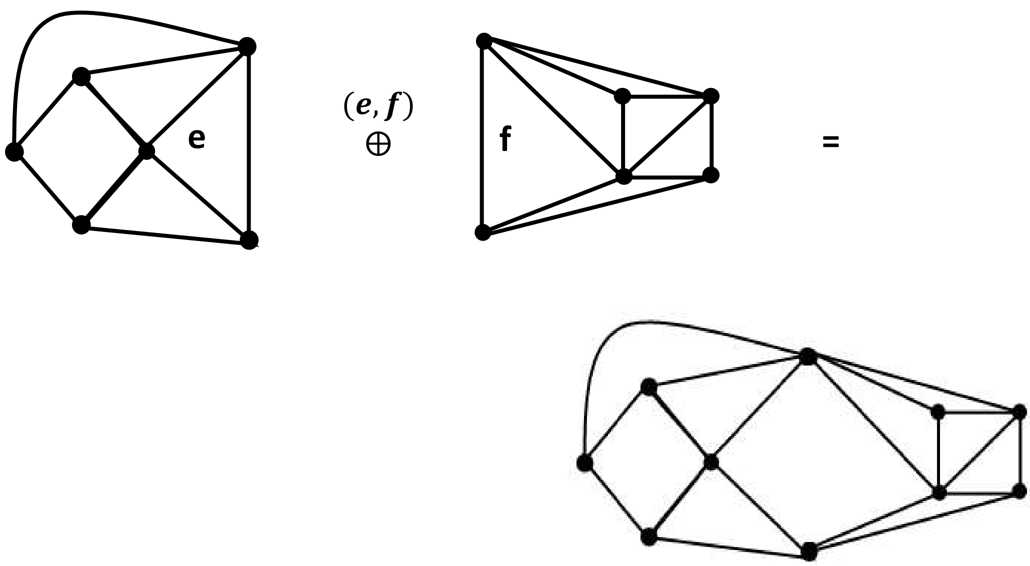


Figure: 4.3

To see that 2-isomorphism is necessary in the above, consider the self-dual graph in Figure:4.3. The map cannot be re-embedded as a self-dual map, nor does it have a 2-sum decomposition described as above, the graph is 2-isomorphic to a self-dual map.

5. Applications

Building the dual graph: the generalization of street segments and the Intersection Continuity Negotiation(ICN) model

A dual representation of street patterns is that a principle can be found that allows to extend the identity of a street over a plurality of edges; this problem can be referred to as one of finding a ‘‘generalization model’’. A generalization model is a process of complexity reduction used by cartographers while reducing the scale of a map; as for the street network, it is a two-steps process: firstly, single street segments are merged into longer ‘‘strokes’’; secondly, those strokes are selected by ‘‘importance’’ for map visualization.

In this context, the first step is relevant as it is about seeking a principle of continuity among different streets/edges, in order to capture the real sense of unity, or unique identity, of an urban street throughout a number of intersections. The question has been solved in Space Syntax substituting the primal graph representation of the network with the axial map ‘‘not properly a graph’’ where the principle of continuity is the linearity of the street spaces (Fig. 1A).

After a first attempt to anchor the representation of street patterns to an actual primal graph, based on characteristic nodes and visibility, Jiang and Claramunt have recently proposed one relevant model that builds a proper dual approach on a different primal representation: under their ‘‘named-street approach’’ (Fig. 1B) the principle of continuity is the street name: two different arcs of the original street network are assigned the same street identity if they share the same street name.

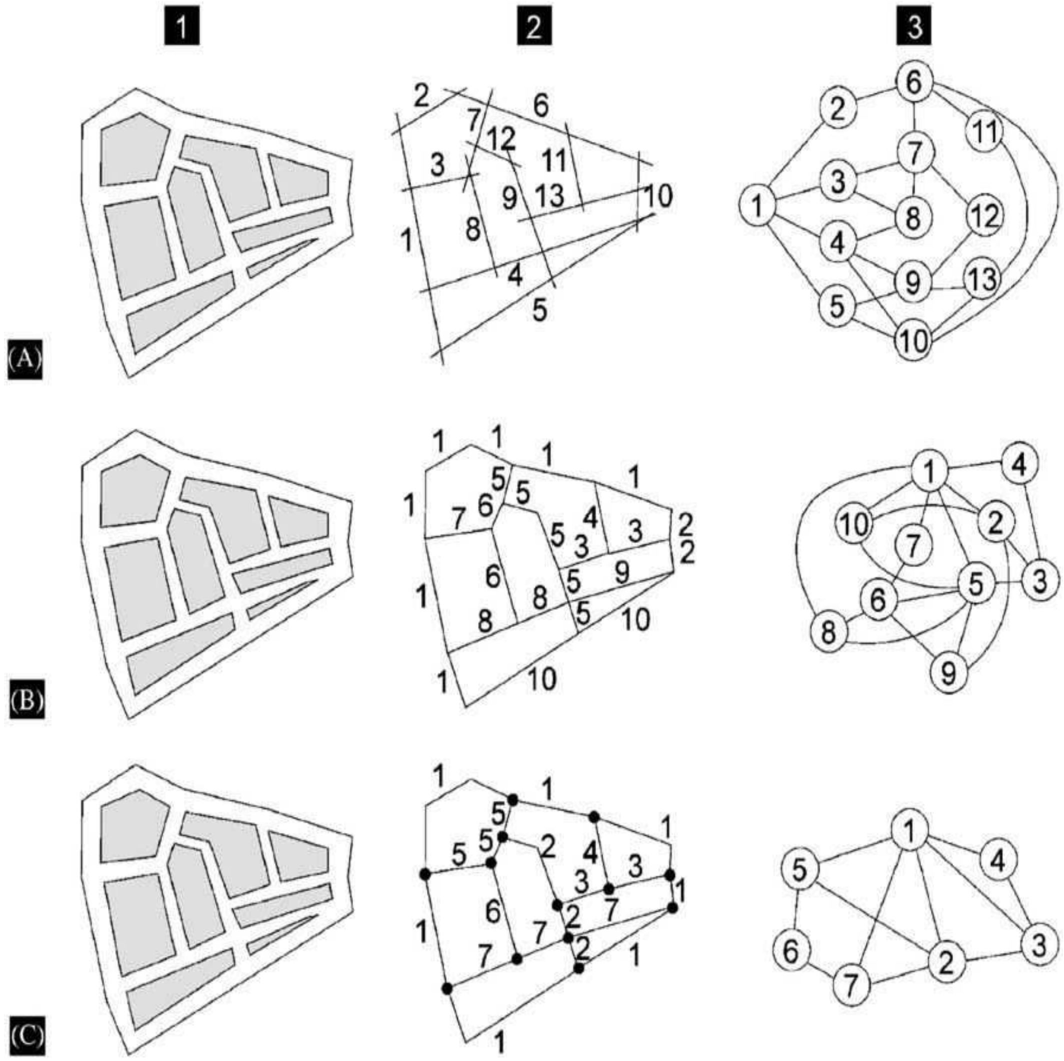


Figure: 5.1

[Figure:5.1. Row A: the Space Syntax way: (1) A fictive urban system; its (2) primal axial map network model; and its (3) dual connectivity graph. Row B: the named street way (street names replaced by numbers): (1) A fictive urban system; its (2) primal network model; and its (3) dual connectivity graph. Row C: the ICN way (street names replaced by numbers): (1) A fictive urban system; its (2) primal graph; and its (3) dual connectivity graph. In this latter proposal, the direct representation of the urban network is properly a graph, where intersections are turned into nodes and street arcs into edges; edges follow the footprint of real mapped streets (a linear discontinuity does not generate a vertex); the ICN process assigns the concatenation of street identities throughout nodes followinga principle of ''good continuation''.]

The main problem with this approach is that it introduces a nominalistic component in a pure spatial context, resulting in a loss of coherence of the process as a whole: street names are not always meaningful in any sense, they are not always reliable as the same street may be termed in different ways by different social groups, or in different contexts, at different scales, in different ages.

Other problems are that street name databases are not easily available for all cases or at all scales, and that the process of embedding and updating street names into GIS seems rather costly for large datasets. However, implemented by Jiang and Claramunt on three real cases, the named-street approach has led to recognize a small-world character in large street networks, but no scale-free behaviour in their degree distribution. In this work we use a generalization model based on a different principle of continuity, one of ‘‘good continuation’’, based on the preference to go straight at intersections, a well known cognitive property of human way finding. The model, which we term ICN is quite simple and purely spatial, in that it excludes anything that cannot be derived by the sole geometric analysis of the primal graph itself (Fig. 5.1).

The model runs in three steps:

(1) All the nodes are examined in turn. At each node, the continuity of street identity is negotiated among all pairs of incident edges: the two edges forming the largest convex angle are assigned the highest continuity and are coupled together; the two edge with the second largest convex angle are assigned the second largest continuity and are coupled together, and so forth; in nodes with an odd number of edges, the remaining edge is given the lowest continuity value.

(2) Beginning with one edge chosen at random in the graph, a street ID code is assigned to the edge and, at relevant intersections, to the adjacent edges coupled in step 1.

(3) The dual graph is constructed by mapping edges coded with the same street ID in the primal graph into nodes of the dual graph, and intersections among each pair of edges in the primal graph into edges connecting the corresponding nodes of the dual graph. Overlaying double edges in the dual graphs are eliminated. Being based on a primal graph, ICN minimizes subjectivity and re-enter the mainstream of the network representation of urban and territorial

patterns. Being based on a pure spatial principle of continuity, it avoids problems of social interpretation within a pure spatial context. Finally, it allows a dual, step-distance representation of urban street networks linking it to a primal graph, which opens to further investigations in geographic-Euclidean space.

A STUDY ON MATRIX GROUPS AND THEIR LIE ALGEBRAS

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P. POOVINSHAL

Reg. No: 19SPMT21

Under the guidance of

Ms. J. JENIT AJITHA M.Sc., M.Phil.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

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CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON MATRIX GROUPS AND THEIR LIE ALGEBRAS"** is submitted to **St.Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **P. POOVINSHAL (Reg. No: 19SPMT21)**

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **" A STUDY ON MATRIX GROUPS AND THEIR LIE ALGEBRAS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. J. Jenit Ajitha M.Sc., M.Phil.,** Assistant professor, Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi.**

P. Poovinshal

Signature of the Student

Station: Thoothukudi

Date: 10. 04. 2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER-1

PRELIMINARIES

Definition: 1.1

A group is a non-empty set G on which there is a binary operation $* : G \times G \to G$ such that

• If a and b belong to G then $a * b$ is also in G (*closure*)

• $a * (b * c) = (a * b) * c$ for all a, b, c in G (*associativity*)

• There is an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$ (*identity*).

e is called the identity element of G.

• If $a \in G$, then there is an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = 1$ (*inverse*).

a^{-1} is called the inverse of a.

Definition: 1.2

A group G is called abelian if the binary operation is commutative, i.e., $a * b = b * a$ for all $a, b \in G$. A group which is not abelian is called a *non-abelian group*.

Examples: 1.1

1. **The additive group of integers:**

- Let $\mathbb{Z}$ be the set of integers.
- Let $+$ be the binary operation of addition in $\mathbb{Z}$.
- $n + 0 = n = 0 + n$ for every $n \in \mathbb{Z}$. Thus $(\mathbb{Z}, +)$ has an identity element.
- If l, m, n are integers, $(l + m) + n = l + (m + n)$
- i.e. $(\mathbb{Z}, +)$ is a semigroup.
- If $n \in \mathbb{Z}$, then $-n$ in $\mathbb{Z}$ has the property $n + (-n) = 0 = (-n) + n$

- i.e. $-n$ is an inverse of n in $(\mathbb{Z}, +)$.

Thus we have shown that $(\mathbb{Z}, +)$ is a group. This group is usually referred to as the additive group of integers. $\mathbb{Z}$ with the addition and 0 as identity is an abelian group.

2. $\mathbb{Z}$ with the multiplication is not a group since there are elements which are not invertible in $\mathbb{Z}$.

Definition: 1.3

Let $*$ be binary operation defined on G. An element $e \in G$ is called *a left identity* if $e * a = a$ for all $a \in G$. e is called *a right identity* if $a * e = a$ for all $a \in G$.

Definition: 1.4

Let $*$ be a binary operation defined on G. Let $e \in G$ be the identity element. Let $a \in G$. An element $a^{-1} \in G$ is called *a left inverse* of a if $a^{-1} * a = e$. a^{-1} is called a *right inverse* of a if $a * a^{-1} = e$.

Definition: 1.5

Let *A* be a finite set. A bijection from *A* to itself is called a *permutation of A.*

Example: 1.2

If $A = \{1,2,3,4\}$, $f : A \rightarrow A$given by $f(1) = 2$, $f(2) = 1$, $f(3) = 4$ and $f(4) = 3$ is a permutation of A. We shall write this permutation as $\begin{pmatrix} 1 & 2 & 34 \\ 2 & 1 & 43 \end{pmatrix}$.

An element in the bottom row is the image of the element just above it in the upper row.

Definition: 1.6

Let A be a finite set containing n elements. The set of all permutations of A is clearly a group under the composition of functions. This group is called the *symmetric group of degree* n and is denoted S_n.

Definition: 1.7

The set of $n \times n$ invertible matrices with real coefficients is a group for the matrix product and identity the matrix I_n. It is denoted by $GL_n(\mathbb{R})$ and called the *general linear group*. It is not abelian for $n \geq 2$.

Definition: 1.8

The order of a group G, denoted by $|G|$, is the cardinality of G, that is the number of elements in G.

Examples: 1.3

1. The trivial group $G = \{0\}$ may not be the most exciting group to look at, but still it is the only group of order 1.

2. The group $G = \{0, 1, 2, \dots, n-1\}$ of integers modulo $n$is a group of order n.

Definition: 1.9

A finite group is a group with a finite number of elements. Otherwise, it is an infinite group.

Definition: 1.10

A subgroup H of a group G is a non-empty subset of G that forms a group under the binary operation of G.

Examples: 1.4

1. If we consider the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo 4, $H = \{0, 2\}$ is a subgroup of G.

2. The set of $n \times n$ matrices with real coefficients and determinant of 1 is a subgroup of $GL_n(\mathbb{R})$, denoted by $SL_n(\mathbb{R})$ and called *the special linear group*.

Definition: 1.11

The order of an element $a \in G$ is the least positive integer n such that $a_n = 1$. If no such integer exists, the order of a is infinite. We denoteit by $|a|$.

Definition: 1.12

A group G is cyclic if it is generated by a single element, which we denote by $G = \langle a \rangle$. We may denote by C_n a cyclic group of n elements.

Example: 1.5

A finite cyclic group generated by $a$is necessarily abelian, and can be written (multiplicatively)

$\{1, a, a_2, \dots, a_{n-1}\}$ with $a_n = 1$

or (additively)

$\{0, a, 2a, \dots, (n - 1)a\}$with $na = 0$.

A finite cyclic group with n elements is isomorphic to the additive group $\mathbb{Z}_n$ of integers modulo n.

Definition: 1.13

Let H be a subgroup of a group G. If $g \in G$, the right coset of Hgenerated by g is $Hg = \{hg, h \in H\}$ and similarly the left coset of Hgenerated by

g is $gH = \{gh, h \in H\}$. In additive notation, we get $H + g$ (which usually implies that we deal with a commutative group where we do not need to distinguish left and right cosets).

Example: 1.6

If we consider the group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$and its subgroup $H = \{0, 2\}$ which is isomorphic to $\mathbb{Z}_2$, the cosets of H in $G$are

$$0 + H = H,$$

$$1 + H = \{1, 3\},$$

$$2 + H = H,$$

$$3 + H = \{1, 3\}.$$

Clearly $0 + H = 2 + H$and $1 + H = 3 + H$.

Definition: 1.14

Let H be a subgroup of G. The number of distinct left(right) cosets of H in G is called *the index of* H *in* G and is denoted by $[G: H]$.

Example: 1.7

In $(\mathbb{Z}_8, \oplus)$, $H = \{0,4\}$ is a subgroup. The left cosets of H are given by

$$0 + H = \{0,4\} = H$$

$$1 + H = \{1,5\}$$

$$2 + H = \{2,6\}$$

$$3 + H = \{3,7\}$$

These are the four distinct left cosets of H. Hence the index of the subgroup H is 4.

Definition: 1.15

Let G be a group and $H \leq G$. We say that H is a normal subgroup of G, or that H is normal in G, if we have $cHc^{-1} = H$, for all $c \in G$.

We denote it $H \trianglerighteq G$, or $H \triangleright G$ when we want to emphasize that H is a proper subgroup of G.

Example: 1.8

Let $GL_n(\mathbb{R})$ be the group of $n \times n$ real invertible matrices, and let $SL_n(\mathbb{R})$ be the subgroup formed by matrices whose determinant is 1. Let us see that $SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$.

Definition: 1.16

Let N be a normal subgroup of G. Then the group G/N is called *the quotient group* (factor group) of G modulo N.

Example: 1.9

$3\mathbb{Z}$ is a normal subgroup of $(\mathbb{Z}, +)$. The quotient group $\mathbb{Z}/3\mathbb{Z} = \{3\mathbb{Z} + 0, 3\mathbb{Z} + 1, 3\mathbb{Z} + 2\}$. Hence $\mathbb{Z}/3\mathbb{Z}$ is a group of order 3.

Definition: 1.17

Given two groups G and H, a group homomorphism is a map $f : G \to H$ such that $f(xy) = f(x)f(y)$ for all $x, y \in G$.

Obviously every isomorphism is a homomorphism and a bijective homomorphism is an isomorphism.

Example: 1.10

The map $f : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$, defined by $f(x) = 2x$ is a group homomorphism.

For, $f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$

Note that f is $1-1$.

Definition: 1.18

Two groups G and $H$are isomorphic if there is a group homomorphism $f : G \to H$ which is also a bijection.

Example: 1.11

If we consider again the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo 4 with subgroup $H = \{0, 2\}$, we have that $H$is isomorphic to $\mathbb{Z}_2$, the group of integers modulo 2.

CHAPTER-II

MATRIX GROUPS

Definition: 2.1

A *binary operation* $*$ on a set S is a function mapping $S \times S$ into S. For each $(a, b) \in S \times S$, we will denote the element $*((a, b))$ of S by $a * b$.

Definition: 2.2

Let $*$ be a binary operation on S and let H be a subset of S. The subset H is *closed under* $*$ if for all $a, b \in H$ we also have $a * b \in H$. In this case, the binary operation on H given by restricting $*$to H is the *induced operation* of $*$ on H.

Definition: 2.3

A group $\langle G,* \rangle$ is a set G, closed under a binary operation $*$, such that the following properties are satisfied.

(1) (*Associativity*) For all $a, b, c \epsilon G$

$a, b, c \in G,\ (a * b) * c = a * (b * c).$

(2) (*Identity*)

There is a unique element e in G such that for all $x \in G$,

$e * x = x * e = x.$

(3) (*Inverse*)

For each $a \in G$, there is a unique element a'in G such that

$a * a' = a' * a = e.$

Definition: 2.4

If a subset H of a group G is closed under the binary operation of G and if $H$with the induced operation from G is a group, then H is a *subgroup* of G.

Notation:

Let $M_n(\mathbb{R})$ denotes the set of all $n \times n$ square matrices with entries in $\mathbb{R}$.

Definition: 2.5

The general linear group over $\mathbb{R}$ is

$GL_n(\mathbb{R})$= $\{A \in M_n(\mathbb{R}) \mid \exists B \in M_n(\mathbb{R})$ with $AB = BA = I_n\}$

Where I_n is the $n \times n$ identity matrix, i.e. $GL_n(\mathbb{R})$ is the collection of all invertible $n \times n$ matrices.

Theorem: 2.6

$GL_n(\mathbb{R})$ is a group with the operation being matrix multiplication.

Proof:

Let $n \in N$ be arbitrary and consider $GL_n(\mathbb{R})$.

Recall that for square matrices $A, B \in M_n(\mathbb{R})$, $det(A)\, det(B) = det(AB)$. Since for all $X, Y \in GL_n(\mathbb{R})$, $det(X) \neq 0 \neq det(Y)$, it follows that

det$(XY) = det(X)det(Y) \neq 0$. Thus $XY \in GL_n(\mathbb{R})$ so $GL_n(\mathbb{R})$ is closed under matrix multiplication.

(*Associativity*)

Let $A = (a_{ij})$, $B = (b_{ij})$, $C = \left(c_{ij}\right) \in M_n(\mathbb{R})$. Then we have

$(A * B) * C = ((a_{ij}) * (b_{ij})) * (c_{ij})$

$= \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right) * (c_{ij})$

$$= \left(\sum_{l=1}^{n} (\sum_{k=1}^{n} a_{ik} b_{kl}) * (c_{lj})\right)$$

$$= \left(\sum_{l=1}^{n} a_{il} * \left(\sum_{k=1}^{n} b_{lk} * c_{kj}\right)\right)$$

$$= \left(a_{ij}\right) * \left(\sum_{k=1}^{n} b_{lk} * c_{kj}\right)$$

$$= (a_{ij}) * ((b_{ij}) * (c_{ij}))$$

$$= A * (B * C).$$

Thus, matrix multiplication is associative, so Associativity holds for $GL_n(\mathbb{R})$ in particular.

(*Identity*)

Let $A \in GL_n(\mathbb{R})$where

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots \ \ddots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

Let I_n be the matrix defined as

$$I_n = \begin{bmatrix} 1 & 0 \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \vdots & \vdots \ \ddots & \vdots \\ 0 & 0 \cdots & 1 \end{bmatrix}$$

It immediately follows that $AI_n = I_nA = A$, and since $det(I_n) = 1$, $I_n \in GL_n(\mathbb{R})$

(*Inverses*)

Since for all $A \in GL_n(\mathbb{R})$, A is invertible by the definition of $GL_n(\mathbb{R})$, so A^{-1} exists. And since A^{-1} is also invertible ($(A^{-1})^{-1} = A$), $A^{-1} \in GL_n(\mathbb{R})$.

Therefore, $GL_n(\mathbb{R})$ is a group under matrix multiplication.

Definition: 2.7

A sequence of real numbers (a_n) converges to a real number a if, for every $\in > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \in$.

Definition: 2.8

Let A_m be a sequence of matrices in $M_n(\mathbb{R})$. We say that A_m *converges* to a matrix A if each entry of A_m converges (as $m \to \infty$ to the corresponding entry of A (i.e. if $(A_m)_{ij}$ converges to $(A)_{ij}$for all $1 \leq i, j \leq n$).

Definition: 2.9

A *matrix group* is any subgroup $G \subset GL_n(\mathbb{R})$ with the following property. If $A_m$is any sequence of matrices in G, and A_m converges to some matrix A, then either $A \in G$, or A is not invertible.

Definition: 2.10

The special linear group over $\mathbb{R}$, denote $SL_n(\mathbb{R})$, is the set of all $n \times n$matrices with a determinant of 1, that is

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) | detA = 1\}.$$

Definition: 2.11

A function $f : A \to \mathbb{R}$ is *continuous* at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.

Definition: 2.12

Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ it follows that $|f(x) - L| < \epsilon$.

Theorem: 2.13

Let $f : A \to \mathbb{R}$and let $c \in A$ be such that there exists some sequence (x_n) where $x_n \in A$ for all $n \in N$ and $x_n \to c$. The function f is continuous at c if and only if any one of the following holds true.

(1)$f(x_n) \to f(c)$,

(2) $lim_{x\to c} f(x) = f(c)$.

Proof:

Let $f : A \to \mathbb{R}$, and let $c \in A$ be such that the sequence (x_n) where $x_n \in A$ for all $n \in N$ has the property that $x_n \to c$. Suppose that $f$is continuous at c and let $\epsilon > 0$ be arbitrary. Since f is continuous at c, there exists some $\delta > 0$ such that whenever $x \in A$ and $|x - c| < \delta$ we are guaranteed that $|f(x) - f(c)| < \epsilon$. Towards contradiction, assume that $\lim_{n\to\infty} f(x_n) \neq f(c)$. Thus there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - c| < \delta$ and $|f(x_n) - f(c)| \geq \epsilon$, which is a contradiction to our assumption that $|f(x) - f(c)| < \epsilon$ for all $x \in A$. Thus $f(x_n) \to f(c)$ by contradiction.

Now suppose that $f$is not continuous at c. This implies that there exists some $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists some $x_0 \in A$ such that $|x_0 - c| < \delta$ and $|f(x_0) - f(c)| \geq \epsilon_0$. For each $n \in \mathbb{N}$, let $\delta_n = 1/n$. This implies that there exists some $x_n \in A$ such that $|x_n - c| > \delta_n$ and $|f(x_n) - f(c)| \geq \epsilon_0$. Clearly, the sequence (x_n) has the property that $x_n \to c$, as for all $\epsilon > 0$, there exists some$N \in \mathbb{N}$ such that for all $n \geq N$, it follows that $|x_n - c| < \delta_n < \epsilon$. Thus, the sequence (x_n) has the property that $x_n \to c$ and for all $N' \in N$there exists some $n_0 \geq N'$such that $|f(x_{no}) - f(c) \geq| \epsilon_0$. This proves that if $x_n \to c$(with $x_n \in A$), then $f(x_n) \to f$(c). Thus, f is continuous at c by the contrapositive. Therefore, statement (1) of Theorem 2.13 holds if and only if f is continuous at c.

(2) We show that statement (1) is equivalent to statement (2). Using Definition 2.12, $lim_{x \to c} f(x) = f(c)$ states that for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ it follows that $|f(x) - f(c)| < \epsilon$. This is equivalent to the statement "if $x_n \to c$ (with $x_n \in A$), then $f(x_n) \to f(c)$" using Definition 2.11. Therefore, statement (1) is equivalent to statement (2), proving Theorem 2.13 in its entirety.

Theorem: 2.14

The determinant function det : $M_n(\mathbb{R}) \to \mathbb{R}$ is continuous.

Proof:

The proof will proceed by induction on n. First, let $n = 1$. Since the determinant of a 1×1 real matrix is simply the entry itself, the determinant function is continuous as it just outputs the entry itself. Thus, thedeterminant function from $M_1(\mathbb{R})$ to $\mathbb{R}$ is continuous. Now, assume that the determinant function from $M_N(\mathbb{R})$ to $\mathbb{R}$ is continuous, with the goal of proving that the determinant function from $M_{n+1}(\mathbb{R})$ to $\mathbb{R}$ is continuous.

Let A $\in M_{n+1}(\mathbb{R})$

Where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & a_{n+1,n+1} \end{bmatrix}$$

By the definition of the determinant, $\det A = \sum_{i=1}^{n+1} (-1)^{i+j} a_{i,j} M_{i,j}$ where $M_{i,j}$ is the minor of the $i - j^{th}$ entry. Since $M_{i,j}$ is the determinant of a $n \times n$ matrix for each

$i, j \in \{1,2,\cdots,n+1\}$, $det\ A$ is simply a sum of continuous functions multiplied by a real number, so det A is continuous. Thus, the determinant function from $M_{n+1}(\mathbb{R})$ to $\mathbb{R}$ is continuous, proving the original statement by induction.

Theorem: 2.15

$SL_n(\mathbb{R})$ is a matrix group.

Proof:

Let $n \in \mathbb{N}$ be arbitrary. We first prove that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$. Let$A, B \in SL_n(\mathbb{R})$. Since $\det(AB) = \det(A)\det(\mathrm{B})$ and $det(A) = 1 = det(B)$. since $A, B \in SL_n(\mathbb{R})$, it follows that $\det(AB) = \det(A)\det(B) = 1(1)$=1.$Thus\ AB \in SL_n(\mathbb{R})$, so $SL_n(\mathbb{R})$is closedunder matrix multiplication. Also, since $det(I_n) = 1$, $I_n \in SL_n(\mathbb{R})$. Lastly, Since $det(AA^{-1}) = det(I_n) = 1 = det(A)det(A^{-1})$ and $det(A)$=1, it follows $det(A^{-1})$=1,so $A^{-1} \in SL_n(\mathbb{R})$. Thus $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

Let (A_m) be a sequence of matrices where $A_m \in SL_n(\mathbb{R})$ for each $m \in N$ and $A_m \to A$. Since $detA_m = 1$ for all $m \in N$ and since the determinant is a continuous function by Theorem 2.14, it follows by Theorem 2.13 that $detA = 1$A=1 as well. Therefore, A $\in SL_n(\mathbb{R})$, so $SL_n(\mathbb{R})$ is a matrix group. To understand the orthogonal group $O_n(\mathbb{R})$, we will first cover what it means to be orthogonal.

Definition: 2.16

The standard inner product on R^n is the function from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle (x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n) \rangle_{\mathbb{R}} = x_1 . y_1 + x_2 . y_2 + \cdots + x_n . y_n$$

Definition: 2.17

The standard norm on R^n is the function from $\mathbb{R}^n \to \mathbb{R}^+$defined by

$|x|_{\mathbb{R}} = \sqrt{\langle x, x\rangle_{\mathbb{R}}}$.

Definition: 2.18

Vectors $x, y \in R^n$ are called *orthogonal* if $\langle x, y\rangle = 0$.

Definition: 2.19

A vector $x \in R^n$ is called *a unit vector* $|x| = 1$.

Definition: 2.20

A matrix $A \in M_n(\mathbb{R})$is said to be orthogonal if the column vectors of A are orthogonal unit vectors.

Note that this definition is equivalent to stating that $\langle xA, yA\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}$. This condition is known as an isometry condition, meaning that an orthogonal matrix is a distance preserving linear transformation. It follows from the above definition alone that for all orthogonal matrices $A \in M_n(\mathbb{R})$, $A^TA = I_n = AA^T$ where A^T is the transpose of matrix A, that is, if $a_{i,j}$ is the entry of A in the i^{th} row and j^{th} column of A, then $a_{i,j}$ is the entry in the jth row and i^{th} column of A^T .The following definitions generalizes orthogonality over different fields.

Definition: 2.21

The orthogonal group over $\mathbb{R}$ is defined as

$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) | \langle xA, yA\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n\}$.

Definition: 2.22

A set $\{x_1, x_2, \dots, x_n\}$ of $\mathbb{R}^n$ is called *orthonormal* if $\langle x_i, x_j\rangle = 1$ when $i = j$ and $\langle x_i, x_j\rangle = 0$ when $i \neq j$.

As an example, an orthonormal set of $\mathbb{R}^n$, is the set

$$\mathcal{B} = \{e_1 = (1,0,\ldots,0), e_2 = (0,1,\ldots,0), \ldots, e_n = (0,0,\ldots,1)\}$$

The set $\mathcal{B}$ is called the standard orthonormal basis for $\mathbb{R}^n$.

Lemma: 2.23

If $A, B \in M_n(\mathbb{R})$, then $AB^T = B^T A^T$.

Proof:

Let $A, B \in M_n(\mathbb{R})$ where

$$A = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots \ \ddots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12}\cdots & b_{1n} \\ b_{21} & b_{22}\cdots & b_{2n} \\ \vdots & \vdots \ \ddots & \vdots \\ b_{n1} & b_{n2}\cdots & b_{nn} \end{bmatrix}$$

The following equalities hold.

$$(AB)^T = \left(\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots \ \ddots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12}\cdots & b_{1n} \\ b_{21} & b_{22}\cdots & b_{2n} \\ \vdots & \vdots \ \ddots & \vdots \\ b_{n1} & b_{n2}\cdots & b_{nn} \end{bmatrix} \right)^T$$

$$= \left(\begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2}\cdots & a_{11}b_{1n} + \cdots + a_{1n}b_{nn} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2}\cdots & a_{21}b_{1n} + \cdots + a_{2n}b_{nn} \\ \vdots & \vdots \qquad \ddots & \vdots \\ a_{n1}b_{11} + \cdots + a_{nn}b_{n1} & a_{n1}b_{12} + \cdots + a_{nn}b_{n2}\cdots & a_{n1}b_{1n} + \cdots + a_{nn}b_{nn} \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{1n} & a_{21}b_{11} + \cdots + a_{2n}b_{n1} \ \cdots & a_{n1}b_{11} + \cdots + a_{nn}b_{n1} \\ a_{11}b_{12} + \cdots + a_{1n}b_{n2} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} \ \cdots & a_{n1}b_{12} + \cdots + a_{nn}b_{n2} \\ \vdots & \vdots \qquad \ddots & \vdots \\ a_{11}b_{1n} + \cdots + a_{1n}b_{nn} & a_{21n}b_{1n} + \cdots + a_{2n}b_{nn}\cdots & a_{n1}b_{1n} + \cdots + a_{nn}b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{21}\cdots & b_{n1} \\ b_{12} & b_{22}\cdots & b_{n2} \\ \vdots & \vdots \ \ddots & \vdots \\ b_{1n} & b_{2n}\cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21}\cdots & a_{n1} \\ a_{12} & a_{22}\cdots & a_{n2} \\ \vdots & \vdots \ \ddots & \vdots \\ a_{1n} & a_{2n}\cdots & a_{nn} \end{bmatrix}$$

$$= B^T A^T.$$

Theorem: 2.24

If $A \in M_n(\mathbb{R})$, then $(A^n)^T = (A^T)^n$.

Proof:

Let $A \in M_n(\mathbb{R})$. We proceed by induction on n. First, let $n = 1$. Clearly, $(A^1)^T = A^T = (A^T)^1$, so this case holds. Now, suppose that $(A^n)^T = (A^T)^n$ holds for some $n \in \mathbb{N}$.

By Lemma 2.23, it follows that,

$(A^T)^{n+1} = (A^T)^n A^T = (A^n)^T A^T = (AA^n)T = (A^{n+1})^T$.

Therefore, $(A^n)^T = (A^T)^n$ is true by the principle of mathematical induction.

Definition: 2.25

If $A \in M_n(\mathbb{R})$, define $\mathbb{R}_A: \mathbb{R}^n \to \mathbb{R}^n$and $L_A: \mathbb{R}^n \to \mathbb{R}^n$such that for all $x \in \mathbb{R}^n$, $\mathbb{R}_A(x) = x. A$and $L_A(x) = (A. x^T)^T$.

Theorem: 2.26

For all $A \in GL_n(\mathbb{R})$, $A \in O_n(\mathbb{R})$ if and only if $A . A^T = I_n$.

Proof:

Let $A = [a_{ij}]_n \in GL_n(\mathbb{R})$ be arbitrary.

($\Rightarrow$) Suppose that $A \in O_n(\mathbb{R})$. Since $\{e_1 = (1,0,\ldots,0), e_2 = (0,1,\ldots,0), \ldots, e_n = (0,0,\ldots,1)\}$ is an orthonormal basis for $\mathbb{R}^n$ and $\langle x . A, y . A\rangle = \langle x, y\rangle$, it follows that

$$\{\mathbb{R}_A(e_1), \mathbb{R}_A(e_2), \ldots, \mathbb{R}_A(e_n)\}$$

is an orthonormal set of vectors.$\{\mathbb{R}_A(e_1), \mathbb{R}_A(e_2), \ldots, \mathbb{R}_A(e_n)\}$ is precisely the set of row vectors of A, where $\mathbb{R}_A(e_i)$ is the i^{th} row of A. Notice that

$$(A \,.\, A^T)_{ij} = (\text{row } i \text{ of } A).(\text{column } j \text{ of } A^T)$$

$$= (\text{row } i \text{ of } A).(\text{row } j \text{ of } A)$$

$$= \langle(\text{row } i \text{ of } A), (\text{row } j \text{ of } A)\rangle.$$

Thus, $(A \,.\, A^T)_{ij} = 1$.

when $i = j$ as $\langle(\text{row } i \text{ of } A), (\text{row } j \text{ of } A)\rangle = \langle\mathbb{R}_A(e_i), \mathbb{R}_A(e_i)\rangle = 1$ and $(A \,.\, A^T)_{ij} = 0$ when $i \neq j$ as $\langle(\text{row } i \text{ of } A), (\text{row } j \text{ of } A)\rangle = \langle\mathbb{R}_A(e_1), \mathbb{R}_A(e_2)\rangle = 0$. Thus, $A \,.\, A^T = I_n$.

($\Leftarrow$) Suppose that $A \,.\, A^T = I_n$. This implies that $\langle\mathbb{R}_A(e_i), \mathbb{R}_A(e_i)\rangle = 1$ and $\langle\mathbb{R}_A(e_i), \mathbb{R}_A(e_j)\rangle = 0$ when $i \neq j$. Let be arbitrary where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.

We see that,

$$\langle x \,.\, A, y \,.\, A\rangle = \langle\mathbb{R}_A(x), \mathbb{R}_A(y)\rangle$$

$$= \langle\textstyle\sum_{i=1}^{n} x_i(\text{row } i \text{ of } A), \sum_{j=1}^{n} y_j(\text{row } j \text{ of } A)\rangle$$

$$=\textstyle\sum_{i=1}^{n} x_i \,\langle(\text{row } i \text{ of } A), (\text{row } j \text{ of } A)\rangle y_i$$

$$= x_1.\, y_1 + x_2.\, y_2 + \cdots + x_n.\, y_n$$

$$= \langle x, y\rangle.$$

Therefore, $A \in O_n(\mathbb{R})$ proving the statement.

CHAPTER-3

TOPOLOGY OF MATRIX GROUPS

Definition: 3.1

A topology on a non-empty set X is a collection $\mathcal{T}$ of subsets of X having the following properties.

(1) $\phi$and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called *a topological space*, denoted $(X, \mathcal{T})$.

Definition: 3.2

If $(X, \mathcal{T})$ is a topological space, we say that a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$. Similarly, if U is an open set containing some point $x \in X$, then we say that U is a *neighborhood* of x.

Definition: 3.3

A subset A of a topological space $(X, \mathcal{T})$ is said to be *closed* if the set $X - A$Ais open in $\mathcal{T}$.

Definition 3.4.

If X is a non-empty set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

(1) For each $x \in X$, there exists some $B \in \mathcal{B}\in$ such that $x \in B$.

(2) If $x \in X$ such that $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

If we have some topology $\mathcal{T}$ and $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{T}$ is the collection of all arbitrary unions of elements of $\mathcal{B}$. A simple example of a topology and a basis is the real numbers, which is explained in the following example.

Example: 3.5

The collection $\mathcal{B}$ of open intervals in $\mathbb{R}$, precisely defined as

$\mathcal{B} = \{(a, b) \mid \text{where } a < b \text{ and } a, b \in \mathbb{R}\}$,

is a basis for a topology on $\mathbb{R}$.

Proof:

Let $\mathcal{B}$ be the collection of all open subsets of $\mathbb{R}$, that is

$$\mathcal{B} = \{(a, b) \mid \text{where } a < b\}.$$

To satisfy condition 1 of a basis, it is easy to see that for any $x \in \mathbb{R}$, the open set $(x - 1, x + 1) \in \mathcal{B}$ contains x. For condition 2, let $x \in X$ be such that $x \in (a_1, b_1) \cap (a_2, b_2)$ for some $(a_1, b_1), (a_2, b_2) \in \mathcal{B}$.

Without loss of generality, assume $a_1 < a_2$that and $b_1 < b_2$. Thus $x \in (a_1, b_1) \cap (a_2, b_2) = (a_2, b_1) \in \mathcal{B}$, so $\mathcal{B}$ is in fact a basis.

The union of the elements of $\mathcal{B}$ gives us the standard topology on $\mathbb{R}$. The standard topology on $\mathbb{R}$ is one of the most fundamental examples of a topology, and will be used to associate matrix groups with topologies in the upcoming sections. Now we look to classify distance within topologies, specifically $\mathbb{R}^n$, through the use of a function called a metric.

Definition: 3.6

A metric ton a non-empty set X is a function

$d : X \times X \to$R having the following properties.

(1) $d(x, y) \geq 0$ for all $x, y \in X$; $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$y)=d(y, x) $for\ all\ x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Definition: 3.7

Let d be a metric on a set X and let $x \in X$. Given $\epsilon > 0$, the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon \text{ and } y \in X\}$$

is called the *ϵ-ball centered at x*.

Definition: 3.8

If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$∈X and $\epsilon > 0$, is a basis for a topology on X, called the *metric topology* induced by d.

One important example of a metric is called the Euclidean metric on $\mathbb{R}^n$, which will be useful when relating real matrix groups of size n with the space $\mathbb{R}^{n^2}$.

Example: 3.9

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function defined as

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

is a metric on $\mathbb{R}^n$ called the *Euclidean metric.*

Lemma: 3.10

For all $x, y \in \mathbb{R}^n$, $|x \,.\, y| \leq \|x\|\|y\|$.

Proof:

First, suppose that $x = \bar{0}$ or $y = \bar{0}$. We see that $|x \,.\, y| = 0 \leq 0 = \|x\|\|y\|$, so our claim holds in this case. Now, suppose that $x \neq \bar{0}$and $y \neq \bar{0}$. Let $a_0 = \frac{1}{\|x\|}$and $b_0 = \frac{1}{\|y\|}$. First note that $0 \leq \|ax \pm by\|$for all $a, b \in \mathbb{R}$.

Through the use of this inequality after squaring both sides, the following inequalities hold.

$$0 \leq \left\|\frac{1}{\|x\|}x \pm \frac{1}{\|y\|}y\right\|^2$$

$$= \left(\sqrt{\left(\frac{1}{\|x\|}x_1 \pm \frac{1}{\|y\|}y_1\right)^2 + \cdots + \left(\frac{1}{\|x\|}x_n \pm \frac{1}{\|y\|}y_n\right)^2}\right)^2$$

$$= \frac{1}{\|x\|^2}x_1^2 \pm \frac{2}{\|x\|\|y\|}x_1y_1 + \frac{1}{\|y\|^2}y_1^2 + \cdots + \frac{1}{\|x\|^2}x_n^2 \pm \frac{2}{\|x\|\|y\|}x_ny_n + \frac{1}{\|y\|^2}y_n^2$$

$$= \frac{1}{\|x\|^2}(x_1^2 + \cdots + x_n^2) \pm \frac{2}{\|x\|\|y\|}(x_1y_1 + \cdots + x_ny_n) + \frac{1}{\|y\|^2}(y_1^2 + \cdots + y_n^2)$$

$$= \frac{1}{\|x\|^2}\|x\|^2 \pm \frac{2}{\|x\|\|y\|}(x \,.\, y) + \frac{1}{\|y\|^2}\|y\|^2$$

$$= 2 \pm \frac{2}{\|x\|\|y\|}(x \,.\, y).$$

This implies that $\mp\frac{1}{\|x\|\|y\|}(x \,.\, y) \leq 1$, which implies that $|x \,.\, y| \leq \|x\|\|y\|$, proving the statement.

Example: 3.9

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function defined as

$$d(x,y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

is a metric on $\mathbb{R}^n$ called the Euclidean metric.

Proof:

Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function defined as

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

We will show that d satisfies the three conditions given in Definition 3.6,

(1) Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. Since $(x_i - y_i)^2 \geq 0$ for all $i \in \{1,2, \dots, n\}$, it immediately follows that $d(x,y) \geq 0$. If $x = y$, then

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

$$= \sqrt{(x_1 - x_1)^2 + (x_2 - x_2)^2 + \cdots + (x_n - x_n)^2}$$

$$= 0$$

If $d(x,y) = 0$, then $(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 = 0$, so $(x_i - y_i) = 0$ for all $1 \leq i \leq n$, implying that $x_i = y_i$for all $1 \leq n \leq n$. Thus $x = y$. We have,

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

$$= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

$$= d(y,x).$$

(2) Let us consider $\|x + y\|^2$. Recall the definition of a standard inner product from definition 2.16, the following equalities hold.

$$\|x + y\|^2 = (x + y) \, . \, (x + y)$$

$$= (x_1 + y_1)^2 + \cdots + (x_n + y_n)^2$$

$$= x_1^2 + 2x_1y_1 + y_1^2 + \cdots + x_n^2 + 2x_ny_n + y_n^2$$

$$=(x_1^2 + \cdots + x_n^2) + 2(x_1y_1 + \cdots + x_ny_n) + (y_1^2 + \cdots + y_n^2)$$

$= (x \,.\, x) + 2(x \,.\, y) + (y \,.\, y)$

$= \|x\|^2 + 2(x \,.\, y) + \|y\|^2.$

Through our knowledge of absolute values and though the use of Lemma 3.10, we see that this implies

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + 2(x \,.\, y) + \|y\|^2 \\ &\leq \|x\|^2 + 2|x \,.\, y| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Taking the square root of both sides of the inequality above,

we get $\|x+y\| \leq \|x\| + \|y\|$.

Now let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$.

We see that,

$$\begin{aligned} d(x,y) = \|x-y\| &= \|x - z + z - y\| \\ &\leq \|x-z\| + \|z-y\| \\ &= \ d(x,z) + d(z,y) \end{aligned}$$

Therefore, d is a metric on $\mathbb{R}^n$.

The metric space induced by the Euclidean metric on $\mathbb{R}^n$ is known as the Euclidean topology on $\mathbb{R}^n$. The Euclidean topology on $\mathbb{R}^n$ is the topology that we need to relate $\mathbb{R}^n$ and $M_m(\mathbb{R})$ with each other.

To relate $\mathbb{R}^n$ and $M_m(\mathbb{R})$ to each other, we can create a one-to-one correspondence between $\mathbb{R}^{n^2}$ and $M_n(\mathbb{R})$ by creating the bijective function

$\phi : \mathbb{R}^{n^2} \to M_n(\mathbb{R})$ defined as

$$\phi(x) = \phi(x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{nn}) = \begin{bmatrix} x_{11} & x_{12}\cdots & x_{1n} \\ x_{21} & x_{22}\cdots & x_{2n} \\ \vdots & \vdots \;\ddots & \vdots \\ x_{n1} & x_{n2}\cdots & x_{nn} \end{bmatrix}$$

Thus ,we can actually talk about the Euclidean space R^{n^2} and still work with matrices, which allows us to study the geometry and topologies of matrix groups through the use of the Euclidean metric and the subspace topology applied to the Euclidean topology.

Definition: 3.11

Let $(X, \mathcal{T})$ be a topological space. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

is a topology on Y , called the *subspace topology*. With this topology, Y is called a subspace of X.

Thus, the topologies of matrix groups are structurally equivalent to subspace topologies of the Euclidean topology.

With an understanding of the topology of matrix groups, we are poised to understand the proof that $O_n(R)$ is a matrix group. The following definition and theorems will be used in the proof that $O_n(\mathbb{R})$ is a matrix group.

Theorem: 3.12

Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y .

Proof:

Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. Let A be closed in Y. Thus, $Y - A \in \mathcal{T}_Y$, so $Y - A = U \cap Y$ where $U \in \mathcal{T}$. Since $X - U$ is closed in X and $A = Y \cap (X - U)$, A is the intersection of Y with a closed set of X.

Now let $A \subset Y$ be such that $A = C \cap Y$ where C is closed in X. Then $X - C \in \mathcal{T}$, so $Y \cap (X - C) \in \mathcal{T}_Y$. Since $(X - C) \cap Y = Y - A, Y - A \in \mathcal{T}_Y$, so A is closed in Y, as desired.

Definition: 3.13

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, be topological spaces. A function $f: X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Theorem: 3.14

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$be topological spaces, let $f: X \to Y$. If f is continuous, then for every closed subset B of Y , the set $f^{-1}(B)$ is closed in X.

Proof:

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and let $f: X \to Y$. Suppose that f is continuous and let B be a closed set of Y . Since $Y - B \in \mathcal{T}'$, and f is continuous, $f^{-1}(Y - B)$ is open in $\mathcal{T}$. Since $f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B) = X - f^{-1}(B)$, it follows that $f^{-1}(B)$ is closed in X, as desired.

Theorem: 3.15

$O_n(\mathbb{R})$ is a matrix group.

Proof:

Let $n \in N$ be fixed, we will first show that $O_n(\mathbb{R})$ is a group. First, note that theidentity matrix I_n is in $O_n(\mathbb{R})$ since for any $x, y \in \mathbb{R}^n$, $\langle xI_n, yI_n \rangle = \langle x, y \rangle$. Second, note that$O_n(\mathbb{R})$ inherits inverses from $GL_n(\mathbb{R})$ since for any $M \in O_n(\mathbb{R})$, $M^{-1} = M^T$, so M^{-1} is orthogonal since,

$$M^{-1}(M^{-1})^T = M^T(M^T)^T = I_n \text{and } (M^{-1})^T M^{-1} = (M^T)^T M^T = I_n.$$

Lastly orthogonal matrices are closed under multiplication since for any $A, B \in O_n(\mathbb{R})$, $(AB)^T = B^T A^T$ and thus $AB(AB)^T = ABB^T A^T = AA^T = I_n$ and $(AB)^T AB = B^T A^T AB = B^T B = I_n$. Thus, $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.Since for all matrices $N, M \in M_n(\mathbb{R})$,$det(N) = det(N^T)\ and\ det(N)det(M) = det(NM)$ it follows that if $A \in O_n(\mathbb{R})$, then $det(A)^2 = det(AA) = det(AA^T\ ) = det(I_n) = 1$, so $det(A) = \pm 1$. Now, define $T : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ by $T(X) = XX^T$for all $X \in GL_n(\mathbb{R})$.

It is clear that T is continuous since, for all $X \in GL_n(X)$ where $X = [x_{ij}]$ the $i - j^{th}$entry of $T(X)$ is simply $\sum_{k=1}^{n} x_{ik} x_{jk}$, which is a polynomial function in $\mathbb{R}$. Thus, since $T^{-1}(\{I_n\}) = O_n(\mathbb{R})$ and one-point sets are closed in $\mathbb{R}^{n^2}$, $\{I_n\}$ is closed in $GL_n(\mathbb{R})$ by Theorem 3.12, so it follows by Theorem 3.14 that $O_n(\mathbb{R})$ is closed in $GL_n(\mathbb{R})$. Therefore, $O_n(\mathbb{R})$ is a matrix group.

CHAPTER-4

LIE ALGEBRAS

Definition: 4.1

Let $M \subset \mathbb{R}^m$ and let x ∈ M. The tangent space to M at x is defined as

$T_xM = \{\gamma'(0) \mid \gamma : (-\epsilon, \epsilon) \to M$ is differentiable with $\gamma(0) = x\}$.

The function $\gamma : (-\epsilon, \epsilon) \to M$ in the previous definition is referred to commonly as a path through the point$x$. Thus, the tangent space to $M \subset \mathbb{R}^m$ at x is the collection of slopes of all paths such that each component function of γ is differentiable from $(-\epsilon, \epsilon)$ to $\mathbb{R}$.

Due to the correlation stated in Section 3 between $M_n(\mathbb{R})$ and $\mathbb{R}^{n^2}$, we are able to consider matrix groups as subsets of the Euclidean space. This gives us the ability to talk about tangent spaces of matrix groups, which gives us the definition of a Lie algebra, given below.

Definition: 4.2

The Lie algebra of a matrix group $G \subset GL_n(\mathbb{R})$ is the tangent space to G at the identity matrix In. We denote the Lie Algebra of G as $\mathfrak{g} := \mathfrak{g}(G) := T_{I_n}G$.

In Theorem 4.4, we prove that the Lie algebras of matrix groups are subspaces of $M_n(\mathbb{R})$.To do so, we will use the product rule for paths in $M_n(\mathbb{R})$, which is the subject of the following theorem.

Theorem 4.3.

If $\gamma, \beta : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$are differentiable, then the product path $(\gamma.\beta)(t) := \gamma(t).\beta(t)$ is differentiable. Furthermore, $(\gamma.\beta)'(t) = \gamma(t).\beta'(t) + \gamma'(t).\beta(t)$

Proof:

Let $\gamma, \beta : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ be differentiable. When $n = 1$, then we have the product rule from calculus. Since $((\gamma.\beta)(t))_{ij} = \sum_{l=1}^{n} \gamma(t)_{il} . \beta(t)_{lj}$ and $\gamma(t)_{il}.\beta(t)_{lj}$is a product of functions from $(-\epsilon, \epsilon)$ to $\mathbb{R}$, it follows that

$((\gamma.\beta)'(t))_{ij}$= $\sum_{l=1}^{n} \gamma(t)_{il} . \beta'(t)_{lj} + \gamma'(t)_{ij}.\beta(t)_{lj}$

$= (\gamma(t).\beta'(t))_{ij} + (\gamma'(t).\beta(t))_{ij}$

Theorem: 4.4

The Lie algebra $\mathfrak{g}$ of a matrix group $G \subset GL_n(\mathbb{R})$ is a real subspace of $M_n(\mathbb{R})$.

Proof:

Let $G \subset GL_n(\mathbb{R})$ be an arbitrary matrix group. To prove that g is a subspace of $M_n(\mathbb{R})$, we need to prove that $\mathfrak{g}$ is closed under scalar multiplication and matrix addition.

Thus, let $\lambda \in R$ and and let $A \in \mathfrak{g}$, so $A = \gamma'(0)$ where $\gamma: (-\epsilon, \epsilon) \to \mathbb{R}^n$ is a differentiable path such that $\gamma(0) = I_n$. Let $\sigma : (-\lambda\epsilon, \lambda\epsilon) \to \mathbb{R}^n$ be the path defined as $\sigma(t) := \gamma(\lambda . t)$ for all $t \in (-\lambda\epsilon, \lambda\epsilon)$. Since $\sigma'(t) = \lambda . \gamma'(\lambda . t)$, it follows that $\sigma'(0) = \lambda . A$. Thus, since $\sigma(0) = \gamma(\lambda . 0) = I_n$, we can conclude that $\lambda . A \in \mathfrak{g}$.

Next, let $A, B \in \mathfrak{g}$. Thus, $A = \gamma'(0)$ and $B = \beta'(0)$ where $\gamma : (-\epsilon_1, \epsilon_1) \to \mathbb{R}^n$ and $\beta : (-\epsilon_2, \epsilon_2) \to \mathbb{R}^n$ are differentiable paths such that $\gamma(0) = \beta(0) = I_n$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, Let $\pi : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be the product path defined as $\pi(t) := \gamma(t).\beta(t)$ for all $t \in (-\epsilon, \epsilon)$ϵ)By Theorem 4.3, we know that $\pi$is a differentiable path that lies in G with

$$\pi'(0) = \gamma(0).\beta'(0) + \gamma'(0).\beta(0) = I_n . B + A . I_n = A + B$$

Therefore $A + B \in \mathfrak{g}$, proving that $\mathfrak{g}$ is a real subspace of $M_n(\mathbb{R})$.

Since Lie algebras are vector space over $\mathbb{R}$, we are able to classify matrix groups and their Lie algebras according to their basis.

Definition: 4.5

The dimension of a matrix group G is the dimension of its Lie algebra.

In order to give examples of Lie algebras $\mathfrak{g}$ of matrix group $G \subset GL_n(\mathbb{R})$, we must construct paths $\gamma_A : (-\epsilon, \epsilon) \to G$ for each $A \in G$ such that $\gamma(0) = I_n$ and $\gamma'(0) = A$. The simplest way to accomplish this is to use a function called matrix exponentiation, which requires a few definitions to understand the beautiful simplicity of the concept.

Definition: 4.6

A vector field is a continuous function $F : \mathbb{R}^m \to \mathbb{R}^m$.

Definition: 4.7

An integral curve of a vector field $F : \mathbb{R}^m \to \mathbb{R}^m$ is a path $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m$ such that $\alpha'(t) = F(\alpha(t))$ for all $t \in (-\epsilon, \epsilon)$.

Intuitively, the vector field $F : \mathbb{R}^m \to \mathbb{R}^m$ gives the value of the tangent vector to every point on the path $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m$ Surprisingly, matrix exponentiation gives us an integral curve for every element in the Lie algebra of a matrix group.

As matrix exponentiation is defined by power series of matrices, we will introduce terms and results that refer to series in $M_n(\mathbb{R})$.

Definition: 4.8

Let $A \in M_n(\mathbb{R})$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots \ \ddots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

The Euclidean norm of A, denoted $|A|$, is defined as

$$|A| = \sqrt{(a_{11})^2 + \cdots + (a_{1n})^2 + (a_{21})^2 + \cdots + (a_{2n})^2 + (a_{n1})^2 + \cdots + (a_{nn})^2}$$

The Euclidean norm of a matrix is better understood as the square root of the sum of squares of the entries of the matrix.

Definition: 4.9

Let $A_i \in M_n(\mathbb{R})$ for all $i \in \mathbb{Z}^*$. We say that the series $\sum_{i=0}^{\infty} A_i = A_0 + A_1 + A_2 + \cdots$converges (absolutely) if, for all $i, j \in \mathbb{Z}^*$, $(A_0)_{ij} + (A_1)_{ij} + (A_2)_{ij} + \cdots$ converges (absolutely) to some $(A)_{ij} \in \mathbb{R}$. This is denoted as $\sum_{i=0}^{\infty} A_i = A$.

Lemma: 4.10

For all $X, Y \in M_n(\mathbb{R})$, $|XY| \leq |X| \,.\, |Y|$.

Proof:

Let $X, Y \in M_n(\mathbb{R})$ be arbitrary. Recall that for all $x, y \in \mathbb{R}^n$, $|\langle x, y\rangle| \leq |x| \,.\, |y|$ (the Schwarz inequality). Using the Schwarz inequality, it follows that for all indices i, j,

$$\left|(XY)_{ij}\right|^2 = \left|\sum_{l=1}^{n} X_{il} Y_{lj}\right|^2$$

$$= |\langle(\text{row } i \text{ of } X), (\text{column } j \text{ of } Y)^T\rangle|^2$$

$$\leq |(\text{row } i \text{ of } X)|^2 . |(\text{column } j \text{ of } Y)^T|^2$$

$$= \left(\sum_{l=1}^{n} |x_{il}|^2\right) . \left(\sum_{l=1}^{n} \left|Y_{lj}\right|^2\right)$$

Thus it follows that$|XY|^2 = \sum_{i,j=1}^{n} \left|XY_{ij}\right|^2$

$$\leq \sum_{i,j=1}^{n} \left(\left(\sum_{l=1}^{n} |x_{il}|^2\right) . \left(\sum_{l=1}^{n} \left|Y_{lj}\right|^2\right)\right)$$

$$= \left(\sum_{l=1}^{n}|x_{il}|^2\right).\left(\sum_{l=1}^{n}|Y_{lj}|^2\right)$$

$$= |X|^2|Y|^2$$

Taking the square root of this equation, we get $|XY| \leq |X|\,.\,|Y|$, as desired.

Theorem: 4.11

Let $f(x) = c_0 + c_1x + c_2x^2 + \cdots = \sum_{i=1}^{n} c_i x^i$ be a power series with coefficients $c_i \in \mathbb{R}$ and a radius of convergence $\mathbb{R}$. If $A \in M_n(\mathbb{R})$ satisfies $|A| < R$, then $f(A) = c_0I_n + c_1A + c_2A^2 + \cdots = \sum_{i=1}^{n} c_i A^i$converges absolutely.

Proof:

Let $f(x) = c_0 + c_1x + c_2x^2 + \cdots$ be a power series with coefficients $c_i \in \mathbb{R}$ with a radius of convergence $\mathbb{R}$. Let $A \in M_n(\mathbb{R})$ be such that $|A| < R$. For any indices i, j, we must show that $\left|(c_0I_n)_{ij}\right| + \left|(c_1A)_{ij}\right| + \left|(c_2A^2)_{ij}\right| + \cdots$converges. For any $k \in \mathbb{N}$, it follows by Lemma 4.10,

$$\left|(c_kA^k)_{ij}\right| \leq |c_kA^k| = |c_k||A^k| \leq |c_k|\,.\,|A|^k$$

Since $|A| < R$, it follows that $\left|(c_0I_n)_{ij}\right| + \left|(c_1A)_{ij}\right| + \left|(c_2A^2)_{ij}\right| + \cdots$ converges, so $f(A) = c_0I_n + c_1A + c_2A^2 + \cdots$ converges absolutely. Through the use of Theorem 4.11, we are able to rigorously define matrix exponentiation.

Definition: 4.12

Let $A \in M_n(\mathbb{R})$. The matrix exponentiation of A is the function

$$e^A = \exp(A) = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{i=1}^{\infty}\frac{1}{i!}A^i$$

Those with sufficient calculus knowledge will recall that the radius of convergence for the power series of e^x is infinite, so e^A converges absolutely for all $A \in M_n(\mathbb{R})$ by Theorem 4.11.

Also, considering the function $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ be defined as $\gamma(t) = e^{tA} = I_n + tA + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \cdots$., it follows that $\gamma(0) = e^{0A} = I_n + 0A + \frac{1}{2!}(0A)^2 + \frac{1}{3!}(0A)^3 + \cdots = I_n$, $\gamma(t) = e^{tA}$ is indeed a path. In fact, $\gamma(t) = e^{tA}$ is one of the most useful paths when trying to define Lie algebras of matrix groups. The following theorems will help us understand the power of matrix exponentiation.

Theorem: 4.13

The path $\gamma(t) = e^{tA} = I_n + tA + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \cdots$, where $A \in M_n(\mathbb{R})$, is differentiable with derivative $\gamma'(t) = A \,.\, e^{tA}$.

Proof:

Let $A \in M_n(\mathbb{R})$ and let the function $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ be defined as $\gamma(t) = e^{tA} = I_n + tA + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \cdots$ for all $t \in (-\epsilon, \epsilon)$

By Theorem 4.11, we know that $\gamma(t)$ is absolutely convergent, so we can take the derivative of $\gamma(t)$. Thus, through term-by-term differentiation, it follows that for all $t \in (-\epsilon, \epsilon)$,

$$\gamma'^{(t)} = \frac{d}{dt}\left(I_n + tA + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \cdots\right) = A + tA^2 + \frac{1}{2!}t^2A^3 + \cdots$$

$$= A \,.\, e^{tA}$$

Theorem: 4.14

Let $A, B \in M_n(\mathbb{R})$. If $AB = BA$, then $e^{A+B} = e^A e^B$.

Proof:

Let $A, B \in M_n(\mathbb{R})$ be such that $AB = BA$

Due to the commutativity of A and B,

$$(A+B)^k = (A+B)(A+B)(A+B)\dots(A+B)$$

$$= (A^2 + AB + BA + B^2)(A+B)\dots(A+B)$$

$$= (A^2 + 2AB + B^2)(A+B)\dots(A+B)$$

$$= (A^3 + A^2B + 2ABA + 2AB^2 + B^2A + B^3)\dots(A+B)$$

$$= (A^3 + 3A^2B + 3AB^2 + B^2)\dots(A+B)$$

$$\vdots$$

$$= A^k + kA^{k-1}B + \binom{k}{2}A^{k-2}B^2 + \cdots + \binom{k}{k-1}AB^{k-1}$$

$$= \sum_{r=0}^{k}\binom{k}{r}A^{k-r}B^r$$

The following equalities hold.

$$e^{A+B} = I_n + A + B + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots$$

$$= \sum_{i=0}^{\infty}\frac{1}{i!}(A+B)^i$$

$$= \sum_{i=0}^{\infty}\frac{1}{i!}\left(\sum_{j=0}^{i}\binom{i}{j}A^{i-j}B^j\right)$$

$$= \sum_{i=0}^{\infty}\frac{1}{i!}\left(\sum_{j=0}^{i}\frac{i!}{(i-j)!\,j!}A^{i-j}B^j\right)$$

$$= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{(i-j)!\,j!}A^{i-j}B^j\right)$$

$$= I_n + A + B + \frac{1}{2!}A^2 + AB + \frac{1}{2!}B^2 + \frac{1}{3!}A^3 + \frac{1}{2!}A^2B + \frac{1}{2!}AB^2 + \frac{1}{3!}B^3 + \cdots$$

$$= \left(\sum_{k=0}^{\infty}\frac{1}{k!}A^k\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!}B^k\right)$$

$= e^A e^B$.

Now, using Theorems 4.13 and 4.14, we will find the Lie algebras for $GL_n(\mathbb{R})$, $O_n(\mathbb{R})$, and $SL_n(\mathbb{R})$. When referring to the Lie algebra of a matrix group, we write the Lie algebra in lower case letters. For example, the Lie algebra of $GL_n(\mathbb{R})$ is typically denoted $gl_n(\mathbb{R})$.

Theorem: 4.15

$M_n(\mathbb{R})$ is the Lie algebra of $GL_n(\mathbb{R})$.

Proof:

Let $A \in M_n(\mathbb{R})$. By Theorem 4.14, $e^A . e^{-A} = e^{A-A} = e^0 = I_n$, so e^A is invertible and thus $e^A \in GL_n(\mathbb{R})$. Let $\gamma : (-\epsilon, \epsilon) \to GL_n(\mathbb{R})$ be defined as $\gamma(t) = e^{tA}$ for all $t \in (-\epsilon, \epsilon)$. By Theorem 4.14, $e^{tA} . e^{-tA} = e^{tA-tA} = e^0 = I_n$, so $e^{tA} \in GL_n(\mathbb{R})$ as well. Since $\gamma(0) = I_n$and $\gamma'(0) = A$, it follows that $A \in gl_n(\mathbb{R})$ and thus $M_n(\mathbb{R}) \subset gl_n(\mathbb{R})$.

For the other direction, since the paths $\gamma(t)$ are all $n \times n$nmatrices, their derivatives at 0 are $n \times n$ matrices as well, so $\mathfrak{g}(GL_n(\mathbb{R})) \subset M_n(\mathbb{R})$. Therefore, by double inclusion, $M_n(\mathbb{R}) = gl_n(\mathbb{R})$.

Notation.

$$O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A + A^T = 0\}$$

Lemma: 4.16

If $A \in o_n(\mathbb{R})$, then $e^A \in O_n(\mathbb{R})$.

Proof:

Let $A \in o_n(\mathbb{R})$. By Theorem 2.24, we see that

$(e^A)^T = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right)^T = \sum_{n=0}^{\infty} \frac{(A^n)^T}{n!} = \sum_{n=0}^{\infty} \frac{(A^T)^n}{n!} = e^{A^T}$

Since $A \in O_n(\mathbb{R})$, $A^T = -A$. Thus,

$$e^A(e^A)^T = e^A e^{A^T} = e^A e^{-A} = e^{A-A} = e^0 = I_n.$$

By Theorem 2.26, it follows that $e^A \in O_n(\mathbb{R})$.

Theorem: 4.17

$O_n(\mathbb{R})$ is the Lie algebra of $O(\mathbb{R})$.

Proof:

First, let $A \in o_n(\mathbb{R})$. By Lemma 4.16, it follows that the path $\gamma(t) = e^{tA} \in O_n(\mathbb{R})$.Since $\gamma(0) = I_n$ and $\gamma'(0) = A$, it follows that $A \in \mathfrak{g}(O_n(\mathbb{R}))$, so $o_n(\mathbb{R}) \subset \mathfrak{g}(O_n(\mathbb{R}))$.

Next, let $B \in \mathfrak{g}(O_n(\mathbb{R}))$. Thus, there exists some path $\sigma : (-\epsilon, \epsilon) \to O_n(\mathbb{R})$, such that $\sigma(t) \in O_n(\mathbb{R})$ for all $t \in (-\epsilon, \epsilon)$,$\sigma(0) = I_n$ and $\sigma'(0) = B$.

Since $\sigma(t) \in O_n(\mathbb{R})$ for all $t \in (-\epsilon, \epsilon)$, $\sigma(t) \,.\, \sigma(t)^T = I_n$by Theorem 2.26. Using the product rule for differentiation, it follows that

$$\frac{d}{dt}(\sigma(t)\,.\,\sigma(t)^T) = \sigma'(t).\sigma(t)^T + \sigma(t)\,.\,\sigma'(t)^T,$$

and since $\sigma(t)\,.\,\sigma(t)^T = I_n$, we get that

$$\frac{d}{dt}(\sigma(t)\,.\,\sigma(t)^T) = \frac{d}{dt}(I_n) = 0$$

When $t = 0$, we get

$$\begin{aligned} 0 &= \frac{d}{d0}(\sigma(0)\,.\,\sigma(0)^T) \\ &= \sigma'(0)\,.\,\sigma(0)^T + \sigma(0)\,.\,\sigma'(0)^T \\ &= B\,.\,I_n + I_n\,.\,B^T \end{aligned}$$

$$= \quad B + B^T$$

Thus, $B \in o_n(\mathbb{R})$, which demonstrates that $\mathfrak{g}(O_n(\mathbb{R})) \subset o_n(\mathbb{R})$. Therefore, $o_n(\mathbb{R})$ is the Lie algebra of $O(\mathbb{R})$. Lemma 4.18 will help in finding the Lie algebra of $SL_n(\mathbb{R})$.

First, we introduce notation that will be used in the proof of Lemma 4.18.

Lemma: 4.18

If $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ is differentiable and $\gamma(0) = I_n$, then

$$\left.\frac{d}{dt}\right|_{t=0} \det\big(\gamma(t)\big) = trace\big(\gamma'(0)\big)$$

where $trace\big(\gamma'(0)\big)$ is the sum of the entries of the main diagonal of $\gamma'(0)$.

Proof:

Let $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ be differentiable with $\gamma(0) = I_n$

$$\left.\frac{d}{dt}\right|_{t=0} \det\big(\gamma(t)\big) = \left.\frac{d}{dt}\right|_{t=0} \sum_{j=1}^{n} (-1)^{j+1} . \gamma(t)_{1j} . \det(\gamma(t)[1,j])$$

$$= \sum_{j=1}^{n} (-1)^{j+1} . \left(\gamma'(0)_{1j} . \det(\gamma(0)[1,j]) + \gamma(0)_{1j} . \left.\frac{d}{dt}\right|_{t=0} \det(\gamma(0)[1,j])\right)$$

$$= \gamma'(0)_{11} . \left.\frac{d}{dt}\right|_{t=0} \det(\gamma(0)[1,1]).$$

Computing $\left.\frac{d}{dt}\right|_{t=0} \det(\gamma(0)[1,1])$ through the same argument n times, we get

$$\left.\frac{d}{dt}\right|_{t=0} \det\big(\gamma(t)\big) = \gamma'(0)_{11} + \gamma'(0)_{22} + \cdots + \gamma'(0)_{nn}$$

$$= trace\big(\gamma'(0)\big).$$

Theorem: 4.19

The Lie algebra of $SL_n(\mathbb{R})$ is

$$sl_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid trace(A) = \ 0\}$$

Proof:

Let $A \in \mathfrak{g}(SL_n(\mathbb{R}))$. Thus, there exists some path $\gamma : (-\epsilon, \epsilon) \to SL_n(\mathbb{R})$ such that γ is differentiable, $\gamma(0) = I_n$, and $\gamma'(0) = A$. By Lemma 4.18, it follows that $trace(\gamma'(0)) = trace(A) = 0$.This show that,

$A \ \in sl_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid trace(A) = \ 0\}$, so $\mathfrak{g}(SL_n(\mathbb{R})) \subset sl_n(\mathbb{R})$.

On the other hand, let $B \in M_n(\mathbb{R})$∈M_n(R) be such that $trace(B) = 0$. Let $\sigma :$ $(-\epsilon, \epsilon) \to SL_n(\mathbb{R})$:(-ϵ,ϵ)⟶SL_n(R) be defined as

$$\sigma(t) = \begin{bmatrix} \frac{ta_{11}+1}{\det(I_n+tB)} & \frac{ta_{12}}{\det(I_n+tB)} \cdots & \frac{ta_{1n}}{\det(I_n+tB)} \\ ta_{21} & ta_{22}+1 \cdots & ta_{2n} \\ \vdots & \vdots \quad \ddots & \vdots \\ ta_{n1} & ta_{n2} \quad \cdots & ta_{nn}+1 \end{bmatrix}$$

Note that $\sigma(0) = I_n$ and

$$\sigma'(t) = \begin{bmatrix} \frac{a_{11}(\det(I_n+tB)) - (ta_{11}+1)\left(\frac{d}{dt}\det(I_n+tB)\right)}{\det(I_n+tB)^2} & \cdots & \frac{a_{1n}(\det(I_n+tB)) - (ta_{1n}+1)\left(\frac{d}{dt}\det(I_n+tB)\right)}{\det(I_n+tB)^2} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Thus, by Lemma 4.18,

$$\sigma'(0) = \begin{bmatrix} \frac{a_{11}(\det(I_n+(0)B)) - ((0)a_{11}+1)(trace(B))}{\det(I_n+(0)B)^2} & \cdots & \frac{a_{1n}(\det(I_n+(0)B)) - ((0)a_{1n}+1)(trace(B))}{\det(I_n+(0)B)^2} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a_{11}(1) - ((0)a_{11}+1)(0)}{1^2} & \cdots & \frac{a_{1n}(1) - ((0)a_{1n}+1)(0)}{1^2} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$= A$

Since$\det(I_n + tB) = \sum_{j=0}^{n} (-1)^{j+1} . (I_n + tB)_{1j} . \det((I_n + tB)[1, j])$, it follows that

$$\det(I_n + tB) = \sum_{j=0}^{n} (-1)^{j+1} . (I_n + tB)_{1j} . \frac{1}{\det(I_n + tB)} . \det\big((I_n + tB)[1, j]\big)$$

$$= \frac{1}{\det(I_n+tB)} . \left(\sum_{j=0}^{n} (-1)^{j+1} . (I_n + tB)_{1j} . \det\big((I_n + tB)[1, j]\big)\right)$$

$$= \frac{1}{\det(I_n+tB)} . \det(I_n + tB)$$

$$= 1.$$

Thus, $\sigma(t) \in SL_n(\mathbb{R})$ for all $t \in (-\epsilon, \epsilon)$, and since $\sigma'(0) = A$, $A \in \mathfrak{g}\big(SL_n(\mathbb{R})\big)$. Therefore, $sl_n(\mathbb{R}) \subset \mathfrak{g}(SL_n(\mathbb{R}))$, so $sl_n(\mathbb{R}) = \mathfrak{g}(SL_n(\mathbb{R}))$.

CHAPTER-5

MANIFOLDS AND LIE GROUPS

Definition: 5.1

A Lie group is a set G with two structures: G is a group and G is a (smooth, real) manifold. These structures agree in the following sense: multiplication and inversion are smooth map.

A morphism of Lie group is a smooth map which also preserves the group operation: $f(gh) = f(g)f(h), f(1) = 1$.

In a similar way, one defines complex Lie groups. However, unless specified otherwise, "Lie group" means a real Lie group.

Theorem: 5.2

Let G be a Lie group. Denote by G^0 the connected component of unity. Then G^0 is a normal subgroup of G and is a Lie group itself. The quotient group is $G/G^0 is$ discrete.

Proof.

We need to how that G^0 is closed under the operations of multiplication and inversion.Since the image of a connected topological space under a continuo map is connected, the inversion map i must take G^0 to one component of G, that which contains $i(1) = 1$, namely G^0. In similar way one show that G^0 is closed under multiplication.

To check that this a normal subgroup, we must show that if g$\in G$ and $h \in G^0$, then $ghg^{-1} \in G^0$. Conjugation by g is continuous and thus will take G^0 to some connected component of G; since it fixes 1, this component is G^0.

This fact that the quotient is discrete is obvious.

This theorem mostly reduces the study of arbitrary Lie groups to the study of finite groups and connected lie groups. In fact, one can go further and reduce the study of connected lie groups to connected simply-connected lie groups.

Theorem: 5.3

If G is a connected lie groups then its universal cover $\tilde{G}$ has a canonical structure of a lie group such that the covering map $p: \tilde{G} \to G$ is a morphism of lie groups, and $Ker\, p = \pi_1(G)$ as a group. Moreover, in this case $Ker\, p$ is a discrete central subgroup in $\tilde{G}$

Proof:

If M,N are connected manifolds, then any continuous map $f: M \to N$ can be lifted to a map $\tilde{f}: \tilde{M} \to \tilde{N}$. Moreover, if we choose m$\in M$, n$\in N$ such that f(m) = n and choose liftings $\tilde{m} \in \tilde{M}$, $\tilde{n} \in \tilde{N}$ such that $p(\tilde{m}) = m, p(\tilde{n}) = n$, there is a unique lifting $\tilde{f}$ of f such that $\tilde{f}(\tilde{m}) = \tilde{n}$.

Now let us choose some element $\tilde{1} \in \tilde{G}$ such that $p(\tilde{1}) = 1 \in G$. Then, by the above theorem there is a unique map $\tilde{\imath}$: $\tilde{G} \to \tilde{G}$ which lifts the inversion map $i: G \to G$ and satisfies $\tilde{\imath}(\tilde{1}) = \tilde{1}$.

In a similar way construct the multiplication map $\tilde{G} \times \tilde{G} \to \tilde{G}$.

Definition: 5.4

A Lie subgroup H of a Lie group G is a subgroup which is also a submanifold.

Remark: 5.5

In this definition, the word “Submanifold” should be understood as “imbedded submanifold”. In particular, this means H is locally closed but not necessarily closed; as we will how below,it will automatically be closed.

Theorem: 5.6

(1) If G is a connected Lie group and U is a neighborhood of I, then U generates G.

(2) Let $f: G_1 \to G_2$ be a morphism of Lie groups, with G_2 connected $f_*: T_1G_1 \to T_1G_2$ is surjective. Then f is surjective.

Proof:

(1) Let H be the subgroup generated by U. Then H is open in G,for any element $h \epsilon H$, set the h . U is a neighborhoods of h in G. Since it is an open subset of manifold, it is a submanifolds, so H is a Lie subgroup.

(2) Given the assumption, the inverse function theorem says that f is Surjective onto some neighborhood U of $1 \epsilon G_2$. Since an image of a group morphism is a subgroup, and U generates G_2, f is surjective.

As in the tLheory of discrete groups, given a subgroup $H \subset G$, we can define the notation of cosets and define the coset space G/H as the set of equivalence classes. The following theorem shows that the coset space is actually a manifolds.

Notation.

Let $B_r := \{W \in M_n(\mathbb{R}) | \, |W| < r\}$.

Theorem: 5.7

Let $G \subset GL_n(\mathbb{R})$ be a matrix group, with Lie algebra $g \subset gL_n(\mathbb{R})$.

1. For all $X \in g$, $e^x \in G$.
2. For sufficiently mall r>0, $V = \exp(B_r \cap g)$ is a neighborhood of I_n in G, and the restriction $\exp : (B_r \cap g) \to V$ is a diffeomorphism.

Proof

The proof of Theorem 5.7 requires delving into the world of analysis and is rather lengthy, and is therefore beyond the scope of this paper. The proof of Theorem 5.1 can be found in Tapp's Matrix Groups for Undergraduates, and is worth studying to understand the inner workings of matrix groups.

Going forwards, we look to define manifolds and prove that all matrix groups are manifolds, the prof of which relies heavily on Theorem 5.7. First we will add some more definitions to our stockpile, specifically those that pertain to functions in a topological space.

Let $U \subset \mathbb{R}^n$ be an open set in the Euclidean topology on $\mathbb{R}^n$. Any function $f : U \to \mathbb{R}^m$ can be thought of as m separate function, that I, $f = (f_1, f_2 \ldots\ldots f_m)$ where $f_i : U \to \mathbb{R}$ for each $i \in \{1,2, \ldots\ldots m\}$. An example of such a function would be the function $h : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $h(x, y) = (xy, x^2 - y^2, x^3 + y)$ for all $x, y \in \mathbb{R}^2$, which is defined by the separate functions $h_1, h_2, h_3 : \mathbb{R}^2 \to \mathbb{R}^3$ where $h_1(x,y) = xy$, $h_2(x,y) = x^2 - y^2$, and $h_3(x,y) = x^3 + y$.

DEFINITION: 5.8

Let $U \subset \mathbb{R}^n$ be an open set in the standard topology on $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ be a function. The directional derivatives of the component functions $\{f_1, f_2, \ldots, f_m\}$ in the directions of the standard orthonormal basis vectors $\{e_1, e_2, \ldots, e_m\}$ of $\mathbb{R}^n$ are called partial derivatives of f and are denoted as

$$\frac{\partial f_i}{\partial x_j}(p) := d(f_i)_p(e_j).$$

Directional derivatives of a function f measure the rates of change of each of the component functions of f. If we fix $i \in \{1,2, \ldots, m\}$ and $j \in \{1,2, \ldots, m\}$ and if $\frac{\partial f_i}{\partial x_i}(p)$ exists for all $p \in U$, then the function $g : U \to \mathbb{R}^m$ defined as $g(p) = \frac{\partial f_i}{\partial x_i}(p)$ is a well-defined function from U to $\mathbb{R}^m$, so we can take the partial derivatives of g. If the partial derivatives of g exist, they are called second order partial derivatives of f. Following in this matter, if we take r partial derivatives of your function f and the partial derivatives exists, then we say that they are the r^{th} order partial derivatives of f.

Definition: 5.9

Let $U \subset \mathbb{R}^n$ be an open set in the standard topology on $\mathbb{R}^n$ and let $f: U \to \mathbb{R}^m$ be a function. The function f is called C^r on U if all r^{th} order partial derivatives exist and are continuous on U, and f is called *smooth* on U if f is C^r on U for all positive integers r.

Similarly, we can define smoothness for any set $X \subset \mathbb{R}^n$, not just open sets.

Definition: 5.10

If $X \subset \mathbb{R}^n$, then $f: X \to \mathbb{R}^n$ is called *smooth* if for all $p \in X$, there exists an open neighborhood U of p in $\mathbb{R}^m$ and a smooth function $\tilde{f}: U \to \mathbb{R}^n$which agrees with f on $X \cap U$.

Using this more general definition of smoothness, we can create a type of similarity between subsets of $\mathbb{R}^n$, which will allow us to define a manifold in $\mathbb{R}^n$.

Definition: 5.11

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are called *diffeomorphic* if there exists a smooth bijective function $f : X \to Y$ whose inverse is also smooth. In this case, f is called a *diffeomorphism.*

Theorem: 5.12

Any matrix group is a manifold.

Proof:

Let $G \subset GL_n(\mathbb{R})$ be a matrix group with Lie algebra $\mathfrak{g}$. Choose a sufficiently small $r > 0$ which is guaranteed by Theorem 5.8. Thus, $V := \exp(B_r \cap \mathfrak{g})$:=(B_r∩g) is a neighborhood of I_n in G, and the restriction map $\exp : B_r \cap \mathfrak{g} \to V$ is a diffeomorphism, so $\exp : B_r \cap \mathfrak{g} \to V$ is a parametrization at I_n.

Next, let $g \in G$ be arbitrary. Define the function $\mathcal{L}_g : G \to G$ as $\mathcal{L}_g(A) = g.A$ for all $A \in G$. $\mathcal{L}_g$ is injective because if $g\,.\,A = g\,.\,B$ for some A,B $\in G$ then $A = B$ through left multiplication by g^{-1}. Also, $\mathcal{L}_g$ is surjective because, for all $C \in G$, $\mathcal{L}_g(g^{-1}.C) = C$, so $\mathcal{L}_g$ is bijective. Since matrix multiplication from $M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ can be thought of as a function with n^2 component functions, it follows that $\mathcal{L}_g$ is smooth as each component function is a polynomial over $\mathbb{R}$, so all r^{th} order partial derivatives exist and are continuous on G.

Also, since G is a group, g^{-1} exists. Thus, the inverse function of $\mathcal{L}_g$ is $(B) = \mathcal{L}_g^{-1}(B) = g^{-1}.B$, which is also smooth through the same reasoning. Thus, $\mathcal{L}_g$ is a diffeomorphism from G to G, so $\mathcal{L}_g(V)$ in particular is a neighborhood of g in G as $\mathcal{L}_g$ maps open neighborhoods to open neighborhoods being a diffeomorphism.

Therefore, $(\mathcal{L}_g \circ \exp) : B_r \cap \mathfrak{g} \to \mathcal{L}_g(V)$ is a parametrization at g as the composition of diffeomorphisms is diffeomorphic, proving that G is a manifold. We are now able to move on to the high point of this section.

Definition: 5.13

A Lie group is a manifold, G, with a smooth group operation $G \times G \to G$ and a smooth inverse map.

Theorem: 5.14

All matrix groups are Lie groups.

Proof:

Let $G \subset GL_n(\mathbb{R})$ be a matrix group. From Theorem 5.12, we know that G is a manifold. Also, from the proof of Theorem 5.12, we know that matrix multiplication over matrix groups is smooth, so the group operation of G is smooth. Further, the inverse map of G is the function $l : G \to G$ defined as $l(A) = \frac{1}{\det(A)}\mathrm{adj}(A)$, which is smooth as this is also just a calculation of polynomials in $\mathbb{R}$ (this is a standard result from linear algebra). This shows that G is a Lie group, proving the statement.

A STUDY ON $\alpha - Q$ –FUZZY SUBGROUPS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

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Submitted by

P. PRIYATHARSINI

Reg. No: 19SPMT22

Under the guidance of

Ms. I. ANBU RAJAMMAL M.Sc., M.Phil., B.Ed., SET,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON $\alpha - Q$ −FUZZY SUBGROUPS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **P. PRIYATHARSINI (Reg. No: 19SPMT22)**

I. Arul

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonom.
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON $\alpha - Q$ –FUZZY SUBGROUPS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. I. Anbu Rajammal M.Sc., B.Ed., M.Phil., SET.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

P. Priyatharsini

Station: Thoothukudi **Signature of the Student**

Date: 10.4.2021

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Place: Thoothukudi

Date: 10.4.2021

CONTENT

CHAPTER 1

PRELIMINARIES

Definition: 1.1

A fuzzy set A of a set X is a function $A: X \longrightarrow [0, 1]$. Fuzzy sets taking the values 0 and 1 are called crisp sets.

Let A and B be two fuzzy subsets of a set X. Then the following expressions are

i. $A \subseteq B$ iff $A(x) \leq B(x)$, for all $x \in X$.

ii. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

iii. $(A \cap B)(x) = \min\{A(x), B(x)\}, \forall\, x \in X$.

iv. $(A \cup B)(x) = \max\{A(x), B(x)\}, \forall\, x \in X$.

Definition: 1.2

A function $A: G \longrightarrow [0, 1]$ is a called fuzzy subgroup (in short FSG) of G if

i. $A(xy) \geq \min\{A(x), A(y)\}$

ii. $A(x^{-1}) \geq A(x), \forall\, x, y \in G$.

It is easy to show that a fuzzy subgroup of a group G satisfies $A(x) \leq A(e)$ and $A(x^{-1}) = A(x)$, for all $x \in G$, where e is the identity element of G.

Definition: 1.3

Let A be a fuzzy subgroup of a group G, then it is called fuzzy normal subgroup (FNSG) of G if $A(xy) = A(yx), \forall\, x, y \in G$.

Definition: 1.4

Let μ a fuzzy subgroups of a group G. For a in G, the fuzzy coset ${}_a\mu$ of G is defined by $({}_a\mu)(x) = \mu(a^{-1}x)$ for all x in G.

Definition: 1.5

Let $A: G \longrightarrow [0,1]$ is a fuzzy normal subgroup of group G. For any $x \in G$, the fuzzy set ${}_xA: G \longrightarrow [0, 1]$ defined by $({}_xA)(y) = A(x^{-1}y), \forall\, y \in G$ is called a left fuzzy coset of A.

Definition: 1.6

Let $A: G \longrightarrow [0,1]$ is a fuzzy normal subgroup of group G. For any $x \in G$, the fuzzy set $A_x: G \longrightarrow [0, 1]$ defined by $(A_x)(y) = A(yx^{-1})$, $y \in G$ is called a right fuzzy coset of A.

Definition: 1.7

Let A be a fuzzy set of a group G. For $t \in [0, 1]$, the upper level subset of A is the set $U(A, t) = \{x \in G: A(x) \geq t\}$.

Clearly, $U(A, 0) = G$ and if $t_1 > t_2$, then $U(A, t_1) \subseteq U(A, t_2)$.

Definition: 1.8

Let A be a fuzzy subgroup of a group G. The subgroup $U(A,t)$, $t \in [0,1]$ with $t \leq A(e)$ are called upper level subgroup of A.

Definition: 1.9

Let $f_1: G_1 \to G_2$ be a homomorphism of a group G_1 into a group G_2. Let A and B be fuzzy subsets of $G_1$and G_2 respectively, then $f(A)$ and $f^{-1}(B)$ are respectively the image of fuzzy set A and the inverse image of fuzzy set B, defined as

$$f(A)(y) = \begin{cases} Sup\,\{A(x): x \in f^{-1}(y)\} & ;\ if\ f^{-1}(y) \neq \emptyset \\ 1 & ;\ if\ f^{-1}(y) = \emptyset \end{cases} \text{, for every } y \in G_2$$

and $f^{-1}(B)(x) = B\big(f(x)\big)$, for every $x \in G_1$.

Definition: 1.10

If A is a fuzzy subset of a set X, then the standard fuzzy complement of A, is the fuzzy subset A^c of X, defined by $A^c(x) = 1 - A(x)$, for all x in X.

Definition: 1.11

If $(G,*)$ and (G', o) are any two groups, then the function $f: G \longrightarrow G'$ is called a homomorphism if $f(x * y) = f(x) o f(y)$, for all x and y in G.

Definition: 1.12

If $(G,*)$ and (G', o) are any two groups, then the function $f: G \longrightarrow G'$ is called a anti-homomorphism if $f(x * y) = f(y) o f(x)$, for all x and y in G.

Definition: 1.13

Let G be a group and θ be a fuzzy subgroup G. Then, θ is a cyclic fuzzy subgroup of G, if θ_s is a cyclic subgroup for all s in [0, 1], and is defined as $\theta_s = \{x/\theta(x) \geq s, for\ all\ x \in G\}$.

CHAPTER 2

α – FUZZY SUBGROUP

Definition: 2.1

Let A be a fuzzy subset of a group G. Let α ∈ [0,1]. Then the fuzzy set A^{α} of G is called the α – fuzzy subset of G (with respect to) and is defined a $A^{\alpha} = \min\ \{A(x), \alpha\}$ for all $x \in G$.

Result: 2.2

1) Let A and B be two fuzzy subsets of X. Then $(A \cap B)^{\alpha} = A^{\alpha} \cap B^{\alpha}$.
2) Let $f: X \to Y$ be a mapping and A and B be two fuzzy subsets of X and Y respectively, then
   a) $f^{-1}(B^{\alpha}) = (f^{-1}(B))^{\alpha}$
   b) $f(A^{\alpha}) = (f(A))^{\alpha}$

Proof:

1) Let A and B be two fuzzy subsets of X.

   To Prove: $(A \cap B)^{\alpha} = A^{\alpha} \cap B^{\alpha}$.

Now, $(A \cap B)^{\alpha} = \min\ \{(A \cap B)(x), \alpha\}$

$$= \min\ \{\min\ \{A\ (x),\ B(x)\},\ \alpha\}$$

$$= \min\{\min\ \{A(x), \alpha\}, \min\ \{B(x), \alpha\}\}$$

$= \min\{A^{\alpha}(x),\ B^{\alpha}(x)\}$

$= (A^{\alpha} \cap B^{\alpha})$(x), for all $x \in X$

Hence $(A \cap B)^{\alpha} = A^{\alpha} \cap B^{\alpha}$.

2) Let $f: X \to Y$ be a mapping and A and B be two fuzzy subsets of X and Y respectively.

a) To prove: $f^{-1}(B^{\alpha}) = (f^{-1}(B))^{\alpha}$

$f^{-1}(B^{\alpha})(x) = B^{\alpha}(f(x))$

$= \min\{B(f(x)), \alpha\}$

$= \min\{f^{-1}(B)(x), \alpha\}$

$= \left(f^{-1}(B)\right)^{\alpha}(x)$, for all $x \in X$

Hence $f^{-1}(B^{\alpha}) = (f^{-1}(B))^{\alpha}$

b) To prove: $f(A^{\alpha}) = (f(A))^{\alpha}$

$f(A^{\alpha})(y) = \sup\{(A^{\alpha})(x) : f(x) = y\}$

$= \sup\{\min\{A(x), \alpha\} : f(x) = y\}$

$= \min\{\sup\{A(x) : f(x) = y\}, \alpha\}$

$= \min\{f(x)(y),\ \alpha\}$

$= (f(A))^{\alpha}$ (y), for all $y \in Y$

Hence $f(A^\alpha) = (f(A))^\alpha$

Definition: 2.3

Let A be a fuzzy subset of a group G. Let $\alpha \in [0,1]$. Then A is called α – fuzzy subgroup (in short α – FSG) of G if A^α is fuzzy subgroup of G i.e. if the following conditions hold

i) $A^\alpha(xy) \geq \min\{A^\alpha(x),\ A^\alpha(y)\}$

ii) $A^\alpha(x^{-1}) = A^\alpha(x),\ for\ all\ x, y \in G.$

Theorem: 2.4

If $A: G \to [0,1]$ is a α – fuzzy subgroup of a group G, then

i) $A^\alpha(x) \leq A^\alpha(e), \forall\ x \in G$, where e is the identity element of G.

ii) $A^\alpha(xy^{-1}) = A^\alpha(e) \implies A^\alpha(x) = A^\alpha(y),\ \forall\ x, y \in G.$

Proof:

Let $A: G \to [0,1]$ is a α – fuzzy subgroup of a group G.

i) To prove: $A^\alpha(x) \leq A^\alpha(e), \forall\ x \in G$, where e is the identity element of G.

$$A^\alpha(e) = A^\alpha(xx^{-1})$$

$$\geq \min\ \{A^\alpha(x),\ A^\alpha(x^{-1})\}$$

$$= \min\ \{A^\alpha(x),\ A^\alpha(x)\}$$

$$= A^\alpha(x)$$

$A^{\alpha}(e) \geq A^{\alpha}(x), \forall\, x \in G$, where e is the identity element of G.

ii) Let $A^{\alpha}(xy^{-1}) = A^{\alpha}(e)$ ---------- (1)

To prove: $A^{\alpha}(x) = A^{\alpha}(y),\ \forall\, x, y \in G$.

$A^{\alpha}(x) = A^{\alpha}(xy^{-1}y)$

$\geq \min\{A^{\alpha}(xy^{-1}),\ A^{\alpha}(y)\}$

$A^{\alpha}(x) \geq A^{\alpha}(y)$ [Since By part (i)] ---------- (2)

$A^{\alpha}(y) = A^{\alpha}(yx^{-1}x)$

$\geq \min\{A^{\alpha}(yx^{-1}),\ A^{\alpha}(x)\}$

$= \min\{A^{\alpha}(yx^{-1})^{-1},\ A^{\alpha}(x)\}$

$= \min\{A^{\alpha}(xy^{-1}),\ A^{\alpha}(x)\}$

$= \min\{A^{\alpha}(e),\ A^{\alpha}(x)\}$ [Since By (1)]

$A^{\alpha}(y) \geq A^{\alpha}(x)$ [Since By part (i)] ---------- (3)

From (2) and (3)

Thus $A^{\alpha}(x) = A^{\alpha}(y),\ \forall\, x, y \in G$

Theorem: 2.5

If A be a fuzzy subgroup of the group G, then A is also α – fuzzy subgroup of G.

Proof:

Let A be a fuzzy subgroup of the group G.

To prove: A is also α – fuzzy subgroup of G.

Let $x, y \in G$ be any elements of the group.

$A^{\alpha}(xy) = \min\ \{ A(xy),\ \alpha\}$

$\geq \min\ \{\min \{A\ (x),\ A(y)\},\ \alpha\ \}\}$

$= \min\{\min\ \{A(x), \alpha\}, \min\ \{B(x), \alpha\}\}$

$= \min\ \{A^{\alpha}(x),\ A^{\alpha}(y)\}$

Thus $A^{\alpha}(xy) \geq \min\ \{A^{\alpha}\ (x), A^{\alpha}(y)\}$

Also, $A^{\alpha}(x^{-1}) = \min\ \{A(x^{-1}),\ \alpha\}$

$= \min\big\{\{A(x), \alpha\}\big\}$

$= A^{\alpha}(x)$

Hence A is a α – fuzzy subgroup of G.

Theorem: 2.6

Let A be a fuzzy subset of a group G such that $A(x^{-1}) = A(x)$ hold for all $x \in G$ Let $\alpha \leq p$, where p = Inf $\{A(x): x \in G\}$. Then A is α – fuzzy subgroup of G.

Proof:

Let A be a fuzzy subset of a group G such that $A(x^{-1}) = A(x)$ hold for all $x \in G$.

Let $\alpha \leq p$, where p = Inf $\{ A(x): x \in G \}$.

To Prove: A is α – fuzzy subgroup of G.

Since $\alpha \leq p \Rightarrow p \geq \alpha$

(i.e) Inf $\{ A(x): x \in G \} \geq \alpha$

$A(x) \geq \alpha$ ---------- (1)

Since A be a fuzzy subset of a group G.

Then, $A^{\alpha}(x) = \min \{A(x) \geq \alpha\}$

(i.e) $A^{\alpha}(x) = \alpha$, for all $x \in G$. [Since By (1)]

Thus, $A^{\alpha}(xy) \geq \min \{A^{\alpha}(x), A^{\alpha}(y)\}$ hold for all $x, y \in G$.

Further, $(A(x^{-1}) = A(x)$ hold for all $x \in G$.

$\Rightarrow A^{\alpha}(x^{-1}) = A^{\alpha}(x)$

Hence A is α – fuzzy subgroup of G.

Theorem: 2.7

Intersection of two α – fuzzy subgroups of a group G is also α – fuzzy subgroup of G.

Proof:

Let A and B be two α – fuzzy subgroups of a group G.

To Prove: $(A \cap B)$ is a α – fuzzy subgroup of G.

Let $x, y \in G$ be any element, then

$(A \cap B)^{\alpha}(xy) = (A^{\alpha} \cap B^{\alpha})(xy)$ [Since By Result: 2.2 (1)]

$= \min \{A^{\alpha}(xy),\ B^{\alpha}(xy)\}$

$\geq \min \{\min \{A^{\alpha}(x),\ A^{\alpha}(y)\},\ \min \{B^{\alpha}(x),\ B^{\alpha}(y)\}\}$

$= \min \{\ \min \{A^{\alpha}(x), B^{\alpha}(x)\}, \min \{A^{\alpha}(y), B^{\alpha}(y)\}$

$= \min \{(A^{\alpha} \cap B^{\alpha})(x), (A^{\alpha} \cap B^{\alpha})(y)\}$

$= \min \{(A \cap B)^{\alpha}(x),\ (A \cap B)^{\alpha}(y)\}$ [Since By Result: 2.2 (1)]

Thus $(A \cap B)^{\alpha}(xy) \geq \min \{(A \cap B)^{\alpha}(x), (A \cap B)^{\alpha}(y)\}$

Also, $(A \cap B)^{\alpha}(x^{-1}) = (A^{\alpha} \cap B^{\alpha})(x^{-1})$ [Since By Result: 2.2 (1)]

$= \min \{A^{\alpha}(x^{-1}),\ B^{\alpha}(x^{-1})\}$

Since A and B be two α – fuzzy subgroups of a group G.

Also, $(A \cap B)^{\alpha}(x^{-1}) = \min \{ A^{\alpha}(x),\ A^{\alpha}(x)\}$

$$= (A^{\alpha} \cap B^{\alpha})(x)$$

$$= (A \cap B)^{\alpha}(x) \quad \text{[Since By Result: 2.2 (1)]}$$

Hence $(A \cap B)$ is a $\alpha -$ fuzzy subgroup of G.

Definition: 2.8

Let A be $\alpha -$ fuzzy subgroup of a group G, where $\alpha \in [0,1]$. For any $x \in G$, define a fuzzy set A_x^{α} of G, called $\alpha -$ fuzzy right coset of A in G as follows

$A_x^{\alpha}(g) = \min \{A(gx^{-1}),\ \alpha\}$, for all $x, g \in G$.

Similarly, we define the $\alpha -$ fuzzy left coset ${}_xA^{\alpha}$ of A in G follows

${}_xA^{\alpha}(g) = \min \{A(x^{-1}g),\ \alpha\}$, for all $x \in G$.

Definition: 2.9

Let A be α – fuzzy subgroup of a group G, where $\alpha \in [0,1]$. Then A is called $\alpha -$ fuzzy normal subgroup (α – FNSG) of G if and only if ${}_xA^{\alpha} = A_x^{\alpha}$, for all $x \in G$.

Theorem: 2.10

If A is a Fuzzy normal subgroup of a group G, then A is also a $\alpha -$ fuzzy normal subgroup of a group G.

Proof:

Let A is a Fuzzy normal subgroup of a group G.

To prove: A is also a α – fuzzy normal subgroup of a group G.

Then for any $x \in G$, we have ${}_xA = A_x$

Therefore, for any $g \in G$, we have $({}_xA)(g) = (A_x)(g)$

(i.e) $A(x^{-1}g) = A(gx^{-1})$

So min $\{A(x^{-1}g), \alpha\} = \min\{A(gx^{-1}), \alpha\}$

(i.e) $({}_xA^\alpha)(g) = (A_x^\alpha)(g)$

So, we have ${}_xA^\alpha = A_x^\alpha$, for all $x \in G$

Hence A is a α – fuzzy normal subgroup of a group G.

Theorem: 2.11

Let A be a α – fuzzy normal subgroup of a group G. Then $A^\alpha(y^{-1}xy) = A^\alpha(x)$ or equivalently, $A^\alpha(xy) = A^\alpha(yx)$, holds for all $x, y \in G$.

Proof:

Let A be a α – fuzzy normal subgroup of a group G.

To prove: $A^\alpha(xy) = A^\alpha(yx)$, holds for all $x, y \in G$.

Since A be a α – fuzzy normal subgroup of a group G.

Therefore, ${}_xA^{\alpha} = A_x^{\alpha}$ hold for all $x \in G$.

$({}_xA^{\alpha})(y^{-1}) = (A_x^{\alpha})(y^{-1})$ hold for all $y^{-1} \in G$.

$\min\{A(x^{-1}y^{-1}), \alpha\} = \min\{A(y^{-1}x^{-1}), \alpha\}$

$A^{\alpha}(x^{-1}y^{-1}) = A^{\alpha}(y^{-1}x^{-1})$

$A^{\alpha}((yx)^{-1}) = A^{\alpha}((xy)^{-1})$

Since A is $\alpha -$ fuzzy subgroup of a group G so $A^{\alpha}(g^{-1}) = A^{\alpha}(g)$, for all $g \in G$.

Therefore, $A^{\alpha}(yx) = A^{\alpha}(xy)$

Theorem: 2.12

Let A be an $\alpha -$ fuzzy subgroup of a group G such that $\alpha \leq p$, where $p = Inf\{A(x): for\ all\ x \in \mathrm{G}\}$. Then A is also a $\alpha -$ fuzzy normal subgroup of a group G.

Proof:

Let A be an $\alpha -$ fuzzy subgroup of a group G such that $\alpha \leq p$, where $p = Inf\{A(x): for\ all\ x \in \mathrm{G}\}$.

To prove: A is also a $\alpha -$ fuzzy normal subgroup of a group G.

Since $\alpha \leq p$

(i.e) $p \geq \alpha$

$Inf\{A(x): for\ all\ x \in \mathrm{G}\} \geq \alpha$

$A(x) \geq \alpha$ for all $x \in G$ ---------- (1)

and so min $\{A(x), \alpha\} = \alpha$ [Since By (1)] ---------- (2)

$A_x^{\alpha}(g) =$ min {A(gx^{-1}), α}

$= \alpha$ [Since By (2)] ---------- (3)

Similarly, ${}_xA^{\alpha}(g)$ = min {A($x^{-1}g$), α}

$= \alpha$ [Since By (2)] ---------- (4)

From (3) and (4)

Thus $A_x^{\alpha}(g) = {}_xA^{\alpha}(g)$, for all $x \in G$

Therefore $A_x^{\alpha} = {}_xA^{\alpha}$, for all $x \in G$.

Hence A is a α – fuzzy normal subgroup of a group G.

Theorem: 2.13

Let A be a α-fuzzy normal subgroup of a group G, then the set $G_{A^{\alpha}} = \{x \in G : A^{\alpha}(x) = A^{\alpha}(e)\}$ is a normal subgroup of G.

Proof:

Let A be a α −fuzzy normal subgroup of a group G.

Let $G_{A^{\alpha}} = \{x \in G : A^{\alpha}(x) = A^{\alpha}(e)\}$ ---------- (1)

To prove: $G_{A^{\alpha}}$ is a normal subgroup of G.

Clearly, $G_{A^\alpha} \neq \emptyset$ for e $\in G_{A^\alpha}$.

Let $x, y \in G_{A^\alpha}$ be any element.

Then we have $A^\alpha(xy^{-1}) \geq \min\ \{A^\alpha(x), A^\alpha(y^{-1})\}$

$\geq \min\ \{A^\alpha(x), A^\alpha(y)\}$

$= \min\ \{A^\alpha(e), A^\alpha(e)\}$ [Since By (1)]

$= A^\alpha(e)$

Therefore, $A^\alpha(xy^{-1}) \geq A^\alpha(e)$ ---------- (2)

But, $A^\alpha(xy^{-1}) \leq A^\alpha(e)$ ---------- (3)

From (2) and (3)

Therefore, $A^\alpha(xy^{-1}) = A^\alpha(e)$

$\Longrightarrow xy^{-1} \in G_{A^\alpha}$

Thus G_{A^α} is a subgroup of G.

Further, Let $x \in G_{A^\alpha}$and $y \in G_{A^\alpha}$, we have

$A^\alpha(y^{-1}xy) = A^\alpha(x)$

$= A^\alpha(e)$ [Since By (1)]

$\Longrightarrow y^{-1}xy \in G_{A^\alpha}$

Hence, G_{A^α} is a normal subgroup of G.

Theorem: 2.14

Let A be a α −fuzzy normal subgroup of a group G, then ${}_xA^{\alpha} = {}_yA^{\alpha}$ iff $x^{-1}y \in G_{A^{\alpha}}$.

Proof:

Let A be a α −fuzzy normal subgroup of a group G.

Let ${}_xA^{\alpha} = {}_yA^{\alpha}$ ---------- (1)

To prove: $x^{-1}y \in G_{A^{\alpha}}$

$A^{\alpha}(x^{-1}y) = \min\ \{A(x^{-1}y, \alpha\}$

$= {}_xA^{\alpha}(y)$

$= {}_yA^{\alpha}(y)$ [Since By (1)]

$= \min\ \{A(y^{-1}y, \alpha\}$

$= \min\ \{A(e), \alpha\}$

$= A^{\alpha}(e)$

Therefore, $A^{\alpha}(x^{-1}y) = A^{\alpha}(e)$

$\Longrightarrow x^{-1}y \in G_{A^{\alpha}}$

Conversely,

Let $x^{-1}y \in G_{A^{\alpha}}$

To prove: ${}_xA^\alpha = {}_yA^\alpha$

Then, $A^\alpha(x^{-1}y) = A^\alpha(e)$ ---------- (2)

Let $z \in G$ be any element.

Now,

$({}_xA^\alpha)(z) = \min\{A(x^{-1}z), \alpha\}$

$= A^\alpha(x^{-1}z)$

$= A^\alpha((x^{-1}y)(y^{-1}z))$

$\geq \min\{A^\alpha(x^{-1}y), A^\alpha(y^{-1}z)\}$

$= \min\{A^\alpha(e), A^\alpha(y^{-1}z)\}$ [Since By (2)]

But, $A^\alpha(y^{-1}z) \leq A^\alpha(e)$

$({}_xA^\alpha)(z) = A^\alpha(y^{-1}z)$

$= ({}_yA^\alpha)(z)$

Therefore, $({}_xA^\alpha)(z) \geq ({}_yA^\alpha)(z)$ ---------- (3)

Interchanging the role of x and y, we get

$({}_yA^\alpha)(z) \geq ({}_xA^\alpha)(z)$ ---------- (4)

From (3) and (4)

Therefore, $({}_xA^\alpha)(z) = ({}_yA^\alpha)(z)$, for all $z \in G$.

Hence, ${}_xA^{\alpha} = {}_yA^{\alpha}$.

Definition: 2.15

The group G/A^{α} of a α – fuzzy coset of the α – fuzzy normal subgroup A of G is called the factor group or the quotient group of G by A^{α}.

Theorem: 2.16

Let $f: G_1 \rightarrow G_2$ be a homomorphism of group G_1 into a group G_2. Let B be α – fuzzy subgroup of group G_2. Then $f^{-1}(B)$ is α – fuzzy subgroup of group G_1.

Proof:

Let $f: G_1 \rightarrow G_2$ be a homomorphism of group G_1 into a group G_2.

Let B be a α – Fuzzy subgroup of group G_2.

To Prove: $f^{-1}(B)$ is α – fuzzy subgroup of group G_1.

Let $x_1, x_2 \in G_1$ be any element.

Then, $(f^{-1}(B))^{\alpha}\ (x_1x_2) = f^{-1}(B^{\alpha})(x_1x_2)$ [Since By Result: 2.2 (2) (a)]

$$= B^{\alpha}((f(x_1x_2)))$$

$$= B^{\alpha}\big(f(x_1)f(x_2)\big)$$

$$\geq \min\ \{B^{\alpha}\big(\mathrm{f}(x_1)\big),\ B^{\alpha}\big(\mathrm{f}(x_2)\big)\}$$

$$= \min\ \{f^{-1}(B^{\alpha})(x_1),\ f^{-1}(B^{\alpha})(x_2)\}$$

$$= \min \{\left(f^{-1}(B)\right)^{\alpha}(x_1), \ \left(f^{-1}(B)\right)^{\alpha}(x_2)\}$$

Thus $(f^{-1}(B))^{\alpha}\ (x_1x_2) \geq \min\ \{\left(f^{-1}(B)\right)^{\alpha}(x_1), \ \left(f^{-1}(B)\right)^{\alpha}(x_2)\}$

Also, $\left(f^{-1}(B)\right)^{\alpha}(x^{-1}) = \ f^{-1}(B^{\alpha})(x^{-1})$ [Since By Result: 2.2 (2) (a)]

$$= B^{\alpha}(\left(f(x^{-1})\right))$$

$$= B^{\alpha}((f(x)^{-1}))$$

$$= B^{\alpha}(\left(f(x)\right))$$

Thus $\left(f^{-1}(B)\right)^{\alpha}(x^{-1}) = f^{-1}(B^{\alpha})(x)$

Hence $f^{-1}(B)$ is α – Fuzzy subgroup of G_1.

Theorem: 2.17

Let $f: G_1 \ \rightarrow \ G_2$ be a homomorphism of a group G_1 into a group G_2. Let B be a α – Fuzzy normal subgroup of group G_2, then $f^{-1}(B)$ is a α – Fuzzy normal subgroup of group G_1.

Proof:

Let $f: G_1 \ \rightarrow \ G_2$ be a homomorphism of a group G_1 into a group G_2.

Let B be a α – Fuzzy normal subgroup of group G_2.

To Prove: $f^{-1}(B)$ is a α – Fuzzy normal subgroup of group G_1.

Let $x_{1\,,}\, x_2 \in \ G_1$ be any element.

Then, $\left(f^{-1}(B)\right)^{\alpha}(x_1x_2) = f^{-1}(B^{\alpha})(x_1x_2)$ [Since By Result: 2.2 (2) (a)]

$= B^{\alpha}(f(x_1x_2))$

$= B^{\alpha}(f(x_1\,)f(\,x_2))$

$= B^{\alpha}(f(x_2\,)f(\,x_1))$

$= B^{\alpha}(f(x_2x_1))$

$= \; f^{-1}(B^{\alpha})(x_2x_1)$

$= \left(f^{-1}(B)\right)^{\alpha}(x_1x_2)$ [Since By Result: 2.2 (2) (a)]

Thus $\left(f^{-1}(B)\right)^{\alpha}(x_1x_2) = f^{-1}(B^{\alpha})(x_2x_1)$

Hence $f^{-1}(B)$ is a α – Fuzzy normal subgroup of group G_1.

CHAPTER 3

Q – FUZZY SUBGROUP

Definition: 3.1

Let Q and G be any two sets. A mapping $\mu: G \times Q \rightarrow [0,1]$ is called a Q – fuzzy set in G.

Definition: 3.2

A Q – fuzzy set μ of a group G is called Q – fuzzy subgroup of G, if for all

$x, y \in G, q \in Q$.

i) $\mu(xy, q) \geq \min\ \{\mu(x, q), \mu(x, q)\}$

ii) $\mu(x^{-1}, q) = \mu(x, q)$

Definition: 3.3

A Q – Fuzzy set μ of a group G is called an anti Q – fuzzy subgroup of G, if for all $x, y \in G, q \in Q$,

i) $\mu(xy,\ q) \leq \max\ \{\mu(x, q), \mu(y, q)\}$

ii) $\mu(x^{-1}, q) = \mu(x, q)$

Definition: 3.4

Let G be a group. A $Q-$ fuzzy subgroup μ of a group G is called $Q-$ fuzzy normal subgroup of G if for all $x, y \in G, q \in Q$, $\mu(xyx^{-1}, q) = \mu(y, q)$ or $\mu(xy, q) = \mu(yx, q)$.

Definition: 3.5

Let $(G,\cdot)$ be a group and Q be a set. A anti $Q-$ fuzzy subgroup μ of G is said to be an anti $Q-$ fuzzy normal subgroup (AQFNSG) of G if, $\mu(xy, q) = \mu(yx, q)$ for all

$x, y \in G, q \in Q$.

Theorem: 3.6

If μ is a $Q-$fuzzy subgroup of a group G if and only if $(\mu^c)^c$ is a $Q-$fuzzy subgroup of a group G.

Proof:

Suppose μ is a Q – fuzzy subgroup of a group G

Then for all $x, y \in G, q \in Q,\ \mu(xy, q) \geq \min\ \{\mu(x, q), \mu(y, q)\}$

Now,

$1 - \mu^c(xy, q) \geq \min\ \{1 - \mu^c(x, q), 1 - \mu^c(y, q)\}$

$\Leftrightarrow\ \mu^c(xy, q) \leq 1 - \min\{\mu^c(x, q), \mu^c(y, q)\}$

$\mu^c(xy, q) \leq \max\{\mu^c(x, q),\ \mu^c(y, q)\}$

Taking complement on both sides

$\Leftrightarrow [\mu^c(xy, q)]^c \leq [\max\{\mu^c(x, q), \mu^c(y, q)\}]^c$

We have $\mu(x, q) = \mu(x^{-1}, q)$, for all $x \in G$ and $q \in Q$.

$\Leftrightarrow 1 - \mu^c(x, q) = 1 - \mu^c(x^{-1}, q)$

$\Leftrightarrow \mu^c(x, q) = \mu^c(x^{-1}, q)$

Taking complement on both sides

$\Leftrightarrow [\mu^c(x, q)]^c = [\mu^c(x^{-1}, q)]^c$

Hence $(\mu^c)^c$ is a Q – fuzzy subgroup of a group G.

Theorem: 3.7

If A is a Q – fuzzy subgroup of a group G then $A(xy^{-1}, q) \geq \min\{A(x, q), A(y, q)\}$ for all $x, y \in G$ and $q \in Q$.

Proof:

Let A be a Q – fuzzy subgroup of a group G.

To prove: $A(xy^{-1}, q) \geq \min\{A(x, q), A(y, q)\}$ for all $x, y \in G$ and $q \in Q$.

Then for all $x, y \in G$ and $q \in Q$, $A(xy, q) \geq \min\{A(x, q), A(y, q)\}$
$A(x^{-1}, q) = A(x, q)$

Now,

$A(xy^{-1}, q) \geq \min \{A(x,q), A(y^{-1}, q)\}$

$A(xy^{-1}, q) \geq \min \{A(x,q), A(y,q)\}$

Therefore,

$A(xy^{-1}, q) \geq \min \{A(x,q), A(y,q)\}$ for all $x, y \in G$ and $q \in Q$.

Theorem: 3.8

If A is an anti $Q-$ fuzzy subgroup of group G then gAg^{-1} is also an anti $Q-$ fuzzy subgroup of group G for all $g \in G$ and $q \in Q$.

Proof:

Let A be an anti $Q-$ fuzzy subgroup of group G.

To prove: gAg^{-1} is also an anti $Q-$ fuzzy subgroup of group G for all $g \in G$ and $q \in Q$.

Then for all $g \in G$ and $q \in Q$.

i. $gAg^{-1}(xy, q) = A(g^{-1}(xy)g, q)$

$= A(g^{-1}(xgg^{-1}y)g, q)$

$= A\big((g^{-1}xg)(g^{-1}yg),\ q\big)$

$\leq \{(A(g^{-1}xg)(g^{-1}yg),\ q)\}$ for all $x, y \in G$ and $q \in Q$

ii. $gAg^{-1}(x,q) = A(g^{-1}xg, q)$

$= A((g^{-1}xg)^{-1},\ q)$

$= A(g^{-1}x^{-1}g,\ q)$

$= gAg^{-1}(x^{-1},\ q)$ for all $x, y \in G$ and $q \in Q$

Hence gAg^{-1} is also an anti Q – fuzzy subgroup of group G for all $g \in G$ and $q \in Q$.

Theorem: 3.9

Let G be a group. Let μ be a Q – fuzzy normal subgroup of a group G then μ^c is an anti Q – fuzzy normal subgroup of a group G.

Proof:

Let G be a group.

Let μ be a Q – fuzzy normal subgroup of a group G.

(i.e) $\mu(xyx^{-1}, q) = \mu(y, q)$

To prove: μ^c is an anti Q – fuzzy normal subgroup of a group G.

Now to show that μ^c is an anti Q – fuzzy subgroup of a group G.

$\mu(xy, q) \geq \min\{\mu(x, q), \mu(y, q)\}$

$1 - \mu^c(xy, q) \geq \min\{1 - \mu^c(x, q), 1 - \mu^c(y, q)\}$

$\mu^c(xy, q) \leq 1 - \min\{1 - \mu^c(x, q), 1 - \mu^c(y, q)\}$

$\mu^c(xy, q) \leq \max\{\mu^c(x, q), \mu^c(y, q)\}$

Hence, μ^c is an anti Q – fuzzy subgroup of a group G.

Since μ be a Q – fuzzy normal subgroup of a group G.

(i.e) $\mu(xyx^{-1}, q) = \mu(y, q)$

$1 - \mu^c(xyx^{-1}, q) = 1 - \mu^c(y, q)$

$\mu^c(xyx^{-1}) = \mu^c(y, q)$

Therefore,

μ^c is an anti Q – fuzzy normal subgroup of a group G.

Theorem: 3.10

Let A be a Q – fuzzy normal subgroup of a group G with identity e. Then $A(xy, q) = A(yx, q)$ for all $x, y \in G$ and $q \in Q$.

Proof:

Let A is a Q – fuzzy normal subgroup of a group G.

That is $A(xyx^{-1}),\ q) = A(y, q)$ ---------- (1)

To prove: $A(xy, q) = A(yx, q)$ for all $x, y \in G$ and $q \in Q$.

Now,

$A(xy, q) = A(xy(xx^{-1}),\ q)$

$= A((xyx)x^{-1},\ q)$

$= A(x(yx)x^{-1},\ q)$

$= A(yx, q)$ [Since By (1)]

Therefore, $A(xy, q\ = A(yx, q)$

Definition: 3.11

Let X be a field. Let F and Q any to fuzzy sets in X. A mapping

$\mu_F: X \times Q \rightarrow [0,1]$ is called Q – fuzzy set in X.

Definition: 3.12

Let μ_F be a Q – fuzzy set in a field X is said to be Q – fuzzy field in X if for all $x, y \in \mu_F$ and $q \in Q$.

i. $\mu_F((x + y), q) \geq \min\{\mu_F(x, q),\ \mu_F(y, q)\}$

ii. $\mu_F(-x, q) \geq \mu_F(x, q)$

iii. $\mu_F((xy), q) \geq \min\{\mu_F(x, q), \mu_F(y, q)\}$

iv. $\mu_F(x^{-1},\ q) \geq \mu_F(x,\ q),\ x \neq 0$ in X.

Theorem: 3.13

If μ_F be a Q – fuzzy field in X and λ_F be a subset of μ_F. Then λ_F is a Q – fuzzy subfield of μ_F in X.

Proof:

Given μ_F is a Q – fuzzy field in X.

Let $x, y \in \lambda_F$ and $q \in Q$.

To prove: λ_F is a Q – fuzzy subfield of μ_F in X.

From the definition,

i. $\mu_F\big((x+y),\ q\big) \ \geq \min\{\mu_F(x,q),\ \mu_F(y,\ q)\}$

ii. $\mu_F(-x,\ q) \ \geq \mu_F(x,\ q)$

iii. $\mu_F\big((xy),\ q\big) \ \geq \min\{\mu_F(x,\ q),\ \mu_F(y,\ q)\,\}$

iv. $\mu_F(x^{-1},\ q) \ \geq \ \mu_F(x,\ q),\ \ x \neq 0$ in X for all $x, y \in \mu_F$ and $q \in Q$.

Hence λ_F is a fuzzy field in X.

Therefore λ_F is a Q – fuzzy subfield of μ_F in X.

CHAPTER 4

$\alpha - Q -$ FUZZY SUBGROUP

Definition: 4.1

If θ is a $Q-$ fuzzy subgroup of a group G, then $C[\theta]$ is the complement of Q – fuzzy subgroup θ, and is defined by $C[\theta(x, q)]$ = 1- $\theta(x, q)$, for all x in G and q in Q.

Definition: 4.2

Let G and Q be any two non-empty sets and $\alpha \in [0,1]$. Then, a mapping $\theta^{\alpha}: G \times Q \rightarrow [0,1]$ is called $\alpha - Q -$ fuzzy subset in , with regard to fuzzy set θ if $\theta^{\alpha}(x, q) = \min\{\theta(x, q), \alpha\}$, for all $x \in G$ and $q \in Q$.

Theorem: 4.3

Let θ^{α} and σ^{α} be two $\alpha - Q -$ fuzzy subset of G. Then $(\theta \cap \sigma)^{\alpha} = \theta^{\alpha} \cap \sigma^{\alpha}$

Proof:

Let θ^{α} and σ^{α} be two $\alpha - Q -$ fuzzy subset of G.

To prove: $(\theta \cap \sigma)^{\alpha} = \theta^{\alpha} \cap \sigma^{\alpha}$

$(\theta \cap \sigma)^{\alpha}(x, q) = \min\ \{(\theta \cap \sigma)(x, q), \alpha\}$

$= \min\ \{\min\{\theta(x, q), \sigma(x, q)\}, \alpha\}$

$= \min\ \{\min\{\theta(x, q), \alpha\},\ \min\{\sigma(x, q), \alpha\}\}$

$$= \min\ \{\theta^{\alpha}(x,q), \sigma^{\alpha}(x,q)\}$$

$$= \ \theta^{\alpha}(x,q) \cap \sigma^{\alpha}(x,q)$$

$= (\theta^{\alpha} \cap \sigma^{\alpha})(x,q)$, for all $x \in G$ and $q \in Q$

Hence, $(\theta \cap \sigma)^{\alpha} = \ \theta^{\alpha} \cap \sigma^{\alpha}$

Theorem: 4.4

Let $f: G \longrightarrow G'$ be a mapping. Let θ^{α} be fuzzy subset of G and σ^{α} be fuzzy subset of G'. Then,

i. $f(\theta^{\alpha}) = (f(\theta))^{\alpha}$
ii. $f^{-1}(\sigma^{\alpha}) = (f^{-1}(\sigma))^{\alpha}$

Proof:

Let $f: G \longrightarrow G'$ be a mapping. Let θ^{α} be fuzzy subset of G and σ^{α} be fuzzy subset of G'.

i. To prove: $f(\theta^{\alpha}) = (f(\theta))^{\alpha}$

$$f(\theta^{\alpha}(x,q)) = \sup\{\theta^{\alpha}(x,q) \mid f(x,q) = (x',q)\}$$

$$= \sup\ \{\min\{\theta(x,q), \alpha\} \mid f(x,q) = (x',q)\}$$

$$= \min\ \{\sup\ \{\theta(x,q) \mid f(x,q) = (x',q)\}, \alpha\}$$

$$= \min\ \{f\big(\theta(x',q)\big), \alpha\}$$

$= \ (f(\theta)(x',q))^{\alpha}$, for all $x \in G$, $x' \in G\,'$ and $q \in Q$

Hence, $f(\theta^{\alpha}) = (f(\theta))^{\alpha}$

ii. To prove: $f^{-1}(\sigma^{\alpha}) = (f^{-1}(\sigma))^{\alpha}$

$$f^{-1}(\sigma^{\alpha}(x,q)) = \sigma^{\alpha}(f(x,q))$$

$$= \min\{\sigma(f(x,q)), \alpha\}$$

$$= \min\{f^{-1}(\sigma^{\alpha}(x,q)), \alpha\}$$

$$= (f^{-1}(\sigma(x,q)))^{\alpha}, \text{ for all } x \in G \text{ and } q \in Q$$

Hence, $f^{-1}(\sigma^{\alpha}) = (f^{-1}(\sigma))^{\alpha}$

Definition: 4.5

Let θ be a Q – fuzzy subgroup of a group G and $\alpha \in [0,1]$. Then, θ^{α} is called $\alpha - Q -$ fuzzy subgroup of G, if for all $x, y \in G$ and $q \in Q$ the following condition hold:

i. $\theta^{\alpha}(xy,q) \geq \min\{\theta^{\alpha}(x,q),\ \theta^{\alpha}(y,q)\}$ and

ii. $\theta^{\alpha}(x^{-1},q) \geq \theta^{\alpha}(x,q)$

Theorem: 4.6

Every $Q -$ fuzzy subgroup of group G is $\alpha - Q -$ fuzzy subgroup of G.

Proof:

Let θ be Q – fuzzy subgroup of G.

Let x, y be any two elements in a group G.

To prove: θ be $\alpha - Q -$ fuzzy subgroup of G.

Then, $\theta^{\alpha}(xy, q) = \min\ \{\theta(xy, q), \alpha\}$

$\geq \min\{\min\ \{\theta(x, q), \theta(y, q)\}\ , \alpha\}$

$= \min\ \{\min\ \{\theta(x, q), \alpha\}, \min\ \{\theta(y, q), \alpha\}\}$

$= \min\ \{\theta^{\alpha}(x, q),\ \theta^{\alpha}(y, q)\}$

Therefore, $\theta^{\alpha}(xy, q) \geq \min\ \{\theta^{\alpha}(x, q),\ \theta^{\alpha}(y, q)\}$

$\theta^{\alpha}(x^{-1}, q) = \min\ \{\theta(x^{-1}, q), \alpha\}$

$\geq \min\ \{\theta(x, q), \alpha\}$

$= \theta^{\alpha}(x, q)$

Therefore, $\theta^{\alpha}(x^{-1}, q) \geq \theta^{\alpha}(x, q)$

Hence, θ is $\alpha - Q -$ fuzzy subgroup of G.

Theorem: 4.7

If θ^{α} is $\alpha - Q -$ fuzzy subgroup of a group G, then

i. $\theta^{\alpha}(e, q) \geq\ \theta^{\alpha}(x, q)$ for all $x \in G$ and $q \in Q$, where e is the identity element of a group G.

ii. A set $K = \{x \in G / \theta^{\alpha}(x, q) =\ \theta^{\alpha}(e, q), for\ q \in Q\}$ is an $\alpha - Q -$ fuzzy subgroup of a group G.

Proof:

Let θ^{α} is $\alpha - Q -$ fuzzy subgroup of a group G.

i. To prove: $\theta^{\alpha}(e,q) \geq \theta^{\alpha}(x,q)$ for all $x \in G$ and $q \in Q$, where e is the identity element of a group G.

$$\theta^{\alpha}(e,q) = \theta^{\alpha}(xx^{-1},q)$$

$$\geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(x^{-1},q)\}$$

$$= \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(x,q)\}$$

$$= \theta^{\alpha}(x,q)$$

Hence, $\theta^{\alpha}(e,q) \geq \theta^{\alpha}(x,q)\ \forall\ x \in G\ and\ q \in Q$.

ii. To prove: K is an $\alpha - Q -$ fuzzy subgroup of the group G.

The set $K \neq \emptyset$ as at least there exists $e \in K$. Let $x, y \in K$ and $q \in Q$.

Then, $\theta^{\alpha}(x,q) = \theta^{\alpha}(y,q) = \theta^{\alpha}(e,q)$

$$\theta^{\alpha}(xy^{-1},q) \geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y^{-1},q)\}$$

$$= \min\ \{\theta^{\alpha}(e,q), \theta^{\alpha}(e,q)\}$$

$$= \min\ \{\theta^{\alpha}(e,q), \theta^{\alpha}(e,q)\}$$

$$= \theta^{\alpha}(e,q)$$

So, $\theta^{\alpha}(xy^{-1},q) \geq \theta^{\alpha}(e,q)$

By part (i), we can show that $\theta^{\alpha}(e, q) \geq \theta^{\alpha}(xy^{-1}, q)$

Therefore, $\theta^{\alpha}(xy^{-1}, q) = \theta^{\alpha}(e, q)$

Hence, K is an $\alpha - Q -$ fuzzy subgroup of the group G.

Theorem: 4.8

Let θ^{α} be $\alpha - Q -$ fuzzy subgroup of a group G. If

$\theta^{\alpha}(xy^{-1}, q) = \theta^{\alpha}(e, q)$, then $\theta^{\alpha}(x, q) = \theta^{\alpha}(y^{-1}, q)$ for all x and y in G and q in Q.

Proof:

Let θ^{α} be a $\alpha - Q -$ fuzzy subgroup of a group G.

Let $\theta^{\alpha}(xy^{-1}, q) = \theta^{\alpha}(e, q)$, for all $x, y \in G$ and $q \in Q$. ---------- (1)

Then, $\theta^{\alpha}(x, q) = \theta^{\alpha}(x(y^{-1}y), q)$

$= \theta^{\alpha}\big((xy^{-1})y, q\big)$

$\geq \min \big\{\theta^{\alpha}\big((xy^{-1}), q\big), \theta^{\alpha}(y, q)\big\}$

$= \min \{\theta^{\alpha}(e, q), \theta^{\alpha}(y, q)\}$ [Since By (1)]

$= \theta^{\alpha}(y, q)$ [Since By theorem: 4.7 (i)]

Therefore, $\theta^{\alpha}(x, q) \geq \theta^{\alpha}(y, q)$ ---------- (2)

$\theta^{\alpha}(y, q) = \theta^{\alpha}(y^{-1}, q)$

$= \theta^{\alpha}\big((xx^{-1})y^{-1}, q\big)$

$= \theta^{\alpha}(x^{-1}(xy^{-1}), q)$

$\geq \min\ \{\theta^{\alpha}(x^{-1}, q), \theta^{\alpha}(xy^{-1}, q)\}$

$= \min\ \{\theta^{\alpha}(x, q), \theta^{\alpha}(e, q)\}$ [Since By (1)]

$= \theta^{\alpha}(x, q)$ [Since By theorem: 4.7 (i)]

Therefore, $\theta^{\alpha}(y, q) \geq \theta^{\alpha}(x, q)$ ---------- (3)

From (2) and (3),

Hence, $\theta^{\alpha}(x, q) = \theta^{\alpha}(y, q)$

Theorem: 4.9

If θ^{α} and σ^{α} is an $\alpha - Q -$ fuzzy subgroup of a group G, then $(\theta \cap \sigma)^{\alpha}$ is also an $\alpha - Q -$ fuzzy subgroup of the group G.

Proof:

Let θ^{α} and σ^{α} is an $\alpha - Q -$ fuzzy subgroup of a group G.

Let x, y be any two elements in G and q in Q.

To prove: $(\theta \cap \sigma)^{\alpha}$ is also an $\alpha - Q -$ fuzzy subgroup of the group G.

Then,

$(\theta \cap \sigma)^{\alpha}(xy, q) = (\theta^{\alpha} \cap \sigma^{\alpha})(xy, q)$ [Since By Result: 2.2 (1)]

$= \min\ \{\theta^{\alpha}(xy, q), \sigma^{\alpha}(xy, q)\}$

$\geq \min\{\min\{\theta^\alpha(x,q),\theta^\alpha(y,q)\},\min\{\sigma^\alpha(x,q),\sigma^\alpha(y,q)\}\}$

$= \min\{\min\{\theta^\alpha(x,q),\sigma^\alpha(x,q)\},\min\{\theta^\alpha(y,q),\sigma^\alpha(y,q)\}\}$

$= \min\{(\theta^\alpha \cap \sigma^\alpha)(x,q),(\theta^\alpha \cap \sigma^\alpha)(y,q)\}$

$= \min\{(\theta \cap \sigma)^\alpha(x,q),(\theta \cap \sigma)^\alpha(y,q)\}$ [Since By Result: 2.2 (1)]

Therefore,

$(\theta \cap \sigma)^\alpha(xy,q) \geq \min\{(\theta \cap \sigma)^\alpha(x,q),(\theta \cap \sigma)^\alpha(y,q)\}$

$(\theta \cap \sigma)^\alpha(x^{-1},q) = (\theta^\alpha \cap \sigma^\alpha)(x^{-1},q)$ [Since By Result: 2.2 (1)]

$= \min\{\theta^\alpha(x^{-1},q),\sigma^\alpha(x^{-1},q)\}$

$\geq \min\{\theta^\alpha(x,q),\sigma^\alpha(x,q)\}$

$= (\theta^\alpha \cap \sigma^\alpha)(x,q)$

$= (\theta \cap \sigma)^\alpha(x,q)$ [Since By Result: 2.2 (1)]

Therefore, $(\theta \cap \sigma)^\alpha(x^{-1},q) \geq (\theta \cap \sigma)^\alpha(x,q)$

Hence, $(\theta \cap \sigma)^\alpha$ is also an $\alpha - Q -$ fuzzy subgroup of the group G.

Theorem: 4.10

If θ^α is an $\alpha - Q -$ fuzzy subgroup of a group G, then $C(\theta^\alpha)$ is an $\alpha - Q -$ fuzzy subgroup of the group G.

Proof:

Let θ^{α} is an $\alpha - Q -$ fuzzy subgroup of a group G.

To prove: $C(\theta^{\alpha})$ is an $\alpha - Q -$ fuzzy subgroup of the group G.

$C[\theta^{\alpha}(xy, q)] = 1 - \theta^{\alpha}(xy, q)$

$\leq 1 - \min\{\theta^{\alpha}(x, q), \theta^{\alpha}(y, q)\}$

$= \max\ \{1 - \theta^{\alpha}(x, q), 1 - \theta^{\alpha}(y, q)\}$

$= \max\ \{C[\theta^{\alpha}(x, q)], C[\theta^{\alpha}(y^{-1}, q)]\}$

Therefore,

$C[\theta^{\alpha}(xy, q)] \leq \max\ \{C[\theta^{\alpha}(x, q)], C[\theta^{\alpha}(y^{-1}, q)]\}$, $\forall\ x \in G$ and $q\ \in\ Q$

$C[\theta^{\alpha}(x^{-1}, q)] = 1 - \theta^{\alpha}(x^{-1}, q)$

$\geq 1 - \theta^{\alpha}(x, q)$

$= C[\theta^{\alpha}(x, q)]$

Therefore, $C[\theta^{\alpha}(x^{-1}, q)] \geq C[\theta^{\alpha}(x, q)]$

Hence, $C(\theta^{\alpha})$ is an $\alpha - Q -$ fuzzy subgroup of the group G.

Theorem: 4.11

Let θ is Q −fuzzy subgroup of a group G. Then, θ^{α} is a $\alpha - Q$ −fuzzy subgroups of groups G iff $\theta^{\alpha}(xy^{-1}) \geq \min\ \{\theta^{\alpha}(x, q), \theta^{\alpha}(y, q)\}$ for all x, y in G and q in Q.

Proof:

Let θ is Q −fuzzy subgroup of a group G.

Let θ^{α} is a $\alpha - Q$ −fuzzy subgroups of groups G.

To prove: $\theta^{\alpha}(xy^{-1}) \geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y,q)\}$ for all x, y in G and q in Q.

Then, $\theta^{\alpha}(xy^{-1}) \geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y^{-1},q)\}$

$\geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y,q)\}$

Therefore, $\theta^{\alpha}(xy^{-1}) \geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y,q)\}$, for all x, y in G and q in Q.

Conversely,

Let $\theta^{\alpha}(xy^{-1}) \geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y,q)\}$ ---------- (1)

for all x, y in G and q in Q.

Let θ is Q −fuzzy subgroup of a group G.

To prove: θ^{α} is a $\alpha - Q$ −fuzzy subgroups of groups G.

Then, $\theta^{\alpha}(xy) = \theta^{\alpha}(x(y^{-1})^{-1}, q)$

$\geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y^{-1},q)\}$ [Since By (1)]

$\geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y,q)\}$

Therefore, $\theta^{\alpha}(xy) \geq \min\ \{\theta^{\alpha}(x,q), \theta^{\alpha}(y,q)\}$ for all x, y in G and q in Q.

$\theta^{\alpha}(x^{-1},q) = \theta^{\alpha}(x^{-1}e,q)$

$\geq \min \{\theta^{\alpha}(x,q), \theta^{\alpha}(e,q)\}$

$= \theta^{\alpha}(x,q)$ [Since By theorem:4.7 (i)]

Therefore, $\theta^{\alpha}(x^{-1},q) \geq \theta^{\alpha}(x,q)$, for all x in G and q in Q.

Where e is the identity element of G.

Hence, θ^{α} is a $\alpha - Q -$fuzzy subgroups of groups G.

Definition: 4.12

Let θ^{α} and σ^{α} be any two $\alpha - Q -$fuzzy subgroups of groups G and G', respectively. Then, $\theta^{\alpha} \times \sigma^{\alpha}$ said to be product of θ^{α} and σ^{α} and is defined as $\theta^{\alpha} \times \sigma^{\alpha}\big((x,x'),q\big) = \min \{\theta^{\alpha}(x,q), \sigma^{\alpha}(x',q)\}$, $\forall\, x \in G$, $x' \in G'$ and $q \in Q$.

Theorem: 4.13

If θ^{α} and σ^{α} be two $\alpha - Q -$ fuzzy subgroups of groups G and G' respectively, then $\theta^{\alpha} \times \sigma^{\alpha}$ is a $\alpha - Q -$ fuzzy subgroup of a group $G \times G'$.

Proof:

Let θ^{α} and σ^{α} be two $\alpha - Q -$ fuzzy subgroups of groups G and G' respectively.

To prove: $\theta^{\alpha} \times \sigma^{\alpha}$ is a $\alpha - Q -$ fuzzy subgroup of a group $G \times G'$.

If $x, y \in G$, and $x', y' \in G'$, then $(x,x'), (y,y') \in G \times G'$,

$\theta^{\alpha} \times \sigma^{\alpha}\big((x,x')(y,y'),q\big) = \theta^{\alpha} \times \sigma^{\alpha}([(xy),(x'y')],q)$

$$= \min\ \{\theta^{\alpha}(xy,q), \sigma^{\alpha}(x'y',q)\}$$

$$\geq \min\ \{\min\ \{\theta^{\alpha}(x,q), \sigma^{\alpha}(y,q)\}, \min\ \{\theta^{\alpha}(x',q), \sigma^{\alpha}(y',q)\}\}$$

$$= \min\ \{\min\ \{\theta^{\alpha}(x,q), \sigma^{\alpha}(x',q)\}, \min\ \{\theta^{\alpha}(y,q), \sigma^{\alpha}(y',q)\}\}$$

$$= \min\ \{\theta^{\alpha} \times \sigma^{\alpha}\big((x,x'),q\big), \theta^{\alpha} \times \sigma^{\alpha}\big((y,y'),q\big)\}$$

Therefore,

$$\theta^{\alpha} \times \sigma^{\alpha}\big((x,x')(y,y'),q\big) \geq \min\ \{\theta^{\alpha} \times \sigma^{\alpha}\big((x,x'),q\big),\ \theta^{\alpha} \times \sigma^{\alpha}\big((y,y'),q\big)\}$$

$$\theta^{\alpha} \times \sigma^{\alpha}((x,x')^{-1},q) = \theta^{\alpha} \times \sigma^{\alpha}\big((x^{-1},(x')^{-1}),q\big)$$

$$= \min\ \{\theta^{\alpha}(x^{-1},q), \sigma^{\alpha}((x')^{-1},q)\}$$

$$\geq \min\ \{\theta^{\alpha}(x,q), \sigma^{\alpha}(x',q)\}$$

$$= \theta^{\alpha} \times \sigma^{\alpha}\big((x,x'),q\big)$$

Therefore, $\theta^{\alpha} \times \sigma^{\alpha}((x,x')^{-1},q) \geq \theta^{\alpha} \times \sigma^{\alpha}\big((x,x'),q\big)$

Hence, $\theta^{\alpha} \times \sigma^{\alpha}$ is a $\alpha - Q -$ fuzzy subgroup of a group $G \times G'$.

Theorem: 4.14

If $\theta^{\alpha}, \sigma^{\alpha}$ and $\theta^{\alpha} \times \sigma^{\alpha}$ are $\alpha - Q -$ fuzzy subgroups of a groups G, G' and $G \times G'$ respectively, then the following statements are true:

i. If $\theta^{\alpha}(e,q) \geq \sigma^{\alpha}(x,q)$ for all $x \in G$ and $q \in Q$, then σ^{α} is an $\alpha - Q -$ fuzzy subgroup of a group G'.

ii. If $\sigma^{\alpha}(e', q) \geq \theta^{\alpha}(x, q)$ for all $x \in G$ and $q \in Q$, then θ^{α} is an $\alpha - Q -$ fuzzy subgroup of a group G.

Where e and e′ are the identity elements of groups G and G', respectively.

Proof:

Let $\theta^{\alpha}, \sigma^{\alpha}$ and $\theta^{\alpha} \times \sigma^{\alpha}$ are $\alpha - Q -$ fuzzy subgroups of a groups G, G' and $G \times G'$ respectively with $x, y \in G$ and $\in Q$.

i. Let $\theta^{\alpha}(e, q) \geq \sigma^{\alpha}(x, q)$ for all $x \in G$ and $q \in Q$.

To prove: σ^{α} is an $\alpha - Q -$ fuzzy subgroup of a group G'.

Since $\theta^{\alpha}(e, q) \geq \sigma^{\alpha}(x, q)$ ---------- (1)

for all $x \in G$ and $q \in Q$.

Then, $\sigma^{\alpha}(xy^{-1}) = \min\ \{\theta^{\alpha}(ee, q), \sigma^{\alpha}(xy^{-1}, q)\}$

$$\sigma^{\alpha}(xy^{-1}) = \theta^{\alpha} \times \sigma^{\alpha}([(ee), (xy^{-1})], q)$$

$$= \theta^{\alpha} \times \sigma^{\alpha}([(e, x), (e, y^{-1})], q)$$

$$\geq \min\ \{\theta^{\alpha} \times \sigma^{\alpha}((e, x), q), \theta^{\alpha} \times \sigma^{\alpha}((e, y^{-1}), q)\}$$

$$= \min\ \{\min\ \{\theta^{\alpha}(e, q), \sigma^{\alpha}(x, q)\}, \min\ \{\theta^{\alpha}(e, q), \sigma^{\alpha}(y^{-1}, q)\}\}$$

$$= \min\ \{\sigma^{\alpha}(x, q), \sigma^{\alpha}(y^{-1}, q)\} \quad \text{[Since By (1)]}$$

Therefore, $\sigma^{\alpha}(xy^{-1}) \geq \min\ \{\sigma^{\alpha}(x, q), \sigma^{\alpha}(y^{-1}, q)\}$

Since σ^{α} are $\alpha - Q -$ fuzzy subgroup of a group G'.

Therefore, $\sigma^{\alpha}(x^{-1}, q) \geq \sigma^{\alpha}(x\ , q)$

Hence, σ^{α} are $\alpha - Q -$ fuzzy subgroup of a group G'.

ii. Let $\sigma^{\alpha}(e', q) \geq \ \theta^{\alpha}(x, q)$ for all $x \in G$ and $q \in Q$.

To prove: θ^{α} is an $\alpha - Q -$ fuzzy subgroup of a group G.

Since $\sigma^{\alpha}(e', q) \geq \ \theta^{\alpha}(x, q)$ ---------- (2)

for all $x \in G$ and $q \in Q$.

Then, $\theta^{\alpha}(xy^{-1}) = \min\ \{\theta^{\alpha}(xy^{-1}, q), \sigma^{\alpha}(e'e', q)\}$

$$\theta^{\alpha}(xy^{-1}) = \theta^{\alpha} \times \sigma^{\alpha}([(xy^{-1}), (e'e')], q)$$

$$= \theta^{\alpha} \times \sigma^{\alpha}([(x, e'), (y^{-1}, e')], q)$$

$$\geq \min\ \{\theta^{\alpha} \times \sigma^{\alpha}((x, e'), q), \theta^{\alpha} \times \sigma^{\alpha}((y^{-1}, e'), q)\}$$

$$= \min\ \{\min\ \{\theta^{\alpha}(x, q), \sigma^{\alpha}(e', q)\}, \min\ \{\theta^{\alpha}(y^{-1}, q), \sigma^{\alpha}(e', q)\}\}$$

$$= \min\ \{\theta^{\alpha}(x, q), \theta^{\alpha}(y^{-1}, q)\} \quad \text{[Since By (2)]}$$

Therefore, $\theta^{\alpha}(xy^{-1}) \geq \min\ \{\theta^{\alpha}(x, q), \theta^{\alpha}(y^{-1}, q)\}$

Since θ^{α} are $\alpha - Q -$ fuzzy subgroup of a group G.

Therefore, $\theta^{\alpha}(x^{-1}, q) \geq \theta^{\alpha}(x\ , q)$

Hence, θ^{α} are $\alpha - Q -$ fuzzy subgroup of a group G.

Definition: 4.15

Let $f: G \to G'$ be any mapping from a group G into a group G'. Let θ^{α} be an $\alpha - Q -$ fuzzy subgroup of a group in $G' = f(G)$, defined by $\sigma^{\alpha}(x', q) = sup\ \theta^{\alpha}(x, q)$, $\forall\ x \in f^{-1}(x'),\ x \in X$ and $x' \in X'$. Then, θ^{α} is called an inverse image of σ^{α} under map f and is denoted by $f^{-1}(\sigma^{\alpha})$.

Theorem: 4.16

Let $f: G \to G'$ be an anti-homomorphism from a group G into the group G', and Q be a non-empty set. If θ^{α} is an $\alpha - Q -$ fuzzy subgroup of G, then the anti-homomorphism image $f(\theta^{\alpha})$ is an $\alpha - Q -$ fuzzy subgroup of G'.

Proof:

Let $f: G \to G'$ be an anti-homomorphism from a group G into the group G'.

Let Q be a non-empty set.

Let θ^{α} be an $\alpha - Q -$ fuzzy subgroup of G and its image $f(\theta^{\alpha})$ be in G'.

To prove: $f(\theta^{\alpha})$ is an $\alpha - Q -$ fuzzy subgroup of G'.

Then for all $f(x), f(y)$ in G, q in Q

$$f(\theta^{\alpha}(f(x)f(y), q)) = f(\theta^{\alpha}(f(yx), q))$$

$$= \theta^{\alpha}(yx, q)$$

$$\geq \min\ \{\theta^{\alpha}(x, q), \theta^{\alpha}(x, q)\}$$

$$= \min \left\{f\left(\theta^{\alpha}(f(x), q)\right), f\left(\theta^{\alpha}(f(y), q)\right)\right\}$$

Therefore, $f(\theta^{\alpha}(f(x)f(y), q)) \geq \min \left\{f\left(\theta^{\alpha}(f(x), q)\right), f\left(\theta^{\alpha}(f(y), q)\right)\right\}$ and for all $f(x)$ in G' and q in Q.

$$f(\theta^{\alpha}(f(x)^{-1}, q)) = f(\theta^{\alpha}(f(x^{-1}), q))$$

$$= \theta^{\alpha}(x^{-1}, q)$$

$$\geq \theta^{\alpha}(x, q)$$

$$= f\left(\theta^{\alpha}(f(x), q)\right)$$

Therefore, $f(\theta^{\alpha}(f(x)^{-1}, q)) \geq f\left(\theta^{\alpha}(f(x), q)\right)$

Hence, $f(\theta^{\alpha})$ is an $\alpha - Q -$ fuzzy subgroup of G'.

A STUDY ON (α, β) AND o-ANTI FUZZY SUBGROUPS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

S. SOWMIYA

Reg. No: 19SPMT23

Under the guidance of

Dr. C. REENA M.Sc., B.Ed., M.Phil., SET., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON (α,β) AND o-ANTI FUZZY SUBGROUPS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **S. SOWMIYA (Reg. No: 19SPMT23)**

Signature of the Guide

Signature of the Coordinator

Joyce
Signature of the Director
Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose
Signature of the Principal
Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON (α, β) AND o-ANTI FUZZY SUBGROUPS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Dr. C. Reena M.Sc., B.Ed., M.Phil., SET., Ph.D.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

S. Sowmiya

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER-1

PRELIMINARIES

Definition: 1.1

A fuzzy set A of a set is a function $A : X \to [0,1]$. Fuzzy sets taking the values 0 and 1 are called **Crisp sets**

Let A and B be two fuzzy subsets of a set X.

- $A \subseteq B$ if and only if $A(x) \leq B(x)$, for all $x \in X$
- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
- The complement of the fuzzy set A is A^c and is defined as $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min\{A(x), B(x)\}, \forall\, x \in X$
- $(A \cup B)(x) = \max\{A(x), B(x)\}, \forall\, x \in X$

Definition: 1.2

A function $A : G \to [0,1]$ is a **fuzzy subgroup (FSG)** of a group G if and only if $A(xy^{-1}) \geq \min\{A(x),\ A(y)\},\ \ \forall\, x, y \in G$

Definition: 1.3

A function $A : G \to [0,1]$ is an **anti fuzzy subgroup (AFSG)** of a group G if and only if $A(xy^{-1}) \leq \max\{A(x), B(x)\}, \forall\, x, y \in G$

Definition: 1.4

A fuzzy subgroup (or anti fuzzy subgroup) A of a group G is called fuzzy **normal subgroup (FNSG) [or anti fuzzy normal subgroup (AFNSG)]** of G if and only if $A(y^{-1}xy) = A(x)$ or equivalently, $A(xy) = A(yx)$, holds for all $x, y \in G$.

Definition: 1.5

Let A be a fuzzy subset of a group G. Then A is called a **Anti fuzzy subgroup**

- $A(xy) \leq \max\{A(x), A(y)\}$
- $A(x^{-1}) \leq A(x), for\ all\ x, y \in G$

It is easy to show that an anti fuzzy subgroup of a group G satisfies $A(x) \geq A(e)$ and $A(x^{-1}) = A(x)$, for all $x \in G$, where e is the identity element of G.

Definition: 1.6

Let A be a fuzzy subset of a group G and $\alpha \in [0,1]$. Then the fuzzy sets A^{α} and A_{α} of G are respectively called **the α-fuzzy subset and α-anti fuzzy subset** of G (w.r.t fuzzy set A) and is defined as $A^{\alpha}(x) = \min\{A(x), \alpha\}$ and

$A_{\alpha}(x) = \max\{A(x), 1 - \alpha\}, \forall x \in G$

Definition: 1.7

Let A be a fuzzy subset of a group G and $\alpha \in [0,1]$. Then A is called **α-fuzzy subgroup (α-FSG)** of G if $A^{\alpha}(xy^{-1}) \geq \min\{A^{\alpha}(x), A^{\alpha}(y)\}; \forall\ x, y \in G.$

Definition: 1.8

A fuzzy subgroup (or anti fuzzy subgroup) A of a group G is called **α-fuzzy normal subgroup (α-FNSG) [or α-anti fuzzy normal subgroup (α-AFNSG)]** of G if and only if

[$A^{\alpha}(y^{-1}xy) = A^{\alpha}(x)$ or equivalently, $A^{\alpha}(xy) = A^{\alpha}(yx)$, holds for all $x, y \in G$

Or

[$A_{\alpha}(y^{-1}xy) = A_{\alpha}(x)$ or equivalently, $A_{\alpha}(xy) = A_{\alpha}(yx)$, holds for all $x, y \in G$]

Definition: 1.9

Let A be a fuzzy subset of a group and $\alpha \in [0,1]$. Then A is called **α-anti fuzzy subgroup (α-AFSG)** of G if $A_\alpha(xy^{-1}) \leq \max\{A_\alpha(x), A_\alpha(y)\}, \forall\, x, y \in G$.

Proposition: 1.10

If A be a FSG of the group G, then A is also α-FSG as well as α-AFSG of G.

Proposition: 1.11

Let $A : G \to [0,1]$ be a α-FSG of a group G, then

- $A^\alpha(x) \leq A^\alpha(e), \forall\, x \in G$, where e is the identity element of G
- $A^\alpha(xy^{-1}) = A^\alpha(e) \Longrightarrow A^\alpha(x) = A^\alpha(y), \forall\, x, y \in G$

Proposition: 1.12

Let $A : G \to [0,1]$ be an α-AFSG of a group G, then

- $A_\alpha(x) \geq A_\alpha(E), \forall\, x \in G$, where E is the identity element of G
- $A_\alpha(xy^{-1}) = A_\alpha(E) \Longrightarrow A_\alpha(x) = A_\alpha(y), \forall\, x, y \in G$

Proposition: 1.13

If A is a FNSG of a group G, then A is also a α-FNSG as well as α-AFNSG of G.

Definition: 1.14

Let X be a nonempty set. A mapping $A : X \to [0,1]$ is called a fuzzy subset of X.

Definition: 1.15

Let A be a fuzzy subset of a universe X and $\delta \in [0,1]$. The set $A^{\delta} = \{x \in X : A(x) \geq \delta\}$ is called **level subset** of a fuzzy set A

Definition: 1.16

Let A be an anti fuzzy subgroup of a group G. For any $x \in G$, the fuzzy set xA defined by $(xA)(y) = A(x^{-1}y)$, for all $y \in G$ is called a **left anti fuzzy coset** of A. The **right anti fuzzy coset** of A may be defined in the same way.

Definition: 1.17

It is quite evident that a group homomorphism f admits the following characteristics:

1. $f(A)f(x) \geq A(x)$, for every element $x \in G_1$
2. If f is bijective map, then $f(A)f(x) = A(x)$, for all $x \in G_1$

Definition: 1.18

Let $f : G_1 \rightarrow G_2$ be a homomorphism of group G_1 into a group G_2. Let A and B be fuzzy subsets of G_1 and G_2 respectively, then $f(A)$ and $f^{-1}(B)$ are respectively the image of fuzzy set A and the inverse image of fuzzy set B, defined as

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in f^{-1}(y)\}; & if\ f^{-1}y \neq \emptyset \\ 1 & ; if\ f^{-1}y = \emptyset \end{cases}, \text{ for every } y \in G_2$$

and $f^{-1}(B)(x) = B(f(x))$, for every $x \in G_1$

Definition: 1.19

Let $s_b : [0,1] \times [0,1] \to [0,1]$ be the bounded sum conorm defined by

$s_b(a,b) = \min(a+b, 1),\ \ 0 \leq a \leq 1, 0 \leq b \leq 1$

Clearly bounded sum conorm satisfies all the axioms of t-conorm.

Remark: 1.20

Clearly, $A^1 = A\ , A^0 = \tilde{0}\ , and\ A_1 = A\ , A_0 = \tilde{1}$

CHAPTER-2

(α, β)-ANTI FUZZY SUBGROUP AND THEIR PROPERTIES

Definition: 2.1

Let A^{α} and A_{β} denote respectively the α-fuzzy set and β-anti fuzzy set of the set X (w.r.t the fuzzy set A). Then the fuzzy set $A_{(\alpha,\beta)}$ defined by

$A_{(\alpha,\beta)} = Max\{(A^{\alpha})^{c}(x), A_{\beta}(x)\}$, for every $x \in X$, is called (α,β)-anti fuzzy set of X (w.r.t. the fuzzy set A), Where $\alpha,\beta \in [0,1]$ such that $\alpha + \beta \leq 1$.

Remark: 2.2

- $A_{(1,0)}(x) = Max\{(A^{1})^{c}(x), A_{0}(x)\} = Max\{A^{c}(x), 1\} = 1$
- $A_{(0,1)}(x) = Max\{(A^{0})^{c}(x), A_{1}(x)\} = Max\{1, A(x)\} = 1$

Definition: 2.3

Let A be a (α,β)-anti fuzzy set of a group G (w.r.t/ the fuzzy set A), then A is called $(\boldsymbol{\alpha},\boldsymbol{\beta})$**-anti fuzzy subgroup** $((\boldsymbol{\alpha},\boldsymbol{\beta}) - \boldsymbol{AFSG})$ of G if the following conditions hold

I. $A_{(\alpha,\beta)}(xy) \leq Max\{A_{(\alpha,\beta)}(x), A_{(\alpha,\beta)}(y)\}$

II. $A_{(\alpha,\beta)}(x^{-1}) = A_{(\alpha,\beta)}(x),\ for\ all\ x, y \in G$

Equivalently, we have

$$A_{(\alpha,\beta)}(xy^{-1}) \leq Max\{A_{(\alpha,\beta)}(x), A_{(\alpha,\beta)}(y)\}, for\ every\ x \in G$$

Remark: 2.4

If A is a (α, β)-anti fuzzy subgroup of a group G, then we have

I. $A_{(\alpha,\beta)}(e) \leq A_{(\alpha,\beta)}(x)$, where e is the identity of group G

II. If $A_{(\alpha,\beta)}(xy^{-1}) = A_{(\alpha,\beta)}(e) \Longrightarrow A_{(\alpha,\beta)}(x) = A_{(\alpha,\beta)}(y)$

Proof:

I. $A_{(\alpha,\beta)}(e) = Max\{(A^{\alpha})^c(e), A_{\beta}(e)\}$

$\leq Max\{(A^{\alpha})^c(x), A_{\beta}(x)\}$

$= A_{(\alpha,\beta)}(x)$

II. $A_{(\alpha,\beta)}(x) = A_{(\alpha,\beta)}(xy^{-1}y)$

$\leq Max\{A_{(\alpha,\beta)}(xy^{-1}), A_{(\alpha,\beta)}(y)\}$

$= Max\{A_{(\alpha,\beta)}(e), A_{(\alpha,\beta)}(y)\}$

$= A_{(\alpha,\beta)}(y)$

$= A_{(\alpha,\beta)}(yx^{-1}x)$

$\leq Max\{A_{(\alpha,\beta)}(xy^{-1}), A_{(\alpha,\beta)}(x)\}$

$= Max\{A_{(\alpha,\beta)}(e), A_{(\alpha,\beta)}(x)\} = A_{(\alpha,\beta)}(x)$

Hence $A_{(\alpha,\beta)}(x) = A_{(\alpha,\beta)}(y)$

Proposition: 2.5

Let A be a α-FSG as well as β-AFSG of a group G, then A is also (α, β)-AFSG of G.

Proof:

Let x, y be any element of the group G, then

$$A_{(\alpha,\beta)}(xy^{-1}) = Max\{(A^{\alpha})^c(xy^{-1}), A_{\beta}(xy^{-1})\}$$

$$\leq Max\{\max\{(A^{\alpha})^c(x), (A^{\alpha})^c(y)\}, max\{A_{\beta}(x), A_{\beta}(y)\}\}$$

$$= Max\{\max\{(A^{\alpha})^c(x), A_{\beta}(x)\}, \max\{(A^{\alpha})^c(y), A_{\beta}(y)\}\}$$

$$= Max\{A_{(\alpha,\beta)}(x), A_{(\alpha,\beta)}(y)\}$$

Thus $A_{(\alpha,\beta)}(xy^{-1}) \leq Max\{A_{(\alpha,\beta)}(x), A_{(\alpha,\beta)}(y)\}$

Hence A is a (α,β)-AFSG of G.

Theorem: 2.6

Let A be a FSG of a group G, then A is also (α,β)-FSG of G.

Proof:

Since A is a FSG of group G.

Therefore A is a α-FSG as well as A is β-AFSG of G [∵proposition 1.10]

Hence the result follows from proposition 2.5

Remark: 2.7

The converse of the theorem (2.6) need not be true

(i.e) A fuzzy set A of a group G can be (α,β)-AFSG of G without being FSG of G .

Example: 2.8

Let $G = \{e, a, b, ab\}$, where $a^2 = b^2 = e$ and $ab = ba$ be the klein four group.

Proof:

Let the fuzzy set A of G be defined as

$$A = \{< e, 0.1 >, < a, 0.3 >, < b, 0.3 >, < ab, 0.4 >\}$$

Clearly, A is not a FSG of G.

Take $\alpha = 0.05\ and\ \beta = 0.6$

Then $A(x) > \alpha, \forall\ x \in G$.

So that $A^{\alpha}(x) = \min\{A(x), 0.05\} = 0.05$

$\Rightarrow (A^{\alpha})^c(x) = 1 - 0.05 = 0.95\ , for\ all\ x \in G$

Also, $A_{\beta}(x) = Max\{A(x), 1 - \beta\}\ \forall\ x \in G$

$$= Max\{A(x), 1 - 0.6\}\ \forall\ x \in G$$

$$= Max\{A(x), 0.4\}\ \forall\ x \in G$$

$$= 0.4$$

Further, $A^{(0.05,0.6)}(x) = Max\{(A^{0.05})^c(x), A_{0.6}(x)\}\ \forall\ x \in G$

$$= Max\{0.95, 0.4\}\ \ = 0.95$$

Thus, $A^{(0.05,0.6)}(xy^{-1}) \leq Max\{A^{(0.05,0.6)}(xy^{-1}), A^{(0.05,0.6)}(xy^{-1})\}\ hold\ for\ all$

$x \in G$.

Hence A is (0.05, 0.6)-AFSG of G.

Proposition: 2.9

Let A be a fuzzy subset of a group G. Let $\alpha \leq p \; and \; \beta \leq 1 - q$, where $p = \inf\{A(x): for \; all \; x \in G\}$ and $q = \sup\{A(x): for \; all \; x \in G\}$. Then A is a (α, β)-AFSG of G.

Proof:

Since $\alpha \leq p \; and \; \beta \leq 1 - q \Rightarrow \alpha + \beta \leq p + 1 - q \leq q + 1 - q = 1$

Now, $\alpha \leq p \Rightarrow p \geq \alpha$

$$\Rightarrow \inf\{A(x): for \; all \; x \in G\} \geq \alpha$$

$$\Rightarrow A(x) \geq \alpha \; \forall \; x \in G$$

$$\therefore A^{\alpha}(x) = \min\{A(x), \alpha\} = \alpha, \forall \; x \in G$$

$$\Rightarrow (A^{\alpha})^{c}(x) = 1 - \alpha, \; \forall \; x \in G.$$

Similarly, as $\beta \leq 1 - q \Rightarrow q \leq 1 - \beta$

$$\Rightarrow \sup\{A(x): for \; all \; x \in G\} \leq 1 - \beta, \forall \; x \in G$$

$$\Rightarrow A(x) \leq 1 - \beta, \forall \; x \in G$$

$$\therefore \; A_{\beta}(x) = \max\{A(x), \; 1 - \beta\} = 1 - \beta, \forall \; x \in G$$

Now, $A_{(\alpha,\beta)}(x) = Max\{(A^{\alpha})^{c}(x), A_{\beta}(x)\}$

$$= Max\{1 - \alpha, 1 - \beta\}, \forall \; x \in G$$

Therefore, $A_{(\alpha,\beta)}(xy^{-1}) \leq Max\{A_{(\alpha,\beta)}(x), A_{(\alpha,\beta)}(y)\} \; hold \; \forall \; x, y \in G$

Hence, A is a (α, β) $-$AFSG of group G.

Remark: 2.10

Intersection of two (α, β) −AFSG's of a group G need not be (α, β) −AFSG of G.

Example: 2.11

Let G = Z, the group of integers under ordinary addition of integers.

Define the two fuzzy sets A and B by

$A(x) = \begin{cases} 0.5\,, & if\ x = 3Z \\ 0.7, & otherwise \end{cases}$ and $B(x) = \begin{cases} 0.6, & if\ x = 2Z \\ 0.8, & otherwise \end{cases}$

Take $\alpha = 0.4\ and\ \beta = 0.35$, then we have

$A^{0.4}(x) = \min\{A(x), 0.4\} = 0.4, \forall x \in Z$ and

$$A_{0.35}(x) = \max\{A(x), 0.65\} = \begin{cases} 0.65, & if\ x = 3Z \\ 0.7, & otherwise \end{cases}$$

$B^{0.4}(x) = \min\{A(x), 0.6\} = 0.6, \forall x \in Z$ and

$$A_{0.35}(x) = \max\{B(x), 0.65\} = \begin{cases} 0.65, & if\ x = 2Z \\ 0.8, & otherwise \end{cases}$$

$$A_{(0.4,0.35)}(x) = Max\{(A^{0.4})^c(x), A_{0.35}(x)\}$$

$$= \begin{cases} 0.65, & if\ x = 3Z \\ 0.7, & otherwise \end{cases} \text{ and}$$

$$B_{(0.4,0.35)}(x) = Max\{(B^{0.4})^c(x), B_{0.35}(x)\} = \begin{cases} 0.65, & if\ x = 2Z \\ 0.8, & otherwise \end{cases}$$

It can be easily verify that A and B are (0.4, 0.35)-AFSG of Z

Now, $(A \cap B)(x) = \min\{A(x), B(x)\} = \begin{cases} 0.5, & if\ x \in 2Z \\ 0.6, & if\ x \in 3Z - 2Z \\ 0.7, & if\ x \notin 2Z\ or\ x \notin 3Z \end{cases}$

$(A \cap B)^{0.4}(x) = \min\{(A \cap B)(x), 0.4\} = 0.4$

$$\Longrightarrow ((A \cap B)^{0.4})^c(x) = 0.6, \forall x \in Z$$

$$(A \cap B)_{0.35}(x) = \max\{(A \cap B)(x), 0.65\} = \begin{cases} 0.65, & if\ x \in 2Z\ or\ x \in 3Z \\ 0.7, & if\ x \notin 2Z\ or\ x \notin 3Z \end{cases}$$

$$(A \cap B)_{(0.4,0.35)}(x) = Max\{((A \cap B)^{0.4})^c(x), (A \cap B)_{0.35}(x)\}$$

$$= Max\{0.6, (A \cap B)_{0.35}(x)\} = \begin{cases} 0.65, & if\ x \in 2Z\ or\ x \in 3Z \\ 0.7, & if\ x \notin 2Z\ or\ x \notin 3Z \end{cases}$$

Take x = 9 and y = 4, we get

$$(A \cap B)_{(0.4,0.35)}(x - y) = (A \cap B)_{(0.4,0.35)}(9 - 4)$$

$$= (A \cap B)_{(0.4,0.35)}(5)$$

$$= 0.7$$

But $(A \cap B)_{(0.4,0.35)}(9) = 0.65\ and\ (A \cap B)_{(0.4,0.35)}(4) = 0.65$

Therefore, $Max\{(A \cap B)_{(0.4,0.35)}(9), \left(A \cap B\right)_{(0.4,0.35)}(4)\}$

$$= Max\{0.65, 0.65\} = 0.65$$

Clearly, $(A \cap B)_{(0.4,0.35)}(9 - 4) > Max\{(A \cap B)_{(0.4,0.35)}(9), (A \cap B)_{(0.4,0.35)}(4)\}$

Hence $A \cap B$ is not (0.4, 0.35)-AFSG of Z.

Remark: 2.12

Union of two (α, β)-AFSG's of a group G need not be (α, β)-AFSG of G.

Example: 2.13

Let G = Z, the group of integers under ordinary addition of integers.

Define the two fuzzy sets A and B as in Example (2.11)

Now, $(A \cup B)(x) = \max\{A(x), B(x)\} = \begin{cases} 0.6, & if\ x \in 2Z \\ 0.7, & if\ x \in 3Z - 2Z \\ 0.8, & if\ x \notin 2Z\ or\ x \notin 3Z \end{cases}$

$(A \cup B)^{0.4}(x) = \min\{(A \cup B)(x), 0.4\} \Longrightarrow ((A \cup B)^{0.4})^c(x) = 0.6\ \forall x \in Z$

$(A \cup B)_{0.35}(x) = \max\{(A \cup B)(x), 0.65\} = \begin{cases} 0.65, & if\ x \in 2Z \\ 0.7, & if\ x \in 3Z - 2Z \\ 0.8, & if\ x \notin 2Z\ or\ x \notin 3Z \end{cases}$

$(A \cup B)_{(0.4,0.35)}(x) = \max\{((A \cup B)^{0.4})^c(x), (A \cup B)_{0.35}(x)\}$

$$= Min\{0.6, (A \cup B)_{0.35}(x)\} = \begin{cases} 0.65, & if\ x \in 2Z \\ 0.7, & if\ x \in 3Z - 2Z \\ 0.8, & if\ x \notin 2Z\ or\ x \notin 3Z \end{cases}$$

Take x = 9, y = 4, we get

$(A \cup B)_{(0.4,0.35)}(x - y) = (A \cup B)_{(0.4,0.35)}(9 - 4)$

$$= (A \cup B)_{(0.4.0.35)}(5)$$

$$= 0.8$$

But $(A \cup B)_{(0.4,0.35)}(9) = 0.7$ and $(A \cup B)_{(0.4,0.35)}(4) = 0.65$

Therefore, $Max\{(A \cup B)_{(0.4,0.35)}(9), (A \cup B)_{(0.4,0.35)}(4)\} = Max\{0.7, 0.65\} = 0.7$

Clearly, $(A \cup B)_{(0.4,0.35)}(9 - 4) > Max\{(A \cup B)_{(0.4,0.35)}(9), (A \cup B)_{(0.4,0.35)}(4)\}$

Hence, $A \cup B$ is not (0.4, 0.35)-AFSG of Z.

CHAPTER-3

$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-ANTI FUZZY COSETS AND NATURAL HOMOMORPHISM

Definition: 3.1

Let A be (α, β)-AFSG of a group G, where $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. For any $x \in G$, define a fuzzy set $(A_{(\alpha,\beta)})_x$ of G, called **$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-anti fuzzy right coset of A** in G as follows

$$(A_{(\alpha,\beta)})_x(g) = A_{(\alpha,\beta)}(gx^{-1}), \text{ for all } x, g \in G.$$

Similarly,

We define a fuzzy set ${}_x(A_{(\alpha,\beta)})$ of G, called **$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-anti fuzzy left coset of A** in G as follows

$${}_x\big(A_{(\alpha,\beta)}\big)(g) = A_{(\alpha,\beta)}(x^{-1}g), \text{ for all } x, g \in G.$$

Definition: 3.2

Let A be (α, β)-AFSG of a group G, where $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$.

Then A is called **$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-anti fuzzy normal subgroup** (**$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-AFNSG) of G** is and only if ${}_x\big(A_{(\alpha,\beta)}\big) = (A_{(\alpha,\beta)})_x$ for all $x \in G$.

Theorem: 3.3

If A is a FNSG of a group G, then A is also a (α, β)-AFNSG of G.

Proof:

Let A be a FNSG of G.

Therefore A is α-FSNG of G as well as A is β-AFNSG of G [By proposition 1.12]

Let $x, g \in G$ be any element.

Then,

$$ {}_x\big(A_{(\alpha,\beta)}\big)(g) = A_{(\alpha,\beta)}(x^{-1}g) $$

$$ = \max\{(A^{\alpha})^c(x^{-1}g), A_{\beta}(x^{-1}g)\} $$

$$ = \max\{(A^{\alpha})^c(gx^{-1}), A_{\beta}(gx^{-1})\} $$

$$ = (A_{(\alpha,\beta)})_x(g) $$

Thus ${}_x\big(A_{(\alpha,\beta)}\big) = (A_{(\alpha,\beta)})_x$, $for\ all\ x \in G$

Hence A is (α, β)-AFNSG of G.

Remark: 3.4

The converse of the above theorem need not be true.

Example: 3.5

Let $G = D_3 = < a,\ b : \ a^3 = b^3 = e,\ ba = a^2b >$ be the dihedral group with six elements.

Define the AFSG A of D_3 by

$$ \begin{cases} 0.8, & if\ x \in < b > \\ 0.7, & otherwise \end{cases} $$

Note that A is not a FNSG of G, for $A\big(a^2(ab)\big) = 0.8 \neq 0.7 = A\big(ab(a^2)\big)$.

Now take $\alpha = 0.6,\ \beta = 0.1,$ then we get

$A^{0.6}(x) = \min\{A(x), 0.6\}\ \ \forall\, x \in G$ [by definition 1.5]

$= 0.6$

$A_{0.1}(x) = \max\{A(x), 0.9\}\ \forall\, x \in G$ [by definition 1.5]

$= 0.9$

$(A^{0.6})^c(x) = 1 - A^{0.6}(x)$

$= 1 - 0.6$

$= 0.4$

Therefore, $A_{(0.6,0.1)}(x) = \max\{(A^{0.6})^c(x), A_{0.1}(x)\}, \forall\, x \in G$

$= \max\{0.4, 0.6\}$

$= 0.9$

Thus, we get $A_{(0.6,0.1)}(xy) = A_{(0.6,0.1)}(yx) = 0.9\ ,\ \forall\, x, y \in G$

Hence A is $(0.6, 0.1)$-AFNSG of G.

Proposition: 3.6

Let A be a (α, β)-AFNSG of a group G. Then $A_{(\alpha,\beta)}(y^{-1}xy) = A_{(\alpha,\beta)}(x)$

or equivalently, $A_{(\alpha,\beta)}(xy) = A_{(\alpha,\beta)}(yx)$, holds for all $x, y \in G$.

Proof:

Let A be a (α, β)-AFNSG of a group G.

Therefore, ${}_x\big(A_{(\alpha,\beta)}\big) = (A_{(\alpha,\beta)})_x$ hold for all $x \in G$.

$\Rightarrow {}_x(A_{(\alpha,\beta)})(y^{-1}) = (A_{(\alpha,\beta)})_x(y^{-1})$ hold for $y^{-1} \in G$.

$\Rightarrow A_{(\alpha,\beta)}(x^{-1}y^{-1}) = A_{(\alpha,\beta)}(y^{-1}x^{-1})$

$\Rightarrow A_{(\alpha,\beta)}((yx)^{-1}) = A_{(\alpha,\beta)}((xy)^{-1})$

$\Rightarrow A_{(\alpha,\beta)}(yx) = A_{(\alpha,\beta)}(xy)$ [by definition 2.3 (ii)]

Next, we show that for some specific values of α, β every (α, β)-AFSG A of G will always be (α, β)-AFNSG of G. In this direction, we have the following:

Proposition: 3.7

Let A be a (α, β)-AFSG of a group G such that $\alpha \leq p$ and $\beta \leq 1 - q$, where $p = Inf\{A(x) : for\ all\ x \in G\}$ and $q = Sup(A(x) : for\ all\ x \in G\}$. Then A is (α, β)-AFNSG of G.

Proof:

From proposition (2.9), we have

$A_{(\alpha,\beta)}(x) = \max\{(A^{\alpha})^c(x), A_{\beta}(x)\}, \quad \forall\, x \in G$

$= \min\{1-\alpha, 1-\beta\}, \ \forall\, x \in G$

$\Rightarrow A_{(\alpha,\beta)}(xy) = \max\{1-\alpha, 1-\beta\} = A_{(\alpha,\beta)}(yx), \forall x, y \in G.$

Hence A is (α, β)-AFNSG of G.

Proposition: 3.8

Let A be a (α, β)-anti fuzzy normal subgroup of G, then the set

$G_{(\alpha,\beta)} = \{x \in G: A_{(\alpha,\beta)}(x) = A_{(\alpha,\beta)}(e)\}$ is a normal subgroup of G.

Proof:

Clearly, $G_{A_{(\alpha,\beta)}} \neq \emptyset$, for $e \in G_{A_{(\alpha,\beta)}}$.

Let $x, y \in G_{A_{(\alpha,\beta)}}$ be any element.

Then $A_{(\alpha,\beta)}(xy^{-1}) \leq \max\{A_{(\alpha,\beta)}(x), A_{(\alpha,\beta)}(y)\}$

$= \max\{A_{(\alpha,\beta)}(e), A_{(\alpha,\beta)}(e)\}$

$= A_{(\alpha,\beta)}(e)$

(i.e) $A_{(\alpha,\beta)}(xy^{-1}) \leq A_{(\alpha,\beta)}(e)$, but $A_{(\alpha,\beta)}(e) \leq A_{(\alpha,\beta)}(xy^{-1})$

Therefore, $A_{(\alpha,\beta)}(xy^{-1}) = A_{(\alpha,\beta)}(e)$

So, $xy^{-1} \in G_{A_{(\alpha,\beta)}}$

Thus, $G_{A_{(\alpha,\beta)}}$ is a subgroup of G.

Further, let $x \in G_{A_{(\alpha,\beta)}}$ and $y \in G$ be any element.

Now, $A_{(\alpha,\beta)}(y^{-1}xy) = A_{(\alpha,\beta)}(x) = A_{(\alpha,\beta)}(e) \Longrightarrow y^{-1}xy \in G_{A_{(\alpha,\beta)}}$

Hence $G_{A_{(\alpha,\beta)}}$ is a normal subgroup of G.

Proposition: 3.9

Let A be a (α, β)-AFNSG of group G, then

(I) ${}_x\big(A_{(\alpha,\beta)}\big) = {}_y(A_{(\alpha,\beta)})$ if and only if $x^{-1}y \in G_{A_{(\alpha,\beta)}}$

(II) $(A_{(\alpha,\beta)})_x = (A_{(\alpha,\beta)})_y$ if and only if $xy^{-1} \in G_{A_{(\alpha,\beta)}}$

Proof:

I. let ${}_x(A_{(\alpha,\beta)}) = {}_y(A_{(\alpha,\beta)})$

$$A_{(\alpha,\beta)}(x^{-1}y) = {}_x(A_{(\alpha,\beta)})(y)$$

$$= {}_y(A_{(\alpha,\beta)})(y)$$

$$= A_{(\alpha,\beta)}(y^{-1}y)$$

$$= A_{(\alpha,\beta)}(e)$$

So, $x^{-1}y \in G_{A_{(\alpha,\beta)}}$

Conversely, let $x^{-1}y \in G_{A_{(\alpha,\beta)}} \Longrightarrow A_{(\alpha,\beta)}(x^{-1}y) = A_{(\alpha,\beta)}(e)$

Let $z \in G$ be any element of G

Then, ${}_x(A_{(\alpha,\beta)})(z) = A_{(\alpha,\beta)}(x^{-1}z)$

$$= A_{(\alpha,\beta)}\{(x^{-1}y)(y^{-1}z)\}$$

$$\leq \max\{A_{(\alpha,\beta)}(x^{-1}y), A_{(\alpha,\beta)}(y^{-1}z)\}$$

$$= \max\{A_{(\alpha,\beta)}(e), A_{(\alpha,\beta)}(y^{-1}z)\}$$

$$= A_{(\alpha,\beta)}(y^{-1}z)$$

$${}_x(A_{(\alpha,\beta)})(z) = {}_y(A_{(\alpha,\beta)})(z)$$

Interchanging the role of x, y we get,

$${}_x(A_{(\alpha,\beta)})(z) = {}_y(A_{(\alpha,\beta)})(z),\ \forall\, z \in G \Longrightarrow {}_x(A_{(\alpha,\beta)}) = {}_y(A_{(\alpha,\beta)})$$

II. This follows similarly as part (I)

Proposition: 3.10

Let A be a (α, β)-AFNSG of a group G and x, y, u, v be any element in G.

If ${}_{x}\big(A_{(\alpha,\beta)}\big) = {}_{u}(A_{(\alpha,\beta)})$ and ${}_{y}\big(A_{(\alpha,\beta)}\big) = {}_{v}(A_{(\alpha,\beta)})$, then ${}_{xy}\big(A_{(\alpha.\beta)}\big) = {}_{uv}(A_{(\alpha,\beta)})$.

Proof:

Since,

${}_{x}\big(A_{(\alpha,\beta)}\big) = {}_{u}(A_{(\alpha,\beta)})$ and ${}_{y}\big(A_{(\alpha,\beta)}\big) = {}_{v}(A_{(\alpha,\beta)})$, then ${}_{xy}\big(A_{(\alpha.\beta)}\big) = {}_{uv}(A_{(\alpha,\beta)})$.

Now, $(xy)^{-1}(uv) = y^{-1}(x^{-1}u)v$

$= y^{-1}(x^{-1}u)(yy^{-1})v$

$= [y^{-1}(x^{-1}u)y](y^{-1}v) \in G_{A_{(\alpha,\beta)}}$

[As $G_{A_{(\alpha,\beta)}}$ is a normal subgroup of G]

So, $(xy)^{-1}(uv) \in G_{A_{(\alpha,\beta)}} \Longrightarrow {}_{xv}\big(A_{(\alpha,\beta)}\big) = {}_{uv}(A_{(\alpha,\beta)})$

Proposition: 3.11

The set $G/A_{(\alpha,\beta)}$ of all α-anti fuzzy cosets of (α, β)-AFNSG A of a group G, form a group under the well defined operations $\otimes$.

Proof:

It is easy to check that the identity element of $G/A_{(\alpha,\beta)}$ is ${}_{e}\big(A_{(\alpha,\beta)}\big)e$, where e is the identity element of the group G, and the inverse of an element

${}_{x}\big(A_{(\alpha,\beta)}\big)\, is\ {}_{x^{-1}}\big(A_{(\alpha,\beta)}\big)$.

Definition: 3.12

The group $G/A_{(\alpha,\beta)}$ of all (α,β)-anti fuzzy coset of the (α,β)-AFNSG A of G is Called the factor group or the quotient group of G by $A_{(\alpha,\beta)}$.

Theorem: 3.13

A natural mapping $f: G \to G/A_{(\alpha,\beta)}$, where G is a group and $G/A_{(\alpha,\beta)}$ is the set of all (α,β)-anti fuzzy cosets of the (α,β)-AFNSG A of G defined by

$f(x) = {}_x(A_{(\alpha,\beta)})$ is an onto homomorphism with $\ker f = G_{A_{(\alpha,\beta)}}$

Proof:

Let $x, y \in G$ be any element.

Then, $f(xy) = {}_{xy}\big(A_{(\alpha,\beta)}\big) = {}_{x}\big(A_{(\alpha,\beta)}\big)\,{}_{y}\big(A_{(\alpha,\beta)}\big) = f(x)f(y)$.

Therefore, f is a homomorphism.

Moreover, f is surjective (obvious)

Now, $\ker f = \{x \in G : f(x) = {}_e(A_{(\alpha,\beta)})\}$

$$= \left\{x \in G: {}_{x}\big(A_{(\alpha,\beta)}\big) = {}_{e}\big(A_{(\alpha,\beta)}\big)\right\}$$

$$= \left\{x \in G : \; e^{-1}x \in G_{A_{(\alpha,\beta)}}\right\}$$

$$= \left\{x \in G : x \in G_{A_{(\alpha,\beta)}}\right\}$$

$$= G_{A_{(\alpha,\beta)}}$$

CHAPTER-4

HOMOMORPHISM OF (α,β)-ANTI FUZZY GROUPS

Lemma: 4.1

Let $f: X \to Y$ be a mapping and A and B be two fuzzy subsets of X and Y respectively, the

(i) $f^{-1}\left(B_{(\alpha,\beta)}\right) = (f^{-1}(B))_{(\alpha,\beta)}$

(ii) $f\left(A_{(\alpha,\beta)}\right) = (f(A))_{(\alpha,\beta)}$

Proof:

(i) $$f^{-1}\left(B_{(\alpha,\beta)}\right)(x) = B_{(\alpha,\beta)}\left(f(x)\right)$$

$$= \max\{(B^{\alpha})^{c}\left(f(x)\right), B_{\beta}\left(f(x)\right)\}$$

$$= \max\{f^{-1}((B^{\alpha})^{c})(x), f^{-1}\left(B_{\beta}\right)(x)\}$$

$$= \max\{[(f^{-1}(B))^{\alpha}]^{c}(x), (f^{-1}(B))_{\beta}(x)\}$$

$$= (f^{-1}(B))_{(\alpha,\beta)}(x)$$

Hence $f^{-1}(B_{(\alpha,\beta)}) = (f^{-1}(B))_{(\alpha,\beta)}$

(ii) $$f\left(A_{(\alpha,\beta)}\right)(y) = \sup\{A_{(\alpha,\beta)}(x): f(x) = y\}$$

$$= \sup\{\max\{\left(A^{\alpha}\right)^{c}(x), A_{\beta}(x)\}: f(x) = y\}$$

$$= \max\{\sup\{(A^{\alpha})^{c}(x): f(x) = y\}, \{A_{\beta}(x): f(x) = y\}\}$$

$$= \max\{f((A^{\alpha})^{c})(y), f\left(A_{\beta}\right)(y)\}$$

$$= \max\{[(f(A))^{\alpha}]^{c}(y), (f(A))_{\beta}(y)\} = (f(A))_{(\alpha,\beta)}(y)$$

Theorem: 4.2

Let $f: G_1 \rightarrow G_2$ be a homomorphism of group G_1 into a group G_2. Let B be (α, β)-AFSG of group G_2. Then $f^{-1}(B)$ is (α, β)-AFSG of group G_1.

Proof:

Let B be an (α, β)-AFSG of group G_2.

Let $x_1, x_2 \in G_1$ be any element.

Then $(f^{-1}(B))_{(\alpha,\beta)}(x_1 x_2) = f^{-1}(B_{(\alpha,\beta)})(x_1 x_2)$

$$= B_{(\alpha,\beta)}((f(x_1 x_2))$$

$$= B_{(\alpha,\beta)}(f(x_1) f(x_2))$$

$$\leq \max\{B_{(\alpha,\beta)}(f(x_1)), B_{(\alpha,\beta)}(f(x_2))\}$$

$$= \max\{f^{-1}(B_{(\alpha,\beta)})(x_1), f^{-1}(B_{(\alpha,\beta)}(x_2)\} \quad \text{[by lemma 4.1]}$$

Thus, $(f^{-1}(B))_{(\alpha,\beta)}(x_1 x_2) \leq \max\{(f^{-1}(B))_{(\alpha,\beta)}(x_1), (f^{-1}(B))_{(\alpha,\beta)}(x_2)\}$

Also, $(f^{-1}(B))_{(\alpha,\beta)}(x^{-1}) = f^{-1}(B_{(\alpha,\beta)})(x^{-1})$

$$= B_{(\alpha,\beta)}(f(x^{-1}))$$

$$= B_{(\alpha,\beta)}(f(x)) = f^{-1}(B_{(\alpha,\beta)})(x)$$

Thus, $(f^{-1}(B))_{(\alpha,\beta)}(x^{-1}) = f^{-1}(B_{(\alpha,\beta)})(x)$

Hence $f^{-1}(B)$ is a (α, β)-AFSG of G_1.

Theorem: 4.3

Let $f : G_1 \to G_2$ be a homomorphism of a group G_1 into a group G_2. Let B be an (α, β)-AFNSG of of group G_2, then $f^{-1}(B)$ is an (α, β)-AFNSG of group G_1.

Proof:

Let B be an (α, β)-AFNSG of of group G_2.

Let $x_1, x_2 \in G_1$ be any element.

Then, $(f^{-1}(B))_{(\alpha,\beta)}(x_1x_2) = f^{-1}\big(B_{(\alpha,\beta)}\big)(x_1x_2)$

$$= B_{(\alpha,\beta)}\big(f(x_1)f(x_2)\big)$$

$$= B_{(\alpha,\beta)}(f(x_2)f(x_2))$$

$$= B_{(\alpha,\beta)}(f(x_2x_1) = (f^{-1}(B))_{(\alpha,\beta)}(x_2x_1)$$

Thus, $(f^{-1}(B))_{(\alpha,\beta)}(x_1x_2) = (f^{-1}(B))_{(\alpha,\beta)}(x_2x_1)$

Hence $f^{-1}(B)$ is an (α, β)-AFNSG of G_1.

Theorem: 4.4

Let $f : G_1 \to G_2$ be a bijective homomorphism of group G_1 onto a group G_2. Let A be an (α, β)-AFSG of group G_1. Then $f(A)$ is an (α, β)-AFSG of group G_2.

Proof:

Let A be an (α, β)-AFSG of group G_1.

Let $y_1,\ y_2 \in G_2$ be any element.

Then there exists unique element $x_1,\ x_2 \in G_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$

$(f(A))_{(\alpha,\beta)}(y_1y_2) = (f\big(A)_{(\alpha,\beta)}\big)(y_1y_2) = \sup\{A_{(\alpha,\beta)}(x_1x_2): f(x_1x_2) = y_1y_2\}$

$$\leq \sup\{\max\{A_{(\alpha,\beta)}(x_1), A_{(\alpha,\beta)}(x_2)\},$$

$$where\ f(x_1) = y_1\ and\ f(x_2) = y_2\}$$

$$= \max\{\sup\{A_{(\alpha,\beta)}(x_1): f(x_1) = y_1\}, \sup\{A_{(\alpha,\beta)}(x_2): f(x_2) = y_2\}\}$$

$$= \max\{\left(f\left(A_{(\alpha,\beta)}\right)\right)(y_1), \left(f\left(A_{(\alpha,\beta)}\right)\right)(y_2)\}$$

$$= \max\{(f(A))_{(\alpha,\beta)}(y_1), (f(A))_{(\alpha,\beta)}(y_2)\} \qquad \text{[by lemma 4.1]}$$

Thus $(f(A))_{(\alpha,\beta)}(y_1y_2) \leq \max\{(f(A))_{(\alpha,\beta)}(y_1), (f(A))_{(\alpha,\beta)}(y_2)\}$

Also, $(f(A))_{(\alpha,\beta)}(y^{-1}) = f(A_{(\alpha,\beta)}(y^{-1}) = \sup\{A_{(\alpha,\beta)}(x^{-1}): f(x^{-1}) = y^{-1}\}$

$$= \sup\{A_{(\alpha,\beta)}(x): f(x) = y\}$$

$$= (f\big(A_{(\alpha,\beta)}\big))(y)$$

Hence $f(A)$ is an (α,β)-AFSG of G_2.

CHAPTER-5

o-anti fuzzy subgroups

5.1 o-anti fuzzy subgroups

Definition: 5.1.1

Let A be a fuzzy subset of a set X and $\delta \in [0,1]$. The fuzzy set A^o of X is called the ***o*-Anti fuzzy subset of X** (w.r.t. fuzzy set A) and is defined as

$$A^o(x) = s_b(A(x), 1-\delta), for\ all\ x \in X.$$

Theorem: 5.1.2

Let A and B be any two fuzzy subsets of X. Then $(A \cup B)^o = A^o \cup B^o$.

Proof:

By definition (5.1.1), we have

$$(A \cup B)^o(x) = s_b((A \cup B)(x), 1-\delta)$$

$$= s_b(\max\big(A(x), B(x)\big), 1-\delta)$$

$$= \max(s_b\big(A(x), B(x)\big), 1-\delta)$$

$$= \max(s_b(A(x), 1-\delta), s_b(B(x), 1-\delta)$$

$$= \max(A^o(x), B^o(x))$$

Hence implies that $A \cup B)^o = A^o \cup B^o$.

Definition: 5.1.3

Let A be a fuzzy subset of a group G and $\delta \in [0,1]$. Then A is called o-anti fuzzy subgroup of G. In other words A is o-anti fuzzy subgroup if A^o admits the following

1. $A^o(xy) \leq \max\{A^o(x), A^o(y)\}$

2. $A^o(x^{-1}) \leq A^o(x), for\ all\ x, y \in G.$

Proposition: 5.1.4

If $A: G \to [0,1]$ is an o-anti fuzzy subgroup of a group G, then

1. $A^o(x) \geq A^o(e)$, for all $x \in G$, where e is the identity element of G.

2. $A^o(xy^{-1}) = A^o(e)$ which implies that $A^o(x) = A^o(y)$, for all $x, y \in G$.

Proof:

1. $A^o(e) = A^o(xx^{-1})$

$\leq \max(A^o(x), A^o(x^{-1}))$

$= \max(A^o(x), A^o(x))$

$= A^o(x).$

Hence $A^o(e) \leq A^o(x)$, for all $x \in G$.

2. $A^o(x) = A^o(xy^{-1}y)$

$\leq \max(A^o(xy^{-1}), A^o(y))$

$= \max(A^o(e), A^o(y))$

$= A^o(y)$

Hence $A^o(x) \leq A^o(y)$, for all $x \in G$. (1)

Similarly, $A^o(y) = A^o(yx^{-1}x)$

$$\leq \max\big(A^o(yx^{-1}), A^o(x)\big)$$

$$= \max\big(A^o(e), A^o(x)\big)$$

$$= A^o(x)$$

Hence $A^o(y) \leq (x)$, for all $x \in G$. (2)

From (1) and (2),

$$A^o(x) = A^o(y), \text{ for all } x, y \in G.$$

Proposition: 5.1.5

Every anti fuzzy subgroup of a group G is an o-anti fuzzy subgroup of G.

Proof:

Let A be anti fuzzy subgroup of a group G.

Let x, y be any two elements in G.

Consider,

$$A^o(xy) = s_b(A(xy), 1-\delta)$$

$$\leq s_b\big(\max\big(A(x), A(y)\big), 1-\delta\big)$$

$$= \max\big(s_b\big(A(x), A(y)\big), 1-\delta\big)$$

$$= \max(s_b(A(x), 1-\delta), s_b(A(y), 1-\delta)$$

$$A^o(xy) \leq \max\big(A^o(x), A^o(y)\big)$$

Further, $A^o(x^{-1}) = s_b(A(x^{-1}), 1 - \delta)$

$$= s_b(A(x), 1 - \delta) = A^o(x).$$

Consequently, A is o-anti fuzzy subgroup of G.

Remark: 5.1.6

The converse of above proposition need not to be true.

Example: 5.1.7

Let $G = \{e, a, b, ab\}$, where $a^2 = b^2 = e$ and $ab = ba$ be the klein four group.

Let the fuzzy set A of G be defined as

$A = \{< e, 0.1 >, < a, 0.3 >, < b, 0.4 >, < ab, 0.5 >\}$

Take $\delta = 0.05$ then

$$A^o(x) = s_b(A(x), 1 - \delta)$$

$$= \min(A(x) + 1 - \delta, 1)$$

$$= \min(A(x) + 1 - 0.05, 1)$$

$A^o(x) = 1$, for all $x \in G$.

This implies that $A^o(xy) \leq \max(A^o(x), A^o(y))$

Further, we have $a^{-1} = a, b^{-1} = b \; and \; (ab)^{-1} = ab$.

Hence we have $A^o(x^{-1}) = A^o(x),$ for all $x \in G$.

This implies that A is o-anti fuzzy subgroup of G.

Clearly A is not anti fuzzy subgroup of G.

Proposition: 5.1.7

The union of two o-anti fuzzy subgroups of a group G is also o-anti fuzzy subgroup.

Proof:

Let A and B be two o-anti fuzzy subgroups of a group G.

Consider, for all $x, y \in G$,

$$
\begin{aligned}
(A \cup B)^o(xy) &= (A^o \cup B^o)(xy) \\
&= \max\big(A^o(xy), B^o(xy)\big) \\
&\leq \max(\max\big(A^o(x), A^o(y)\big), \max\big(B^o(x), B^o(y)\big)) \\
&= \max(\max\big(A^o(x), B^o(x)\big), \max\big(A^o(y), B^o(y)\big)) \\
&= \max((A \cup B)^o(x), (A \cup B)^o(y))
\end{aligned}
$$

Thus $(A \cup\cup B)^o(xy) \leq \max((A \cup B)^o(x), (A \cup B)^o(y))$

Moreover,

$$
\begin{aligned}
(A \cup B)^o(x^{-1}) &= (A^o \cup B^o)(x^{-1}) \\
&= \max(A^o(x^{-1}), B^o(x^{-1})) \\
&= \max(A^o(x), B^o(x))
\end{aligned}
$$

$$(A \cup B)^o(x^{-1}) = (A \cup B)^o(x)$$

Consequently, $(A \cup B)$ is o-anti fuzzy subgroup of G.

Corollary: 5.1.8

The union of any finite number of o-anti fuzzy subgroups of a group G is also o-anti fuzzy subgroup of G.

Remark: 5.1.9

The intersection of two o-anti fuzzy subgroups of a group G need not be o-anti fuzzy subgroup of G.

Example: 5.1.10

Consider the group of integers Z. Define the two fuzzy subsets A and B of Z as follows

$$A(x) = \begin{cases} 0.5, & if\ x = 3Z \\ 1, & otherwise \end{cases}$$

and

$$B(x) = \begin{cases} 0.8, & if\ x = 2Z \\ 0.83, & otherwise \end{cases}$$

It can be easily verified that A and B are o-anti fuzzy subgroups of Z.

Now, $(A \cap B)(x) = \min(A(x), B(x))$

Therefore, $(A \cap B)(x) = \begin{cases} 0.5, & if\ x \in 3Z \\ 0.8, & if\ x \in 2Z - 3Z \\ 0.83 & otherwise \end{cases}$

Take $x = 9\ and\ y = 4$

Then, $(A \cap B)(x) = 0.5\ and\ (A \cap B)(y) = 0.8$

But $(A \cap B)(x - y) = (A \cap B)(9 - 4)$

$$= (A \cap B)(5)$$

$$= 0.83$$

And $\max((A \cap B)(x), (A \cap B)(y) = \max(0.5, 0.8)$

Clearly, $(A \cap B)(x - y) > \max((A \cap B)(y))$

Consequently, $A \cap B$ is not o-anti fuzzy subgroup of G.

Hence, the intersection of two o-anti fuzzy subgroups of G need not be o-anti fuzzy subgroup of G.

5.2 *o*-anti fuzzy normal subgroups

Definition: 5.2.1

Let Abe an o-anti fuzzy subgroup of a group G and $\delta \in [0,1]$. The **right *o*-anti fuzzy coset** of A in G is denoted by $A^o x$ and its defined as

$$A^o x(g) = s_b(A(gx^{-1}), 1 - \delta), \text{ for all } x, y \in G.$$

Similarly, we define **the *o*-anti fuzzy left coset** xA^o of G as follows

$$xA^o(g) = s_b(A(x^{-1}g), 1 - \delta), \text{ for all } x, y \in G.$$

Definition: 5.2.2

Let A be an o-anti fuzzy subgroup of a group G and $\delta \in [0,1]$. Then A is called o-anti fuzzy normal subgroup of G if and only if $xA^o = A^o x$, for all $x \in G$.

Proposition: 5.2.3

Every anti fuzzy normal subgroup of a group G is an o-anti fuzzy normal subgroup of G.

Proof:

Let A be anti fuzzy normal subgroup of a group G.

Then for any $x \in G$, we have $xA = Ax$

$$\Longrightarrow xA(g) = Ax(g),\ for\ all\ x \in G.$$

Then we have,

$$A(x^{-1}g) = A(gx^{-1}),$$

$$\Longrightarrow s_b(A(x^{-1}g), 1-\delta)$$

$$= s_b(A(gx^{-1}), 1-\delta)$$

Hence, $xA^o = A^o x, for\ all\ x \in G.$

Consequently, A is o-anti fuzzy normal subgroup of G.

Remark: 5.2.4

The converse of the above proposition need not to be true.

Example: 5.2.5

Consider the dihedral group of degree 3 with finite presentation

$G = D_3 =< a, b: a^3 = b^2 = e, ba = a^2 b >$. Define the anti fuzzy subgroup of D_3 by

$$A(x) = \begin{cases} 0.1, & if\ x \in < b > \\ 0.2, & otherwise \end{cases}$$

Take $\delta = 0.6$, we have

$$xA^o = xA^o(g) = s_b(A(x^{-1}g), 1-\delta)$$

$$= s_b(A(x^{-1}g), 0.4)$$

$= s_b(A(gx^{-1}), 0.4)$

$= s_b(A(gx^{-1}), 1-\delta)$

$= A^o x(g) \ = A^o x$

This shows that A is o-anti fuzzy normal subgroup of G.

$A(a^2(ab) = A(a^3 b) = A(b) = 0.1$

$A\big((ab)a^2\big) = A(a(ba)a) = A(a(a^2 b)a) = A(a^3 ba) = A(ba) = 0.2$

This implies that A is not anti fuzzy normal subgroup of G.

Proposition: 5.2.6

Let A be an o-anti fuzzy normal subgroup of a group G. Then $A^o(y^{-1}xy) = A^o(x)$ or equivalently, $A^o(xy) = A^o(yx)$, hold for all $x, y \in G$.

Proof:

Since A be an o-anti fuzzy normal subgroup of a group G.

Therefore, $xA^o = A^o x$, holds for all $x \in G$.

This implies that

$$xA^o(y^{-1}) = A^o x(y^{-1}), y \in G. \quad (1)$$

By definition (5.2.1),

(1) becomes, $s_b(A(x^{-1}y^{-1}), 1-\delta) = s_b(A(y^{-1}x^{-1}), 1-\delta)$

Which implies that, $A^o((yx)^{-1}) = A^o((xy)^{-1})$.

Consequently, $A^o(xy) = A^o(yx)$.

Definition: 5.2.7

Let A be an o-anti fuzzy normal subgroup of a group G. we define a set

$G_{A^o} = \{x \in G: A^o(x) = A^o(e)\}$, where e the identity element of G.

The following result illustrate that the set G_{A^o} is infect a normal subgroup of G.

Proposition: 5.2.8

Let A be an o-anti fuzzy normal subgroup of a group G. Then G_{A^o} is a normal subgroup of G.

Proof:

Obviously, $G_{A^o} \neq \emptyset, for\ e \in G_{A^o}$

Let $x, y \in G_{A^o}$ be any element.

Then we have, $A^o(xy^{-1}) \leq \max(A^o(x), A^o(y^{-1}))$

$$= \max(A^o(x), A^o(y))$$

$$= \max(A^o(e), A^o(e))$$

$$= A^o(e)$$

This implies that

$$A^o(xy^{-1}) \leq A^o(e), but\ A^o(xy^{-1}) \geq A^o(e)$$

Therefore, $A^o(xy^{-1}) = A^o(e)$, which implies that $xy^{-1} \in G_{A^o}$

Hence G_{A^o} is a subgroup of G.

Further, let $x \in G_{A^o}$ and $y \in G$.

We have $A^o(y^{-1}xy) = A^o(x) = A^o(e)$.

This implies that $y^{-1}xy \in G_{A^o}$.

Consequently, G_{A^o} is normal subgroup of G.

Proposition: 5.2.9

Let A be an o-anti fuzzy normal subgroup of G, then

1. $xA^o = yA^o$ if and only if $x^{-1}y \in G_{A^o}$.

2. $A^o x = A^o y$ if and only if $xy^{-1} \in G_{A^o}$.

Proof:

1. Suppose that $xA^o = yA^o$, for $x, y \in G$.

By definition (5.15), the above relation becomes,

$$A^o(x^{-1}y) = s_b(A(x^{-1}y), 1-\delta)$$

$$= (xA^o)(y)$$

$$= (yA^o)(y)$$

$$= s_b(A(y^{-1}y), 1-\delta)$$

$$= s_b(A(e), 1-\delta) = A^o(e).$$

This implies that $x^{-1}y \in G_{A^o}$.

Conversely, let $x^{-1}y \in G_{A^o}$, which implies that $A^o(x^{-1}y) = A^o(e)$.

For any element $z \in G_{A^o}$.

$$(xA^o)(z) = s_b(A(x^{-1}z), 1-\delta)$$

$= A^o(x^{-1}z)$

$= A^o((x^{-1}y)(y^{-1}z))$

$\leq \max(A^o((x^{-1}y), A^o(y^{-1}z))$

$= \max(A^o(e), A^o(y^{-1}z)) \quad = A^o(y^{-1}z)$

$(xA^o)(z) = (yA^o)(z)$

Interchanging the roles of $x\ and\ y$, we get $(xA^o)(z) = (yA^o)(z), for\ all$

$z \in G.$

Consequently, $(xA^o) = (yA^o)$.

2. one can prove this part analogous to (1).

Proposition: 5.2.10

Let A be ab o-anti fuzzy normal subgroup of a group G and x, y, u, v be any element in G. If $xA^o = uA^o$ and $yA^o = vA^o$ then $xyA^o = uvA^{o.}$

Proof:

Given that $xA^o = uA^o$ and $yA^o = vA^o$,

Which implies that $x^{-1}u$ and $y^{-1}v \in G_{A^o}$.

Consider, $(xy^{-1})uv = y^{-1}(x^{-1}u)(yy^{-1})v \in G_{A^o}$.

$= [y^{-1}(x^{-1}u)y](y^{-1}v) \in G_{A^o}$.

This implies that $(xy)^{-1}uv \in G_{A^o}$.

Consequently, $xyA^o = uvA^o$.

Definition: 5.2.11

Let A be an o-anti fuzzy normal subgroup of a group G. The set of all o-anti fuzzy cosets of A denoted by G/A^o forms a group under the binary operation * defined as follow.

Let $xA^o, yA^o \in G/A^o$,$xA^o * yA^o = (x * y)A^o, x, y \in G.$

This group is called the factor group or the quotient group of G with respect to o-anti fuzzy normal subgroup A^o.

Theorem: 5.2.12

The set G/A^o defined in proposition (5.2.9) forms a group under the above stated binary operation *.

Proof:

Let $A^o x_1 = A^o x_2 \ and \ A^o y_1 = A^o y_2, for \ x_1, x_2, y_1, y_2 \in G.$

Let $g \in G$ be any element of G.

$$[A^o x_1 * A^o y_1](g) = (A^o x_1 y_1)(g)$$

$$= s_b(A(g(x_1 y_1)^{-1})1 - \delta)$$

$$= s_b(A(g{y_1}^{-1}{x_1}^{-1}), 1 - \delta)$$

$$= s_b(A(g{y_1}^{-1}){x_1}^{-1}, 1 - \delta)$$

$$= A^o x_1(g{y_1}^{-1})$$

$$= A^o x_2(g{y_1}^{-1})$$

$$= s_b(A(g{y_1}^{-1}){x_2}^{-1}, 1 - \delta)$$

$$= s_b(A(x_2{}^{-1}g)y_1{}^{-1}, 1-\delta)$$

$$= A^o y_1(x_2{}^{-1}g)$$

$$= A^o y_2(x_2{}^{-1}g)$$

$$= s_b(A(x_2{}^{-1}g)y_2{}^{-1}, 1-\delta)$$

$$= s_b(A(y_2{}^{-1}x_2{}^{-1})g, 1-\delta)$$

$$= s_b(A(x_2y_2)^{-1}g, 1-\delta)$$

$$= s_b(Ag(x_2y_2)^{-1}, 1-\delta)$$

$$= (A^o x_2 y_2)(g).$$

This implies that * is well defined.

Obviously, the set G/A^o admits closure and associative properties with respect to the binary operation *.

Moreover, $A^o * xA^o = eA^o * xA^o = (e * x)A^o = xA^o$, which implies that A^o is identity of G/A^o.

It is easy to note that inverse of each element of G/A^o exist as if for $xA^o \in G/A^o$, there exist

$x^{-1}A^o \in G/A^o$ such that $(x^{-1}A^o) * (xA^o) = (x^{-1} * x)A^o = A^o$.

Consequently, (G/A^o) is a group under *.

Theorem: 5.2.13

Let A^o be an o-anti fuzzy normal subgroup of a group G. Then there exists a natural epimorphism between G and G/A^o which may be defined as $x \to A^o x, x \in G$, where G_{A^o} is kernel of this homomorphism.

Proof:

f is homomorphism as if for $x, y \in G$.

We have, $f(xy) = A^o x A^o y = f(x)f(y)$.

Obviously f is surjective as well.

Consequently, f is an epimorphism from G to G/A^o.

Further, $kerf = \{x \in G : f(x) = A^o e\}$

$$= \{x \in G : A^o x = A^o e\}$$

$$= \{x \in G : xe^{-1} \in G_{A^o}\}$$

$$= G_{A^o}.$$

Theorem: 5.2.14

Let A^o be an o-anti fuzzy normal subgroup of a group G. Then show that $G/A^o \cong G/G_{A^o}$.

Proof:

By theorem (5.2.12), G/G_{A^o} is well defined.

Define a map $f: G/A^o \to G/G_{A^o}$

$$f(xA^o) = xG_{A^o}, x \in G$$

f is well defined because if $xA^o = yA^o$, which implies that $xG_{A^o} = yG_{A^o}$.

This implies that $f(xA^o) = f(yA^o)$

f is injective as if $f(xA^o) = f(yA^o)$, which implies that $xG_{A^o} = yG_{A^o}$.

Hence, $xA^o = yA^o$.

f is surjective as for each $xG_{A^o} \in G/G_{A^o}$, there exist $xA^o \in G/A^o$ such that

$f(xA^o) = xG_{A^o}$

f is homomorphism as for each $xA^o, yA^o \in G/A^o$

$$
\begin{aligned}
f(xA^o yA^o) &= f((xy)A^o) \\
&= xyG_{A^o} \\
&= xG_{A^o} yG_{A^o} \\
&= f(xA^o)f(yA^o).
\end{aligned}
$$

Consequently, there is an isomorphism between G/A^o and G/G_{A^o}.

5.3 HOMOMORPHISM OF A *O*-FUZZY SUBGROUP

Theorem: 5.3.1

Let $f: G_1 \to G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and B be an o-fuzzy subgroup of group G_2. Then $f^{-1}(B)$ is an o-fuzzy subgroup G_1.

Proof:

Given that B is an o-anti fuzzy subgroup of group G_2.

Let $x_1, x_2 \in G_1$ be any element.

To prove:

1. $(f^{-1}(B))^{o}(x_1x_2) \leq \max\{(f^{-1}(B))^{o}(x_1), (f^{-1}(B))^{o}(x_2)\}$

2. $(f^{-1}(B))^{o} = f^{-1}(B^{o})(x)$

Then,

$$\left(f^{-1}(B)\right)^{o}(x_1x_2) = f^{-1}(B^{o})(x_1x_2)$$

$$= B^{o}(f(x_1x_2))$$

$$= B^{o}(f(x_1)f(x_2)$$

$$\leq \max\{B^{o}\left(f(x_1)\right), B^{o}\left(f(x_2)\right)\}$$

$$= \max\{f^{-1}(B^{o})(x_1), f^{-1}(B^{o})(x_2)\}$$

$$= \max\{(f^{-1}(B))^{o}(x_1), (f^{-1}(B))^{o}(x_2)\}$$

Thus,

$$(f^{-1}(B))^{o}(x_1x_2) \leq \max\{(f^{-1}(B))^{o}(x_1), (f^{-1}(B))^{o}(x_2)\}$$

Also,

$$(f^{-1}(B))^{o} = f^{-1}(B^{o})(x^{-1})$$

$$= B^{o}(f(x^{-1}))$$

$$= B^{o}(f(x)^{-1})$$

$$= B^{o}(f(x))$$

$$(f^{-1}(B))^{o} = f^{-1}(B^{o})(x)$$

Consequently, $f^{-1}(B)$ is an *o*-anti fuzzy subgroup of group G_1.

Theorem: 5.3.2

Let $f: G_1 \rightarrow G_2$ be an isomorphism form a group G_1 to group G_2 and B be an o-anti fuzzy normal subgroup of a group G_2. Then $f^{-1}(B)$ is an o-anti fuzzy normal subgroup of group G_1.

Proof:

Given that B is an o-anti fuzzy normal subgroup of group G_2.

Let $x_1, x_2 \in G_1$ be any element.

Then,

$$(f^{-1}(B))^o(x_1x_2) = f^{-1}(B^o)(x_1x_2)$$

$$= B^o(f(x_1x_2))$$

$$= B^o(f(x_1)f(x_2))$$

$$= B^o(f(x_2x_1))$$

$$(f^{-1}(B))^o(x_1x_2) = f^{-1}(B^o)((f^{-1}(B))^o(x_2x_1)$$

Consequently, $f^{-1}(B)$ is o-anti fuzzy normal subgroup of group G_1.

Theorem: 5.3.3

Let $f: G_1 \rightarrow G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and A be an o-anti fuzzy subgroup of group G_1. Then $f(A)$ is an o-anti fuzzy subgroup of group G_2.

Proof:

Given that A is an o-anti fuzzy subgroup of group G_1.

Let $y_1, y_2 \in G_2$ be any element.

Then there exist unique element $x_1, x_2 \in G_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Consider,

$$(f(A))^o(y_1y_2) = s_b(f(A)(y_1y_2), 1-\delta)$$

$$= s_b(f(A)f(x_1)f(x_2), 1-\delta)$$

$$= s_b(A(x_1x_2), 1-\delta) \qquad [\because f(A)f(x) = A(x)]$$

$$= A^o(x_1x_2)$$

$$\leq \max(A^o(x_1), A^o(x_2), \text{ for all } x_1, x_2 \in G_1$$

$$\leq \max(\min\{A^o(x_1): f(x_1) = y_1\}, \min\{A^o(x_2): f(x_2) = y_2\})$$

$$= \max(f(A^o)(y_1), f(A^o)(y_2))$$

$$= \max((f(A))^o(y_1), (f(A))^o(y_2))$$

Further,

$$(f(A))^o(y^{-1}) = f(A^o)(y^{-1})$$

$$= \min\{A^o(x^{-1}): f(x^{-1}) = y^{-1}\}$$

$$= \min\{A^o(x): f(x) = y\}$$

$$= (f(A))^o(y)$$

Consequently, $f(A)$ is o-anti fuzzy subgroup of G_2.

Theorem: 5.3.4

Let $f: G_1 \rightarrow G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and A be an o-anti fuzzy normal subgroup of G_1. Then $f(A)$ is an o-anti fuzzy normal subgroup of group G_2.

Proof:

It is sufficient to show that $f(A)^o$ is anti fuzzy normal in G_2.

Given that A is o-anti fuzzy normal subgroup of group G_1.

Let $y_1, y_2 \in G_2$ be any element.

Then there exists unique elements $x_1, x_2 \in G_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Consider,

$$
\begin{aligned}
(f(A))^o(y_1y_2) &= s_b(f(A)(y_1y_2), 1-\delta) \\
&= s_b(f(A)f(x_1)f(x_2), 1-\delta) \\
&= s_b(f(A)f(x_1x_2), 1-\delta) \\
&= s_b(A(x_1x_2), 1-\delta) \\
&= A^o(x_1x_2) \\
&= A^o(x_2x_1) \\
&= s_b(A(x_2x_1), 1-\delta) \\
&= s_b(f(A)f(x_2x_1), 1-\delta) \\
&= s_b(f(A)f(x_2)f(x_1), 1-\delta) \\
&= s_b(f(A)(y_2y_1), 1-\delta)
\end{aligned}
$$

$$= (f(A))^o(y_2y_1).$$

Consequently, $f(A)$ is a o-anti fuzzy normal subgroup of G_2.

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Submitted by

K. SUBASHINI

Reg.No: 19SPMT24

Under the guidance of

Ms. M. KANAGA M.Sc., B.Ed., SET.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

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CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON CONVERGENCE OF FUZZY FILTERS AND FUZZY NETS** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **K. SUBASHINI (Reg. No: 19SPMT24)**

M. Kanagu

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON CONVERGENCE OF FUZZY FILTERS AND FUZZY NETS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. M. Kanaga M.Sc., B.Ed., SET.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

K. Subashini

Signature of the Student

Station: Thoothukudi

Date: 10. 04. 2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

INTRODUCTION

In a topological space filter is an important tool to study many properties. The closure of a set A can be characterized using convergent filters. The continuity of a function from one topological space to another can be characterized using convergent filters. In the year 1965 Lotfi A. Zadeh [14] introduced the concept of fuzzy sets and fuzzy logic. In the year 1968 C. L. Chang [3] introduced Fuzzy topological spaces. In the year 2014 We [3] introduced fuzzy sequences in a metric space. Also We [3] introduced fuzzy nets in topological spaces.

In 1994 H. Maki and K. Balachandran [6] introduced α generalized closed set. In 1996 H. Maki and others [6] introduced generalized pre closed set. In the year 1997 A. Csaszar [5] introduced the concept of generalized open sets. S. Palaniammal [11] introduced tri topological space. She introduced a new tool named generalized filter and defined convergence of generalized filters.

In the year 2009, R. Shen [12] studied connectedness in generalized topological spaces. In the same year R.X. Shen [13] made some remarks on product of generalized topological spaces. In 2010 R. Baskaran, M. Murugalingam and D. Sivaraj [2] introduced some new concepts in generalized topological spaces.

After the introduction of generalized topology by A. Csaszar [4] in the need for generalizing the limit concept in generalized topology was felt and hence R. Baskaran, M. Murugalingam, D. Sivaraj [2] introduced generalized nets in generalized topological spaces and studied the convergence of generalized nets. R. Baskaran and

M. Murugalingam [2] modified the concept of connectedness in generalized topological space.

CHAPTER 1

PRELIMINARIES

Definition: 1.1

Let X be a non empty set and let τ be a collection of subsets of X then

1. $\Phi \in \tau$ and $X \in \tau$

2. The intersection of any two sets in τ belong to τ

3. The union of any number of sets in τ belong to τ

the τ is called a topology on X. The pair (X, τ) is called a topological space.

Definition: 1.2

The collection consisting of only the set X and Φ is the topology on X which is known as **indiscrete topology** (or) trivial topology.

Definition: 1.3

Let X be a non empty set. $F \subset P(X)$ is called a **filter** on X if

1. $\Phi \notin F$

2. $F_1, F_2 \in F \Rightarrow F_1 \cap F_2$

3. $\boldsymbol{F} \in F$ and $\boldsymbol{F} \subset G \Rightarrow G \in F$

Definition: 1.4

Let (X, T) be a topological space and let $x \in X$. N_x is a filter on X. N_x is called the **neighbourhood filter** on X.

Definition: 1.5

Let X be a non empty set. Let $a \in X$. $F = \{A \subset X / a \in A\}$, F is a filter. This filter is called the **principal filter** at a.

Definition: 1.6

Let X be a non empty set. Let $F = \{X\}$. Then F is a filter on X. This F is called the **Indiscrete filter** on X.

Definition: 1.7

Let X be a non empty set and $S \subset X$. $F = \{A / S \subset A\}$. This filter is called **principal filter** at S.

Definition: 1.8

Let (X, T) be a topological space and F be a filter on X. F is said to converge to x, if $N_x \subset F$. X is called a **limit** of F.

Definition: 1.9

Let X be a non empty set. $A \subset P(X)$ is called a **generalized filter** if

1. $\Phi \notin$ **A**

2. $A \in \boldsymbol{A}$ and $B \subset A \Rightarrow B \in \boldsymbol{A}$.

Definition: 1.10

Let A be a generalized filter in a generalized topological space X. A is said to converge to an element x X if every open set containing x belongs to A. we write $A \rightarrow a$, a is called a limit of A.

Definition: 1.11

Let X be a non empty set and let A be a generalized filter on $X. S \subset P(X)$ is called a **bases** for the generalized filter A if $S \subset A$ and every element of A is a superset of some elements of S. A is called the generalized filter generated by S.

Definition: 1.12

Let X be a generalized topological space. Let S be the collection of all open sets containing a fixed point x. The generalized filter generated by S is called the **neighbourhood of filter of $\boldsymbol{x}$** and it is denoted by $N(X)$.

Definition: 1.13

Let F_1 and F_2 be fuzzy filter on X. F_2 is said to be **finer** than F_1 if $F_1(A) \leq F_2(A)$ for all $A \in P(X)$.

Definition: 1.14

Let (X, T) be a topological space. Let $\boldsymbol{A}$ be a filter on X. Let $a \in X. \boldsymbol{A}$ is called a fuzzy neighbourhood filter at a if $\boldsymbol{A}(A) = 1$ if A is an open set containing a.

Definition: 1.15

Let (X, T) be a topological space Let $\boldsymbol{A}$ be a filter on X. Let $a \in X$. $\boldsymbol{A}$ is called a fuzzy neighbourhood filter at a if $\boldsymbol{A}(A) \geq \alpha$ if A is an open set containing a.

Definition: 1.16

Let X be a non empty set. A function $A: X \to [0,1]$ is called a **fuzzy set** on X.

Definition: 1.17

Let A and B be fuzzy sets on X.

$A \cup B: X \to [0,1]$ is defined as $A \cup B(x) = \max\{A(x), B(x)\}$

$A \cap B: X \to [0,1]$ is defined as $A \cap B(x) = \min\{A(x), B(x)\}$

Definition: 1.18

A generalized filter base S is said to converge to a point X if the generalized filter generated by S converges to x. we write S → x.

Definition: 1.19

Let X be a non empty set. Let $a \in X$. Define $\boldsymbol{A}: P(X) \to [0,1]$ as $\boldsymbol{A}(A) = 1$ if $a \in A$ and $\boldsymbol{A}(A) = 1$ if $a \notin A$. This $\boldsymbol{A}$ is called the fuzzy principal filter at a.

Definition: 1.20

Let X be a non empty set. A fuzzy set A on $N \times X$ is called a **fuzzy sequence** in X. That is., $A: N \times X \to [0,1]$ is called a fuzzy sequence in X

Definition: 1.21

Let X be an infinite set.$F = \{A \subset X \,/\, A^c$ is finite$\}$. A is the complement of a finite set and A is called a cofinite set. F is the collection of all cofinite subsets of X.

Definition: 1.22

A topological space X is said to be Hausdroff space if for each distinct pair of $x \neq y$ in X, there exists neighbourhoods U and V of X and Y respectively such that $U \cap V = \emptyset$

CHAPTER 2

FUZZY FILTERS

2.1 Fuzzy Filters

Definition: 2.1.1

First we recall the concept of filter in a topological space. Let X be a non empty set $F \subset P(X)$ is called a filter (Crisp filter) on X if

1. $\Phi \notin F$
2. F is closed under finite intersection.

   (i.e) $A, B \in F \Rightarrow A \cap B \in F$.
3. $B \in F$ and $B \subset A \Rightarrow A \in F$.

Now we define Fuzzy Filter.

Definition: 2.1.2

Let X be a non empty set. A fuzzy set f on $P(X)$ is called a fuzzy filter $f: P(X) \to [0,1]$ if

1. $f(\Phi) = 0$
2. $f(A \cap B) \geq \min\{f(A), f(B)\}$
3. $B \subset A \Rightarrow f(B) \leq f(A)$

Example: 2.1.3

Let $X = \{1,2,3\}$. Define $f: P(X) \to [0,1]$ as

$f(\Phi) = 0$, $f(\{1\}) = 1$, $f(\{2\}) = 0$, $f(\{3\}) = 0$, $f(\{1,2\}) = 1$, $f(\{1,3\}) = 5$,

$f(\{2,3\}) = 0$, $f(X) = 1$.

f is a fuzzy filter on X.

Theorem: 2.1.4

Let f be a fuzzy filter on X. Then $f(A \cap B) = \min\{f(A), f(B)\}$

Proof:

Let f be a fuzzy filter.

Then $f(A \cap B) \geq \min\{f(A), f(B)\}$ .... (1)

Since $(A \cap B) \subset A$,

We have $f(A \cap B) \leq f(A)$.

Also $(A \cap B) \Rightarrow f(A \cap B) \leq f(B)$.

Now $f(A \cap B) \leq f(A)$ and $f(A \cap B) \leq f(B)$

Hence $f(A \cap B) \leq \min\{f(A), f(B)\}$ ……………..(2)

From (1) and (2) we have

$f(A \cap B) = \min\{f(A), f(B)\}$.

Theorem: 2.1.5

Every crisp filter is a fuzzy filter.

Proof:

Let F be a crisp filter on X.

Then $F \subset P(X)$ where

1. $\Phi \notin F$

2. $A, B \in F \Rightarrow A \cap B \in F$

3. $B \subset A, B \in F \Rightarrow A \in F$

Now define $f: P(X) \to [0,1]$ as $f(A) = \begin{cases} 1 \text{ if } A \in F \\ 0 \ otherwise \end{cases}$

Claim:

1. F is crisp filter

Hence $\Phi \notin F$

$\therefore f(\Phi) = 0$.

2. Take $A, B \in P(X)$

Case: 1

Let $A \cap B \in F$.

$A \cap B \subset A$ and $A \cap B \subset B$.

Hence $A, B \in F$.

Now $f(A \cap B) = 1$

$f(A) = 1$

$f(B) = 1$

Hence have $f(A \cap B) = \min\{f(A), f(B)\}$.

Case: 2

Let $A \cap B \notin F$.

Hence $f(A \cap B) = 0$

If $A \in F$ and $B \in F$ then $A \cap B \in F$.

Therefore either $A \notin F$ or $B \in F$.

Hence$f(A) = 0 \; or \; f(B) = 0$.

Hence $\min\{f(A), f(B)\} = 0$.

Therefore $f(A \cap B) = \min\{f(A), f(B)\}$.

3. Let $B \subset A$.

Case: 1

Let $B \in F$.

Then $f(B) = 1$.

Now $B \subset A, B \in F$ implies $A \in F$ and hence $f(A) = 1$.

Therefore $f(B) \leq f(A)$

Case: 2

Let $B \notin F$.

Then $f(B) = 0$

Now $f(A) = 0 \text{ or } 1$.

Hence $f(B) \leq f(A)$.

Hence f is a fuzzy filter.

The given crisp filter F can be identified with the fuzzy filter f.

Every crisp filter can be considered as a fuzzy filter.

Note: 2.1.6

The converse is not true.

A fuzzy filter need not be a crisp filter.

for example,

Let $X = \{1,2,3\}$.

Define $f: P(X) \to [0,1]$ as

$f(\Phi) = 0$, $f\{1\} = 1$, $f\{2\} = 2$, $f\{3\} = 0$, $f\{1,2\} = 1$, $f\{1,3\} = 2$, $f\{2,3\} = 0$,

$f(X) = 1$,

Clearly f is a fuzzy filter.

F takes value other than 0 and 1.

Hence f is not a crisp filter.

Theorem: 2.1.7

Intersection of two fuzzy filters on X is a fuzzy filter on X.

Proof:

Let f and g be two fuzzy filters on X.

$f: P(X) \rightarrow [0,1]$, $g: P(X) \rightarrow [0,1]$.

Let $h = f \cap g$.

h is defined as $h: P(X) \rightarrow [0,1]$ where $h(A) = \min\{f(A), g(A)\}$.

1. f and g are fuzzy filters.

   Hence $f(\Phi) = 0, g(\Phi) = 0$.

   Now $h(\Phi) = \min\{f(\Phi), g(\Phi)\} = \min\{0,0\} = 0$.

2. Let $A, B \subset X$

   f is a fuzzy filter.

Therefore $f(A \cap B) \geq \min\{f(A), f(B)\}$.

$\Rightarrow f(A \cap B) \geq \min\{f(A), f(B), g(A), g(B)\}$

Hence $f(A \cap B) \geq \min\{\min f(A), g(A), \min f(B), g(B)\}$

Therefore $f(A \cap B) \geq \min\{h(A), h(B)\}$ …………….. (1)

Similarly $g(A \cap B) \geq \min\{h(A), h(B)\}$ ……………….. (2)

From (1) and (2)

$\min\{f(A \cap B), g(A \cap B)\} \geq \min\{h(A), h(B)\}$.

Hence $h(A \cap B) \geq \min\{h(A), h(B)\}$.

3. Let $B \subset A$.

f and g are fuzzy filters.

Therefore $f(B) \leq f(A)$

Hence $g(B) \leq g(A)$

Hence $\min\{f(B), g(B)\} \leq \min\{f(A), g(A)\}$

Hence $f \cap g(B) \leq f \cap g(A)$

Hence $h(B) \leq h(A)$

From 1,2,3 h is a fuzzy filter.

Definition: 2.1.8

We recall the definition of α cut of a fuzzy set. Let A be a fuzzy set on X. Let $\alpha \in [0,1]$. Then α cut of A denoted as αA is defined as $\alpha A = \{x \,/\, A(x) \geq \alpha\}$. Now we see that α cut of a fuzzy filter is a crisp filter.

Theorem: 2.1.9

Let F be a fuzzy filter on a non empty set X. Let $\alpha \in Im\, F$. Let $\alpha \neq 0$. Then α cut of F is a crisp filter on X.

Proof:

Let F be a fuzzy filter on a non empty set $\boldsymbol{X}$.

$F: P(X) \rightarrow [0,1]$ is a fuzzy filter.

$\alpha F = \{A \in P(X)/F(A) \geq \alpha\}$

We claim that αF is a crisp filter on X.

Here $\alpha \neq 0$.

1. $F(\Phi) = 0 \Rightarrow F(\Phi)$ is not greater than or equal to α.

   Hence $\Phi \notin \alpha F$.

2. Let $A, B \in \alpha F$.

   Then $F(A) \geq \alpha$ and $F(B) \geq \alpha$.

   Then implies $\min\{F(A), F(B)\} \geq \alpha$.

Now $F(A \cap B) \geq \min\{F(A), F(B)\} \geq \alpha$.

Hence $A \cap B \in \alpha F$.

Therefore $A, B \in \alpha F \Rightarrow A \cap B \in \alpha F$.

3. Let $A \in \alpha F$ and $A \subset B$.

$A \in \alpha F$ implies $F(A) \geq \alpha$.

$A \subset B$ and F is a fuzzy filter.

Therefore $F(B) \geq F(A)$.

Hence $F(B) \geq \alpha$.

Therefore $B \in \alpha F$.

Hence $A \in \alpha F$ and $A \subset B \Rightarrow B \in \alpha F$.

Therefore αF is a crisp filter on X.

Note: 2.1.10

Converse is not true.

Let $F: P(X) \to [0,1]$ be a fuzzy set on X. αF is a crisp filter. Then F need not be a fuzzy filter.

for example,

Let $X = \{a, b\}$

Define $f: P(X) \to [0,1]$ as

$f(\Phi) = 0$, $f\{a\} = .4$, $f\{b\} = .1$, $f\{c\} = .1$, $f\{a, b\} = .1$, $f\{a, c\} = .1$,

$f\{b, c\} = .5$, $f(X) = 1$.

Take $\alpha = .5$

$\alpha f = \{\{b,c\},X\}$.

αf is a crisp filter.

Now $\{a\} \subset \{a,b\}, f\{a\} = .4$ and $f\{a,b\} = .1$.

Hence f is not a fuzzy filter.

Therefore α cut of a fuzzy filter is a crisp filter does not imply that f is a fuzzy filter.

2.2 CONVERGENCE OF FUZZY FILTERS

Definition: 2.2.1

Let X be a topological space. Let F be a fuzzy filter on X. Let $a \in X$. F is said to be converges to a if for every open neighbourhood U of a , $F(U) = 1$. 'a' is called limit of F.

Definition: 2.2.2

Let X be a topological space. Let F be a fuzzy filter on X. Let $\in X$. Let $\alpha \in (0,1]$. F is said to converge to 'a' at level α, if for every neighbourhood U of a, $F(U) \geq \alpha$. 'a' is called α −level limit of F.

Example: 2.2.3

Let $X = \{a,b,c\}$.

$T = \{\Phi,\{a\},X\}$.

$F: P(X) \to [0,1]$ is defined as

$F(\Phi) = 0$, $F\{a\} = 1$, $F\{b\} = 0$, $F\{c\} = 0$, $F\{a, b\} = 1$, $F\{a, c\} = 1$,

$F\{b, c\} = 0$, $F(X) = 1$.

The neighbourhood of a are $\{a\}$ and X.

$F\{a\} = 1 \; and \; F(X) = 1$.

Hence F converges to a.

The neighbourhood of b is X only and $F(X) = 1$.

Hence $F \to b$.

The neighbourhood of c is X only and $F(X) = 1$.

Hence $F \to c$.

Result: 2.2.4

The above example shows that limit of a fuzzy use not be unique. But may be unique also.

Example: 2.2.5

$X = \{a, b\}$

$T = P(X)$

$F: P(X) \to [0,1]$ is defined as

$F(\Phi) = 0$, $F\{a\} = 1$, $F\{b\} = 0$, $F(X) = 1$.

Hence F converges to a.

F does not converges to b.

So in the example limit of F is unique.

Example: 2.2.6

Let $X = \{a, b, c\}$

$T = \{\Phi, \{a\}, \{b\}, \{a, b\}, X\}$

$F: P(X) \rightarrow [0,1]$ is defined as

$F(\Phi) = 0$, $F\{a\} = .6$, $F\{b\} = .4$, $F\{a, b\} = .6$, $F\{a, c\} = .6$, $F\{b, c\} = .4$,

$F(X) = 1$.

Take $\alpha = .6$.

Consider the neighbourhood of a. $\{a\}, \{a, b\}, X$

$F\{a\} \geq \alpha, F\{a, b\} \geq \alpha, F(X) \geq \alpha$.

Hence F converges to a at level $\alpha = .6$

F does not converges to b at level $\alpha = .6$

The neighbourhood of b are $\{b\}, \{a, b\}, X$.

Here $F\{b\} = .4$ is not greater than or equal to α.

Hence F does not converges to b at level α.

The neighbourhood of c is X only.

$F(X) = 1 \geq \alpha$.

Hence F converges to c at level $\alpha = .6$.

Result: 2.2.7

Let F be a fuzzy filter and let $\beta \leq \alpha$. Then if F converges to a at level α, then F converges to a at level β

Theorem: 2.2.8

Let F be a fuzzy filter on X. Let $\alpha \in (0,1]$. Let $a \in X$. Then the fuzzy filter F converges to a at level α iff the crisp filter α −cut of F converges to a.

Proof:

Let F converges to a at level α.

$\alpha F = \{A/F(A) \geq \alpha\}$

Let U be a neighbourhood of a.

Since F converges to a, $F(U) \geq \alpha$.

Hence $U \in \alpha F$.

The crisp filter αF contains every neighbourhood of a.

Hence αF converges to a.

Conversely, Suppose αF converges to a.

We claim that the fuzzy filter F converges to a at level α.

Let U be a neighbourhood of a.

Since αF converges to a, $U \in \alpha F$.

Hence $F(U) \geq \alpha$.

Therefore the fuzzy filter F converges to a at level α.

Definition: 2.2.9

Let X be an infinite set. A function $F: P(X) \to [0,1]$ is called a fuzzy co finite filter, if $F(A) = \begin{cases} 1 \text{ if } A^c \text{ is finite} \\ 0 \quad \text{otherwise} \end{cases}$

Theorem: 2.2.10

Fuzzy co finite filter is a fuzzy filter.

Proof:

Let $F: P(X) \to [0,1]$ be defined as F(A) =$\begin{cases} 1 \text{ if } A^c \text{ is finite} \\ 0 \quad \text{otherwise} \end{cases}$

Now consider $\Phi \in P(X)$

$\Phi^c = X$ which is not finite.

Hence F(Φ) $= 0$.

Take $A, B \in P(X)$

If F(A) = 0 and F(B) = 0 then whatever be the value of $F(A \cap B)$.

We have $F(A \cap B) \geq \min\{F(A), F(B)\}$.

Now suppose F(A) = 1 and F(B) = 1, then A^c is finite and B^c is finite.

Hence $A^c \cup B^c$ is finite which implies $(A \cap B)^c$ is finite.

Hence $F(A \cap B) = 1$.

Hence have $F(A \cap B) \geq \min\{F(A), F(B)\}$.

Now suppose as $F(A) = 0$ and $F(B) = 1$, then min{F(A),F(B)} = 0.

Hence $F(A \cap B) \geq \min\{F(A), F(B)\}$.

Let $A \subset B$.

If $F(A) = 0$ then whatever be the value of F(B), we have $F(B) \geq F(A)$.

If $F(A) = 1$ then A^c is finite.

This implies that B^c is finite.

Hence $F(B) = 1$.

Therefore $F(A) \leq F(B)$.

Hence $A \subset B \Rightarrow F(A) \leq F(B)$.

Hence F is a fuzzy filter.

CHAPTER 3

FUZZY GENERALIZED FILTER (FUZZY G FILTER)

3.1 Fuzzy Generalized Filter:

Definition: 3.1.1

Let X be a non empty set $F: P(X) \to [0,1]$ is called fuzzy generalized filter if

1. $F(\Phi) = 0$.
2. $A \subset B \Rightarrow F(A) \leq F(B)$ for all A, B in $P(X)$.

Example: 3.1.2

Let $X = \{1,2,3\}$. Define $F: P(X) \to [0,1]$ as

$\Phi \to 0$, $\{1\}\to 0.4$, $\{2\}\to 0.4$, $\{3\}\to 0.4$, $\{1,2\}\to 0.5$, $\{1,3\}\to 0.5$, $\{2,3\}\to 0.5$,

$\{1,2,3\}\to 0.6$,

F is a fuzzy g filter.

Theorem: 3.1.3

Every crisp g filter is a fuzzy g filter.

Proof:

Let $\boldsymbol{A}$ be a g filter on a non empty set X.

Define $F: P(X) \to [0,1]$ as $F(A) = 1$ if A$\in \boldsymbol{A}$ and $F(A) = 0$ if A$\notin \boldsymbol{A}$.

We claim that F is a fuzzy g filter on X.

1.$\Phi \notin \boldsymbol{A}$ and hence $F(\Phi) = 0$.

2. Let $A \subset B$. Let $F(A) = 0$. Now $F(B) = 0$ or $F(B) = 1$.

Therefore $F(A) \leq F(B)$.

Let $F(A) = 1$. Then A$\in \boldsymbol{A}$.

Since $\boldsymbol{A}$ is a g filter and $A \subset B$, and we have $B \in \boldsymbol{A}$.

$\Rightarrow F(B) = 1$.

Hence $F(A) \leq F(B)$.

Therefore F is a fuzzy g filter.

The crisp filter can be identified uniquely with the fuzzy g filter.

Hence every crisp g filter is a fuzzy g filter.

Note: 3.1.4

Converse is not true.

A fuzzy g filter need not be a crisp g filter.

for example,

Let $X = \{1,2,3\}$. Define $F: P(X) \to [0,1]$ as $\{\Phi\} \to 0$.

$\{1\} \to 0.5$, $\{2\} \to 0.6$, $\{1,2\} \to 0.7$, $\{1,3\} \to 0.6$, $\{2,3\} \to 0.7$, $\{1,2,3\} \to 0.9$,

F is a fuzzy g filter.

F is not a crisp g filter.

Theorem: 3.1.5

The $\alpha -$ cut of a fuzzy filter is a crisp g filter.

Proof:

Let $X \neq \Phi$,

Let F be a fuzzy g filter on X.

Let $\alpha F = \alpha$ cut of F.

We claim that αF is a crisp g filter.

1. Since F is a fuzzy g filter, $F(\Phi) = 0$

   Therefore $F(\Phi)$ is not greater than or equal to α.

   Hence $\Phi \notin \alpha F$.

2. Let $A \in \alpha F$ and $A \subset B$.

   $A \in \alpha F \Rightarrow F(A) \geq \alpha$.

   Now $A \subset B$ and F is a fuzzy g filter.

   Hence $F(B) \geq F(A)$.

   Therefore $F(B) \geq \alpha$.

   This implies $B \in \alpha F$.

Hence αF is a crisp g filter.

Therefore α cut of a fuzzy g filter is a crisp g filter.

Note: 3.1.6

Converse is not true.

That is, Let $F: P(X) \to [0,1]$ be a function. Let $\alpha \in (0,1]$. $\alpha -$ cut of F is a crisp g filter does not imply that F is fuzzy g filter.

For example,

Let $X = \{1,2,3\}$.

Define $F: P(X) \to [0,1]$ as

$\{\Phi\} \to 0$, $\{1\} \to 0.4$, $\{2\} \to 0.4$, $\{3\} \to 0.4$, $\{1,2\} \to 0.3$, $\{1,3\} \to 0.5$, $\{2,3\} \to 0.5$, $\{1,2,3\} \to 0.5$.

Take $\alpha = 0.5$

$\alpha -$ cut of $F = \big\{\{1,3\},\{2,3\},\{1,2,3\}\big\}$

$\alpha -$ cut of F is a crisp g filter. $F(\{2\}) = 0.4$ $F(\{1,2\}) = 0.3$

Hence, F is not a fuzzy g filter.

Theorem: 3.1.7

Let F be function $P(X) \rightarrow [0,1]$. If for each $\alpha \in Im\, F$ and $\alpha \neq 0$, α −cut of F, is a g-filter then F is a fuzzy g filter.

Proof:

Let $F: P(X) \rightarrow [0,1]$ be a function.

For each $\alpha \in Im\, F$ and $\alpha \neq 0$, $\alpha -$ cut of F is a crisp g filter on X.

First we prove that $F(\Phi) = 0$.

Let $F(\Phi) = \alpha$.

If $\alpha \neq 0$, then by assumption, $\alpha -$ cut of F is a crisp g filter.

Since $F(\Phi) = \alpha$, Φ belongs to $\alpha -$ cut of F.

This shows that the crisp g filter $\alpha -$ cut of F contains Φ.

This is a contradiction.

Hence α = 0.

Therefore $F(\Phi) = 0$.

Let $A \subset B$.

We prove that $F(A) \leq F(B)$.

Let $F(A) = \alpha$.

If $\alpha = 0$ then $F(B) \geq F(\alpha)$.

If $\alpha \neq 0$ consider $\alpha -$ cut of F which is a crisp g filter by assumption.

Now $F(A) \geq \alpha \Rightarrow A \in \alpha -$ cut of F.

Now $A \in \alpha -$ cut of F, $A \subset B$ and $\alpha -$ cut of F is a crisp g filter.

Hence $B \in \alpha -$ cut of F.

Hence $F(B) \geq \alpha$.

This implies $F(B) \geq F(\alpha)$.

Hence $A \subset B \Rightarrow F(A) \leq F(B)$.

Therefore F is a fuzzy g filter.

Example: 3.1.8

Let $X = \{1,2,3\}$.

Define $F: P(X) \rightarrow [0,1]$ as

$\{\Phi\} \rightarrow 0$, $\{1\} \rightarrow 0.4$, $\{2\} \rightarrow 0.5$, $\{3\} \rightarrow 0.5$, $\{1,2\} \rightarrow 0.6$, $\{2,3\} \rightarrow 0.7$,

$\{1,3\} \rightarrow 0.6$, $X \rightarrow 0.7$.

Consider values of $\alpha \in Im\ F$ and $\alpha \neq 0$.

We have $\alpha_1 = 0.4, \alpha_2 = 0.5, \alpha_3 = 0.6, \alpha_4 = 0.7$.

α_1 cut of $F = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},X\}$.

Clearly α_1 cut of F is a crisp g filter

α_2 cut of $F = \{\{2\},\{3\},\{1,3\},\{2,3\},X\}$. Clearly α_2 cut of F is a crisp g filter.

α_3 cut of $F = \{\{1,2\},\{1,3\},\{2,3\},X\}$. Clearly α_3 cut of F is a crisp g filter.

α_4 cut of $F = \{\{2,3\},X\}$. Clearly α_4 cut of F is a crisp g filter.

For all possible values of α,α cut of F is a crisp g filter.

Clearly F is a fuzzy g filter.

Theorem: 3.1.9

Every fuzzy filter is a fuzzy g filter.

Proof:

Let X be a non empty set.

Let F be a fuzzy filter on X.

Since F is a fuzzy filter, $F(\Phi) = 0$ and $A \subset B \Rightarrow F(A) \geq F(B)$.

Hence F is a fuzzy g filter.

Note: 3.1.10

Converse is not true.

A fuzzy g filter need not be a fuzzy filter.

for example,

Let $X = \{1,2,3\}$. Define $F: P(X) \to [0,1]$ as $\{\Phi\} \to 0$.

$\{1\} \to 0.3$, $\{2\} \to 0.3$, $\{3\} \to 0.3$, $\{1,2\} \to 0.4$, $\{1,3\} \to 0.4$, $\{2,3\} \to 0.4$, $X \to 1$.

Clearly F is a fuzzy g filter.

Take $A = \{1,2\}, B = \{1,3\}, A \cap B = \{1\}, F(A \cap B) = 0.3$

$\min\{F(A), F(B)\} = \min\{0.4, 0.4\} = 0.4$.

$F(A \cap B) < \min\{F(A), F(B)\}$.

Hence F is not a fuzzy filter.

Theorem: 3.1.11

Union of two fuzzy g filters on a non empty set X is a fuzzy g filter on X.

Proof:

Let X be a non empty set.

Let F and H be a two fuzzy g filters on X.

1. Since F and H are fuzzy g filters, $F(\Phi) = 0$ and $H(\Phi) = 0$.

   Hence $\max\{F(\Phi), H(\Phi)\} = 0$.

   Therefore $(F \cup H)\Phi = 0$.

2. Let $A \subset B$.

   Since F is a fuzzy g filter, we have $F(A) \leq F(B)$.

Since H is a fuzzy g filter, we have $H(A) \leq H(B)$.

Now $F(A) \leq F(B)$ and $H(A) \leq H(B)$

$\Rightarrow \max\{F(A), H(A)\} \leq \max\{F(B), H(B)\}$

This implies $(F \cup H)(A) \leq (F \cup H)(B)$.

Hence $F \cup H$ is a fuzzy g filter.

3.2 Convergence of Fuzzy G Filter:

Definition: 3.2.1

Let F be a fuzzy g filter on a topological space X. Let $a \in X$. F is said to converge to a if for every neighbourhood U of a. $F(U) = 1$. We write $F \rightarrow a, a$ is called limit of F.

Example: 3.2.2

Let $X = \{a, b, c\}$. $T = \left\{\Phi, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\right\}$

Define $F: P(X) \rightarrow [0,1]$ as

$F\{\Phi\} = 0$, $F\{a\} = 0.1$, $F\{b\} = 0.1$, $F\{c\} = 0.1$, $F\{a, b\} = 1$, $F\{b, c\} = 0.6$, $F\{a, c\} = 0.5$, $F(X) = 1$.

F converges to b. F does not converges to a. F does not converges to c.

Result: 3.2.3

Limit of a fuzzy g filter need not be unique.

Example: 3.2.4

Let $X = \{a, b, c\}$. $T = \{\Phi, \{a\}, \{a, b\}, \{a, c\}, X\}$

Define $F: P(X) \to [0,1]$ as

$F\{\Phi\} = 0$, $F\{a\} = 1$, $F\{b\} = 0.1$, $F\{c\} = 0.2$, $F\{a, b\} = 1$, $F\{b, c\} = 1$

$F\{a, c\} = 1$, $F(X) = 1$.

Hence $F \to a, F \to b$ and $F \to c$.

Result: 3.2.5

Limit of a g filter in a topological space need not be unique even if space is haussdroff.

Example: 3.2.6

$X = \{a, b, c\}$. $T = P(X)$

Define $F: P(X) \to [0,1]$ as

$F(A) = 1$, if $A \neq \Phi$, $F(A) = 1$ if $A = \Phi$.

Theorem: 3.2.7

Let F be a fuzzy g filter on a topological space X. If $F \to a$ then $\alpha -$ cut of F converges to a for any $\alpha \in (0,1], \alpha \in$ Im F.

Proof:

Let F be a Fuzzy g filter converges to a.

Consider $\alpha F = \alpha -$ cut of F.

Let U be a neighbourhood of a.

Since F converges to a, $F(U) = 1$

Therefore $F(U) \geq \alpha$.

Hence $U \in \alpha F$.

Therefore αF contains all neighbourhood of a.

Hence F converges to a.

Note: 3.2.8

Converse is not true.

If one $\alpha -$ cut of F converges to a then we can say that the fuzzy g filter F converges to a.

for example,

Let $X = \{1,2,3\}$. Define $F: P(X) \to [0,1]$ as

$\{\Phi\} \to 0$, $\{1\} \to 0.5$, $\{2\} \to 0.3$, $\{1,2\} \to 0.7$, $\{1,3\} \to 0.6$, $\{2,3\} \to 0.7$, $X \to 0.7$

The topology T on X is given by $T = \{\Phi, \{1\}, \{1,2\}, \{1,3\}, X\}$.

Take $\alpha = 0.5$, Now $\alpha -$ cut of $F = \{\{1\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$.

$\alpha -$ cut of F converges to 1.

But F does not converge to 1.

Theorem: 3.2.9

Let F be a fuzzy g filter on a topological space (X, T). If the 1-cut of F converges to a then F converges to a.

Proof:

Let $F: P(X) \to [0,1]$ be a fuzzy g filter on X. (X, T) is a topological space.

Let 1-cut of F converge to a.

Now we claim that the fuzzy g filter F converges to a.

Let U be an open set containing a.

Since the crisp g filter 1-cut of F converges to a.

$U \in 1 -$ cut of F.

Hence $F(U) = 1$.

This is true for all open sets containing to a.

Hence the fuzzy g filter F converges to a.

Theorem: 3.2.10

Let X be a generalized topological space. Let S be a g filter base. $S \to x$ iff for every open set O containing x, there exists an element A in S such that $A \subset O$

Proof:

Let $S \to x$. Then the g filter A Which is generated by S converges to x. If O is any open set containing x

Since A converges to x, $O \in A$. Since A is generated by the g filter base $S,$ there exists $A \in S$ such that $A \subset O$.

Conversely,

Let A be the filter generated by S. Let O be any open set Containing x,

Then there exists $A \in S$ such that $A \subset O$ which implies $O \in A$.

Hence $A \to x$ which gives $S \to x$.

CHAPTER 4

FUZZY NETS

4.1 Fuzzy Nets:

Definition: 4.1.1

Let (X, T) be a topological space. Let D be a directed set. A function $f: D \times X \rightarrow [0,1]$ is called a fuzzy net on X.

We recall the definition of net on set X. A function from $D \rightarrow X$ Where D is a directed set is called a net. Hereafter we call this net as crisp net.

Theorem: 4.1.2

Every crisp net in X induces a fuzzy net in X.

Proof:

Let X be a non empty set.

Let D be a directed set and let A be a crisp net on X.

That is, $A: D \rightarrow X$ is a function.

This function A can be associated with a function from $D \times X \rightarrow [0,1]$.

Define $B: D \times X \rightarrow [0,1]$

Take $\lambda \in D$, $x \in X$.

If $A(\lambda) = x$ then we define $B(\lambda, x) = 1$.

If $A(\lambda) \neq x$ then we define $B(\lambda, x) = 0$

Therefore we get $B: D \times X \to [0,1]$ using the given crisp net A on X.

Hence, any crisp net in X induces a fuzzy net on X.

Theorem: 4.1.3

Every fuzzy sequence in X is a fuzzy net in X.

Proof:

Let A be a fuzzy sequence in X.

Then A is a function from $N \times X \to (0,1]$

That is, $A: N \times X \to (0,1]$

It is clear that $(N, \leq)$ is an ordered set.

Hence, A is a fuzzy net on X.

Therefore, every fuzzy sequence is a fuzzy net.

Note: 4.1.4

Converse is not true.

A fuzzy net on X need not be a fuzzy sequence in X.

for example,

Let $D = P(X)$ where $X = \{1,2,3\}$

Consider zero function $A: D \times X \to [0,1]$

It is clear that A is a fuzzy net.

Since the directed set $D \neq N$, A is not a fuzzy sequence.

Hence, a fuzzy net need not be a fuzzy sequence.

Theorem: 4.1.5

Let $A: D \times X \to (0,1]$ be a fuzzy net. If for each $\lambda \in D$, there exists $x \in X$ such that $A(\lambda, x) = 1$ then A induces a crisp net.

Proof:

Let $A: D \times X \to (0,1]$ be a fuzzy net.

Now we define a function $f: D \to X$ as $f(\lambda) = x$ if $A(\lambda, x) = 1$.

Clearly f is a crisp net which is induced by the fuzzy net A.

Theorem: 4.1.6

Let $A: D \times X \to (0,1]$ be a fuzzy net. If for each $\lambda \in D$, there exists unique $x \in X$ such that $A(\lambda, x) = 1$ and $A(\lambda, y) = 0$ for $y \neq x$, then A is a crisp net.

Proof:

Let $A: D \times X \to (0,1]$ be a fuzzy net.

Define $f: D \to X$ as $A(\lambda) = x$ where $A(\lambda, x) = 1$.

Clearly f is a crisp net.

Now the fuzzy net induced by f is given fuzzy net A. F and A are the same.

Hence, A is a crisp net.

4.2 CONVERGENCE OF FUZZY NET

Let (X, T) be a topological space and $f: D \to X$ where $f(\lambda) = x_\lambda$ be a net in X. We say the net (x_λ) converges to x_0 if for every neighbourhood U of $x_0, y_0 \in D$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

Definition: 4.2.1

Let (X, T) be a topological space. Let $A: D \times X \to [0,1]$ be a fuzzy net. Let $x_0 \in X$. Let $\alpha \in (0,1]$. The fuzzy net A is said to be converges to x_0 at level α if

1. For each $\lambda \in D$, there exists atleast one x in X such that $A(\lambda, x) \geq \alpha$

2. For each neighbourhood U of x_0, there exists $\lambda_0 \in D$, such that $x \in U\ \forall\ \lambda \geq \lambda_0$ and $A(\lambda, x) \geq \alpha$.

Definition: 4.2.2

Let X be a non empty set. Let D be a directed set. Let $x_0 \in X$. Define $A: D \times X \to [0,1]$ as $A(\lambda, x_0) = 1$ and $A(\lambda, x) = 0$ if $x \neq x_0$. This is called constant net and is denoted by A_{x_0}.

Theorem: 4.2.3

Let (X, T) be a topological space. The constant fuzzy net A_{x_0} converges to x_0 at level $\alpha > 0$.

Proof:

Let $A_{x_0}: D \times X \to [0,1]$ be defined as $A(\lambda, x) = 1$ if $x = x_0$ and 0 otherwise.

Let U be a neighbourhood of x_0. Take any $\alpha \in (0,1]$. Take any $\lambda_0 \in D$

Now $\lambda \geq \lambda_0$ and $A(\lambda, x) \geq \alpha \Rightarrow A_{x_0}(\lambda, x) = 1 \Rightarrow x = x_0 \Rightarrow x \in U$.

Therefore, for any neighbourhood U, there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ and $A_{x_0}(\lambda, x) \geq \alpha$.

Therefore A_{x_0} converges to x_0

Definition: 4.2.4

Let (X, T) be a topological space. Let A be a fuzzy net in X. A is called an attractive fuzzy net if A converges to every point of X.

Example: 4.2.5

$X = \{a, b, c\}, T = \{\Phi, \{a, b\}, X\}, D = (P(X), \subset)$

Let $P(X) = \{A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ where

$A_0 = \{\Phi\}, A_1 = \{a\}, A_2 = \{b\}, A_3 = \{c\}, A_4 = \{a, b\}, A_5 = \{a, c\}$

$A_6 = \{b, c\}, A_7 = X$

Define $A: D \times X \to [0,1]$ as $A(A_0, c) = \frac{1}{4}$, $A(A_1, c) = 1$,

$A(A_2, a) = \frac{1}{4}$, $A(A_3, c) = 1$, $A(A_4, b) = 1$, $A(A_5, c) = 1$,

$A(A_6, b) = 1$, $A(A_7, a) = 1$.

We have seen that A converges to a, b and c.

Hence, A is an attractive fuzzy net.

Theorem: 4.2.6

Let (X, T) be a topological space. The constant net A_a is an attractive net iff every non empty open set contains a.

Proof:

Let A_a be an attractive net and let U be a non empty open set.

Take $b \in U$. Now A_a converges to b.

Hence, there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ and $A(\lambda, x) \geq \alpha \Rightarrow x \in U$

$\lambda \geq \lambda_0$ and $A_a(\lambda, x) = 1$ implies $x \in U$.

Now $A_a(\lambda, x) = 1$ implies $x = a$.

Hence, $a \in U$.

Hence, every non empty open set contain a.

Conversely, suppose every non empty open set contain a.

Let $b \in X$

Let U be a neighbourhood of b.

Take any $\lambda_0 \in D$

Now $\lambda \geq \lambda_0$ and $A_a(\lambda, x) \geq \alpha$ implies $\lambda \geq \lambda_0$ and $A_a(\lambda, x) = 1$ implies $x = a$.

Hence, $x \in U$

Therefore, for every neighbourhood U of b there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ and $A_a(\lambda, x) \geq \alpha$ implies $x \in U$.

Hence A_a converges to b.

Therefore, A_a converges to every point of X.

Hence A_a is an attractive net.

Definition: 4.2.7

Let $A: D \times X \to [0,1]$ be a fuzzy net. Let $\lambda_0 \in D$ and $\alpha \in [0,1]$. We define $t(\lambda_0, \alpha) = \{x \,/\, \lambda \geq \lambda_0 \text{ and } A(x, \lambda) \geq \alpha\}$. $t(\lambda_0, \alpha)$ is called a tail of the fuzzy net A. It is called the (λ_0, α) tail of A.

Theorem: 4.2.8

Let A be a fuzzy net in topological space X. If A converges to $a \in X$ at level α then for every neighbourhood U of a, there exists $\lambda_0 \in D$ such that U contains $t(\lambda_0, \alpha)$ and $t(\lambda_0, a) \neq \Phi$

Proof:

Let A converges to a at level α then

1. For each $\lambda \in D$, there exists atleast one x in X such that $A(\lambda, x) \geq \alpha$.

2. For each neighbourhood U of a, there exists $\lambda_0 \in D$ such that $x \in U$ for all $\lambda \geq \lambda_0$ and $A(\lambda, x) \geq \alpha$.

Now consider the tail $t(\lambda_0, \alpha)$.

Since for each $\lambda \in D$, there exists atleast one x in X such that $A(\lambda, x) \geq \alpha$, we have $t(\lambda_0, \alpha) \neq \Phi$.

Claim: $t(\lambda_0, \alpha) \subset U$.

$x \in t(\lambda_0, \alpha)$ implies $A(\lambda, x) \geq \alpha$ for all $\lambda \geq \lambda_0$.

Now $\lambda \geq \lambda_0$ and $A(\lambda, x) \geq \alpha$ implies $x \in U$.

Therefore $t(\lambda_0, \alpha) \subset U$.

Hence, there exists λ_0 such that U contains $t(\lambda_0, \alpha)$ and $t(\lambda_0, a) \neq \Phi$.

Theorem: 4.2.9

Let A be a fuzzy net in X such that for each $\lambda \in D$, there exists $x \in X$ with $A(\lambda, x) \geq \alpha$. If the every neighbourhood U of a there exists $\lambda_0 \in D$ such that U contains $t(\lambda_0, \alpha)$ and $t(\lambda_0, a) \neq \Phi$ then A converges to a.

Proof:

Let $A: D \times X \to [0,1]$ be a fuzzy net. It is given that

1. For each $\lambda \in D$, there exists $x \in X$ such that $A(\lambda, x) \geq \alpha$

2. Let U be a neighbourhood of a ,there exists $\lambda_0 \in D$, such that $t(\lambda_0, \alpha) \subset U$ $\lambda \geq \lambda_0$ and $A(\lambda, x) \geq \alpha$ implies $x \in$ $t(\lambda_0, \alpha)$ which implies $x \in$ U.

Hence , A converges to a.

Definition: 4.2.10

Let $A: D \times X \to [0,1]$ be a fuzzy net. A is called a s-fuzzy net if for each $\lambda \in D$ there exists unique x in X where $A(\lambda, x) = 1$.

Definition: 4.2.11

Let $A: D \times X \to [0,1]$ be a s-fuzzy net. Define $f: D \to X$ as $f(\lambda) = x$ if $A(\lambda, x) = 1$. The crisp net f is called the induced crisp net.

Theorem: 4.2.12

Let X be a topological space. Let A be a s-fuzzy net and let f be the induced crisp net. If A converges to a, at level α then f converges to a.

Proof:

Let A be a s-fuzzy net. $A: D \times X \to [0,1]$

$f: D \to X$ is defined as $f(\lambda) = x$ where $A(\lambda, x) = 1$.

Now A is converges to a.

We claim that f converges to a.

Let U be a neighbourhood of a.

Since A converges to a, there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ and $A(\lambda, x) \geq \alpha$ implies $x \in U$.

Take $\lambda \geq \lambda_0$ consider $f(\lambda)$

$A(\lambda, f(\lambda)) = 1$ and hence $A(\lambda, f(\lambda)) \geq \alpha$.

Therefore $f(\lambda) \in U$.

Hence for every neighbourhood of a, there exists $\lambda_0 \in D$ such that $f(\lambda) \in U$ for all $\lambda \geq \lambda_0$.

Therefore the crisp net F converges to a.

Theorem: 4.2.13

Every crisp net is a s-fuzzy net.

Proof:

Let f be a crisp net in X.

Then $f: D \to X$ is a function.

Consider f as a fuzzy net.

It is $A: D \times X \to (0,1]$ defined as $A(\lambda, x) = 1$ if $x = f(\lambda)$ and 0 otherwise.

Clearly for each $\lambda \in X$ there exists unique $x = f(\lambda)$ such that $A(\lambda, x) = 1$.

Hence A is a s-fuzzy net.

Result: 4.2.14

If a s-fuzzy net takes only two values 0 and 1 then it is a crisp net.

Result: 4.2.15

Let f be a crisp net. Then it can be considered as a fuzzy net. Let it be A. Then A is a s-fuzzy net. Hence A induces a crisp net. The crisp net induced by the s-fuzzy net A is the given crisp net f.

CONCLUSION

In this project I have discuss the concept of fuzzy filters, fuzzy generalized filters, fuzzy nets and fuzzy generalized nets. These tools will be useful to study many topological properties. I have also given some concepts in various generalizations also various convergence of fuzzy filter and fuzzy net can be introduced and properties can be studied.

REFERENCES

[1] Baskaran R, " Ph.D Thesis A Study on the properties of Generalized Topology" Manonmaniam Sundaranar University, October 2010.

[2] Baskaran R, Murugalingam M and Sivaraj D, Separated set in generalized topological spaces, J.Adv.Res. Pure Maths, 2(1)(2010), 74-83.

[3] Chang C.L , Fuzzy Topological Space, J Math, Annal. Appl.,24 (1968) 182-190.

[4] Csaszar A, Generalized topology, Generalized continuity, Acta Math. Hungar., 96 (2002) 233-254.

[5] Csaszar A, Generalized open set, Acta Math. Hungar., 75 (1-2) (1997), 65-87.

[6] Maki H, Balachandran K, Generalized $\alpha -$ closed sets in topology, Bull Fukuoka Univ.Ed.Part III, 42 (1993), Page 13-21.

[7] Murugalingam M, "Ph.D Thesis A Study of Semi Generalized Topology" Manonmaniam Sundaranar University, December 2004.

[8] Muthukumari M, "Fuzzification of Filters" Mathematical Sciences International Research Journal Volume 3 Issue 2 (2004), Page 669-671.

[9]Muthukumari M, "Convergence of Fuzzy Filters" International Journal of Mathematical Sciences and Applications, Vol 5, no 2 (2015), Page 207-210.

[10] Muthukumari M, "Convergence of Fuzzy Generalized Filters", Proceedings of the UCG Sponsored Two Day National Conference, Special Issue-Enrich, Page 44-51.

[11] Palaniammal S, “ Ph.D Thesis A Study on Tri topological space”, Manonmaniam Sundaranar University 2011.

[12] Shen R, A note on generalized connectedness, Acta Math. Hunger 122 (3) (2009), 231-235.

[13] Shen R.X, Remark on products of generalized topologies, Acta Math. Hunger 124 (4) (2009), 363-369.

[14] Zadeh L.A, “Fuzzy Set”, Information and Control, 8(1965) 338-353.

A STUDY ON NEAR-RINGS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

S.SUBHALAKSHUMI

Reg. No: 19SPMT25

Under the guidance of

Ms. K.AMBIKA M.Sc., B.Ed., S.E.T.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON NEAR-RINGS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **S.SUBHALAKSHUMI (Reg. No: 19SPMT25)**

Signature of the Guide

Signature of the Coordinator

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

CERTIFICATE

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K. Amutha

Signature of the Guide

Signature of the Coordinator

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON NEAR-RINGS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. K.Ambika M.Sc., B.Ed., S.E.T.,** Assistant Professor of Mathematics, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi.**

Station: Thoothukudi **Signature of the Student**

Date : 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CHAPTER 1

PRELIMINARIES

Definition 1.1:

A right near-ring is a set N together with two binary operations '+' and '.' such that

1. (N, +) is a group.
2. (N, ·) is a semi group.
3. For all x,y,z in N, (x+y)z = xz + yz for each x,y,z in N and x·0 = 0 for every x in N.

Notation 1.2:

Near-ring are usually denoted by N,N^1, N_1 or similar symbols. We abbereviate (N,+, ·) by N. Multiplication will in most cases be indicated by juxtapositions so we write$n_1 n_2$. In dealing with general near-rings the neutral element of (N,+) will be denoted by 0. |N| will be the order of the near-ring N. The term "near- ring" will often be abbreviated by "**nr**". The class of all near-ring will be denoted by ŋ .

Definition 1.3:

A near-ring which is not a ring is called a **non-ring.**

Definition 1.4:

If $|N| < \infty$ we say the near-ring is **finite** if $|N| = \infty$ we call N to be an **Infinite near-ring.**

Remark 1.5:

Every ring is a near-ring.

Definition 1.6:

A subgroup M of a near-ring N with N. $M \subseteq N$ is called a **sub near-ring.**

Example 1.7:

Let $(z,+,\cdot)$ be a near-ring. $(2z,+,\cdot)$ is a sub near-ring.

Definition 1.8:

Let N be a near-ring. If $(N,+)$ is abelian, we call N an abelian near-ring.

Definition 1.9:

Let N be a near-ring. If $(N,\cdot)$ is commutative we call N itself a **commutative near-ring.**

Definition 1.10:

An element a in N is said to be **distributive**, if $a(b+c) = ab + ac$ for all b and c in N.

Definition 1.11:

Let $(P,+)$ be a group with 0 and let N be a near-ring. Let $\mu: N \times P \to P$; (P,μ) is called an **N-group** if for all $p \in P$ and for all $n, n_1 \in N$ we have $(n + n_1)p = np + n_1p$ and $(n\, n_1)p = n(n_1)p$. N^p stands for N-groups.

Definition 1.12:

A subgroup S of N^p with NS $\subset$ S is a N-subgroup of P.

Definition 1.13:

Let N be a near-ring and P be a N-group. A normal subgroup I of (N,+) is called an **ideal** of N if

(i) IN $\subseteq$ I

(ii) For all n, $n_1 \in$ N and for all i $\in$ I, n(n_1+ i) - $nn_1 \in$ I .

Definition 1.14:

N is called a **near-field** if it contains an identity and each non-zero element has a multiplicative inverse.

Definition 1.15:

An element n $\in$ N is called **nilpotent** if $n^k = 0$ for some positive integer k.

Definition 1.16:

An element e in N is called **idempotent**, if e^2= e.

Definition 1.17:

An idempotent a in N is called a **central** if ax = xa for all in N.

Definition 1.18:

N is **integral** if N has no non-zero zero divisors.

Definition 1.19:

N_0 = {n∈ N/ n0=0} is called the **zero symmetric part of N**.

Definition 1.20:

N_c = {n∈ N/ n0=n} = {n∈ N / nn' = n, ∀ n'∈ N} is called the **constant part** of N.

Definition 1.21:

If N = N_0, then N is called **constant near-ring**.

Note 1.22:

(i) η_0 stands for the class of all zero symmetric near-rings.

(ii) η_c stands for the class of all constant near-rings.

Definition 1.23:

N is **regular** if for every x in N there is some y in N such that x=xyz.

Definition 1.24:

A map f: N → N is called a mate function if for all x in N, x=xf(x)x. f(x) is called a **mate** of x.

Note 1.25:

N^* denotes the set of all non-zero elements of N, (i.e.) N^*=N-{0}.

Definition 1.26:

The zero symmetric part of N is {n ∈ N/ n0=0} and is denoted by N_0. N is called **zero symmetric** if N =N_0.

Definition 1.27:

N is subdirectly irreducible if and only if the intersection of any family of non-zero ideals is again non-zero.

Definition 1.28:

N has **IFP (Insertion of Factors property)** if for $x,y \in N$, $xy=0 \Rightarrow xny=0$ for all $n \in N$.

Definition 1.29:

N is said to be **Property P_4** if for all ideals I of N, $xy \in I \Rightarrow yx \in I$.

Definition 1.30:

N has **Strong IFP** if for all ideals I of N $ab \in I \Rightarrow anb \in I$ for all $a,b,n \in N$.

Note 1.31:

Every simple near-ring is subdirectly irreducible.

Theorem 1.32:

The following are equivalent for a zero symmetric near-ring.

1) N is a near-field.

2) $N_d \neq \{0\}$ and for all $n \in N-\{0\}$, $Nn = N$

Definition 1.33:

N is called **Boolean** if and only if $a^2 = a$ for all $a \in N$.

Definition 1.34:

N is called a **nil near-ring** if every element of N is nilpotent.

Definition 1.35:

N is said to be **subcommutative** if Na = aN for all a ∈ N.

Definition 1.36:

N is said a$\mathbf{P_k}$ **near-ring** ($\mathbf{P'_k}$ **near-ring)** if there exists a positive integer k such that $x^kN = xNx$.

Definition 1.37:

Clearly {0} and N are ideals of N. These are called **trivial** ideals.

Definition 1.38:

N is **simple** if and only if N has no non-trivial ideals.

Definition 1.39:

N is called **weak commutative** if abc = acb ∀ a,b,c ∈ N.

Definition 1.40:

If N is a strong S_1 near-ring then N is zero symmetric.

Definition 1.41:

N is a strong S_1 near-ring if and only if axa = xa for all a,x ∈ N

CHAPTER 2: S_1- NEAR RING AND ITS SUBSET

Section 2.1: S_1- NEAR RING

Definition 2.1.1:

N is called an **S_1- NEAR RING** if for every $a \in N$, there exists $x \in N^*$ such that $axa = xa$.

Theorem 2.1.2:

Let N be an S_1- near ring

1) If $ax = 0$ then $xa = 0$

2) If $ax \in E$ then $xa \in E$

3) If the right cancellation law is valid in N then $xa \in E$ implies $ax \in E$, for all $a \in N$ and for some $x \in N^*$

Proof:

Let $a \in N$,

Since N is an S_1- near ring there exists $x \in N^*$ such that $axa = xa$ ⇢(1)

1) If $ax = 0$ then from (1) we get, $xa = 0a = 0$.

Thus $xa = 0$

2) If $ax \in E$ then $(xa)^2 = ax$ ⇢(2)

Now $(xa)^2 = (xa)xa = (axa)xa$ (by(1))

$= (ax)^2a = (axa)$ (by(2))

$= axa = xa$. That is $(xa)^2 = xa$

3) Now $(ax)^2a = (ax)a$.

Since the right cancellation law is valid in N, $(ax)^2 = ax$

Thus $ax \in E$.

Theorem 2.1.3:

Let N be an S_1- near ring without non-zero zero divisors. If N is commutative then N is Boolean.

Proof:

Let $a \in N$

Since N is an S_1- near ring, there exists $x \in N^*$ such that $axa = xa$.

Since N is commutative, $a(ax) = ax \Rightarrow (a^2-a)x = 0$

Since N has no non-zero zero divisors, $a^2-a = 0$.

Consequently N is Boolean.

Theorem 2.1.4:

Let N be a nil near-ring, then N is an S_1- near ring if and only if N is zero symmetric.

Proof:

For the only if part,

We take $a \in N$.

Since N is an S_1- near ring, there exists $x \in N^*$ such that $axa = xa \dashrightarrow (1)$

We shall prove that $ax^ka = x^ka \dashrightarrow (2)$ for all positive integers k.

We use induction on k, eqn (1) demands (2) is true for k =1

Assume that the result is true for k = s-1.

If k= s then $ax^sa = ax^{s-1}(xa) = ax^{s-1}(axa)$ (by(1)) $= (ax^{s-1}a)xa = (x^{s-1}a)xa =$ $x^{s-1}(axa) = x^{s-1}(xa) = x^sa$.

Thus $ax^ka = x^ka$ for all positive integers k.

Since $ax^ta = x^ta$, $a0a = 0a$

$\Rightarrow a0 = 0$.

Thus N is zero symmetric.

For the if part,

Let $a \in N$

Since N is nil, there exists a positive integer k > 1 such that $a^k = 0$

N is zero symmetric. This implies xa = 0, where $x = a^{k-1}$.

Therefore axa = a(xa) = a0 = 0

$[\because N=N_0]$ = xa.

Thus N is an S_1- near ring.

Theorem 2.1.5:

Let N be a Boolean near-ring. Each of the following statements implies that N is an S_1- near ring.

(i) N is zero symmetric

(ii) N is an IFP near-ring with identity

(iii) Na = aNa for all a∈ N (N is a P_1' near-ring)

(iv) N is subcommutative

(v) N is distributive.

Proof:

(i) Let N be a zero symmetric near-ring.

Let a ∈ N

If a ≠ 0, we take x = a.

Then axa = a^2a = aa [since N is Boolean] = xa.

That is axa = xa.

If a = 0 then for any x ∈ N^*, axa = 0 = xa [since N=N_0].

Consequently N is an S_1- near ring.

(ii) Let N be an IFP near-ring with identity '1' and let a ∈ N. Since N is Boolean, a^2 = a ⇒ a^2- a = 0 ⇒ (a-1)a = 0.

Since N has IFP, (a-1)xa = 0 for all x ∈ N.

In particular (a-1)xa = 0 for any x ∈ N^* ⇒ axa-xa = 0 ⇒ axa = xa.

Thus N is an S_1- near ring.

(iii) Let a ∈ N

Since Na = aNa, for any x ∈ N, there exists y ∈ N such that xa = aya.

Therefore axa = a(xa) = a(aya) = a^2ya = aya

[Since N is Boolean] = xa and (iii) follows.

(iv) Let a ∈ N

Since N is subcommutative, Na = aN.

Therefore for any x ∈ N, there exists y ∈ N such that xa = ay.

Therefore axa = a(xa) = a(ay) = a^2y = ay [since N is Boolean] = xa.

That is axa = xa for all x ∈ N.

In particular axa = xa for any x ∈ N^*.

Thus N is an S_1- near ring.

(v) Let N be a distributive near-ring. Since every distributive near-ring is zero symmetric, the result follows from (i).

Section 2.2: THE SUBSET OF S_1 NEAR-RING N_{S_1}(a), a ∈ N

Definition 2.2.1:

N is called an **S_1- near ring** if for every a ∈ N, there exists x ∈ N^* such that axa = xa.

Notation 2.2.2:

For any a ∈ N, we can denote {x ∈ N^*/axa = xa} by N_{S_1}(a).

Remark 2.2.3:

It follows that N is an $\mathbf{S_1}$**- near ring** if and only if $N_{S_1}(a) \neq 0$ for all a ∈ N.

Example 2.2.4:

Let (N,+,·) be the near-ring where (N,+) is the klein's four group N = {0,a,b,c} and the semigroup operation '·' is defined as follows

·	0	a	b	c
0	0	0	0	0
a	0	a	a	A
b	0	b	b	B
c	0	c	c	C

Clearly this is an S_1- near ring. We observe that $N(x) \neq \emptyset$ for all x ∈N.

$(N_{S_1}(0) = \{a,b,c\}$, $N_{S_a}(a) = \{a\}$, $N_{S_b}(b) = \{b\}$, $N_{S_c}(c) = \{c\}$

Theorem 2.2.5:

Let N be an S_1- near ring. If a ∈ $N_{S_1}(a)$ for all a ∈N then N is regular.

Proof:

Let a ∈N.

By hypothesis, a = xa for some x ∈ $N_{S_1}(a)$.

Since, x∈ $N_{S_1}(a)$, axa = xa.

Therefore a = axa.

Thus N is regular.

Remark 2.2.6:

Converse of theorem 2.2.5 is not valid. Consider the near-ring (N,+,·) where (N,+) is the group of integers modulo 6 and '·' is defined as follows.

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

This S_1- near ring is regular. But $2 \notin N_{S_1}(2)$ and $5 \notin N_{S_1}(5)$.

Lemma 2.2.7:

Let N be an S_1- near ring. Then N_{S_1}(a) has no non-zero zero divisors if and only if N_{S_1}(a) is a multiplicative system.

Proof:

Since N is a S_1- near ring, $N_{S_1}(a) \neq \emptyset$ for all $a \in N$

Now, let x,y $\in N_{S_1}(a)$. Then x, y $\in N^*$

Then axa = xa, aya = ya

Then, it follows a(xy)a = ax(ya) = ax(aya) = (axa)ya = (xa)ya = x(aya) = x(ya) = (xy)a

Further, $N_{S_1}(a)$ has no non-zero zero divisors, $xy \neq 0$

Consequently, $xy \in N_{S_1}(a)$

Thus, $N_{S_1}(a)$ is a multiplicative system.

Conversely, let $x,y \in N_{S_1}(a)$

Since, $N_{S_1}(a)$ is a multiplicative system, $xy \in N_{S_1}(a)$

$\therefore N_{S_1}(a) \subset N^*$, it follows that $xy \neq 0$

Hence, $N_{S_1}(a)$ has no non-zero divisor.

Theorem 2.2.8:

N is zero symmetric if and only if $N^* = N_{S_1}(0)$

Proof:

Let $x \in N^*$

Since $N = N_0$

$x0 = 0 \Rightarrow 0x0 = 0 = x0$

$\Rightarrow x \in N_{S_1}(0)$

Clearly, $N^* = N_{S_1}(0)$

Conversely, $N^* = N_{S_1}(0)$

$\Rightarrow 0x0 = x0$ for all $x \in N^*$

$\Rightarrow x0 = 0 \therefore$ N is zero symmetric.

CHAPTER 3 : STRONG S_1-NEAR RINGS

Definition 3.3.1:

A near-ring N is said to be **STRONG S_1-NEAR RING** if $N^* = N_{S_1}(a)$ for all a∈N.

Example 3.3.2:

Let (N,+) be the symmetric group of degree 3 with N={0,a,b,c,x,y} and we define '·' as follows

·	0	A	b	c	x	y
0	0	0	0	0	0	0
a	0	A	b	c	0	0
b	0	A	b	c	0	0
c	0	A	b	c	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

This near-ring is a strong S_1-near ring. It is worthy nothing that it is not regular.

Theorem 3.3.3:

N is a strong S_1-near ring if and only if axa = xa for all a ∈ N and for all x∈ N^*

Proof:

By defn, $N^* = N_{S_1}(a)$

axa = xa

Converse part, axa = xa

By theorem 2.2.5, Let a ∈ N

By hypothesis, $a \in N_{S_1}(a)$

$axa = xa \Rightarrow a = xa$ for all $x \in N_{S_1}(a)$

Since, $x \in N_{S_1}(a)$

$axa = xa \Rightarrow axa = a$ (∵xa = a)

N is regular.

⇒ N is zero symmetric ($N = N_0$)

$\Leftrightarrow N^* = N_{S_1}(a)$

∴ N is Strong S_1-near ring.

Corollary 3.3.4:

Every Strong S_1-near ring is an S_1-near ring.

Proof:

By above theorem.

Theorem 3.3.5:

If N is a Strong S_1-near ring then N is zero symmetric.

Proof:

Since N is a Strong S_1-near ring

From theorem 3.3.3

axa = xa for all a ∈ N and for all x ∈ N^*

Putting a = 0,

0x0 = x0 for all x ∈ N^*

⇒ x0 = 0 for all x ∈ N^*

By defn, zero symmetric of N is {x ∈ N / x0=0}

∴ N is zero symmetric.

We furnish below the characterization of Strong S_1-near rings.

Theorem 3.3.6:

N is Strong S_1-near ring if and only if axa = xa for all a,x ∈ N.

Proof:

If N is a Strong S_1-near ring

Then from theorem 3.3.5

N is zero symmetric

⇒ a0 = 0 for all a ∈ N

⇒ a0a = 0 = 0a for a ∈ N

axa = xa

$\Rightarrow 0a0 = 0a$ for all $a \in N$

Theorem 3.3.7:

Let N be a Strong S_1-near ring. Then

(i) ab and ba $\in$ E for all a,b $\in$ N.

(ii) N has (*IFP).

(iii) N has Strong IFP.

(iv) N has Property (P_4).

Proof:

Let N be a Strong S_1-near ring. Then it follows from Theorem 3.3.8 that $axa = xa$ for all $a,x \in N$ ⇢ (1)

(i) Let a,b $\in$ N. Now (1) implies that $ab = bab = (ba)b = (aba)b = (ab)^2$

$\Rightarrow ab \in E$.

Again (1) implies that $ba = aba = (ab)a = (bab)a = (ba)^2$

$\Rightarrow ba \in E$.

(ii) Suppose $xy = 0$ for $x,y \in N$ ⇢ (2)

Now, $yx = xyx$ (by 1)

$\Rightarrow (xy)x = 0x = 0$

Also, for every $n \in N$, $xny = x(ny) = x(yny) = (xy)ny = 0ny = 0$

That is $xny = 0$

∴ N has (*IFP)

(iii) Let I be an ideal of N

Suppose ab ∈ I for a,b ∈ N

By theorem 3.3.5,

N is zero symmetric and

∴ $NI \subset I$ ⇢ (3) and $IN \subset I$ ⇢ (4)

Now, for any n∈ N, anb = (an)b = (nan)b = na(nb) = na(bnb)

= n(ab)nb ∈ NIN = (NI)N ⊂ IN ⊂ I

That is anb ∈ I

∴ N has Strong IFP.

(iv) Let I be an ideal of N

Suppose xy ∈ I for x,y ∈ N

By (iii), IN ⊂ I and NI ⊂ I

Now, $(yx)^2$ = yxyx = y(xy)x ∈ NIN = (NI)N ⊂ IN ⊂ I

$(yx)^2 \in I$

From (i) we get yx = $(yx)^2 \in I$

(i.e) yx ∈ I

Consequently, N has (P_4)

Theorem 3.3.8:

Let N be a Boolean near-ring. Each of the following statements implies that N is a Strong S_1-near ring,

1) N is commutative

2) N is an IFP near-ring with identity

3) N is a P_1' near-ring

4) N is sub commutative

Proof:

1) Let N be a commutative near-ring and let a,b ∈ N

Now aba = a(ba)

= a(ab) (∵ N is commutative)

= a^2b

= ab (since N is Boolean)

= ba

2) Let N be an IFP near-ring with identity '1' and let a ∈ N.

Since N is Boolean, $a^2 = a \Rightarrow a^2 - a = 0$

⇒ (a-1)a = 0

Since N has IFP, (a-1)xa = 0 for all x ∈ N,

⇒ axa-xa = 0 ⇒ axa = xa.

Thus N is an Strong S_1-near ring.

3) Let $a \in N$,

Since $Na = aNa$, for any $x \in N$, there exists $y \in N$ such that $xa = aya$

Therefore, $axa = a(xa)$

$= a(aya)$

$= a^2ya$

$= aya$ (N is Boolean)

$= xa$

4) Let $a \in N$,

Since $Na = aN$, for any $x \in N$, there exists $y \in N$ such that $xa = ay$

Therefore, $axa = a(xa)$

$= a(ay)$

$= a^2y$

$= ay$ (N is Boolean)

$= xa$

$\Rightarrow axa = xa$.

Hence N is an Strong S_1-near ring.

Theorem 3.3.9:

Let N be a strong S_1-near ring. If N has no non-zero zero divisors then we have the following:

1) N is simple

2) N is sub directly irreducible

3) N is distributive

4) N is a near-field

5) N is Boolean

6) N is regular

Proof:

1) Suppose N has a non-trivial ideal I

Let i be a non-zero element of I. Now let $y \in N$. Since N is a Strong S_1-near ring by theorem 3.3.8, $iyi = yi \Rightarrow (iy-i)I = 0$

Since N has no non-zero zero divisor, $iy-y = 0$

$\Rightarrow iy = y$. That is $y = iy \in IN \subset I$ [I is an ideal of N]

$\Rightarrow y \in I$. Therefore $N \subset I$ and hence $I = N$.

Thus N is simple.

2) Follows from (1) and from theorem 1.31

3) Let $x,y \in N$.

Clearly, $0 \in N_d$. Let $b \in N^*$.

Since N is strong S_1, $b(x+y)b = (x+y)b = xb+yb = bxy + byb = (bx+by)b$

That is $b(x+y)b = (bx+by)b \Rightarrow (b(x+y)-(bx+by))b = 0$

Since N has no non-zero zero divisors, $b(x+y)-(bx+by) = 0$

$\Rightarrow b(x+y) = bx+by$. Consequently N is distributive.

4) Let $n \in N$.

Since $ana = na$ for all $a \in N^*$, $(an-a)a = 0$

Since N has no non-zero zero divisors, $an-n = 0$

That is $na \subset an \in Nn$.

Consequently $N = Nn \Rightarrow$ N is a near-field (by (iii) and theorem 1.32)

5) Let $a \in N^*$

Since $aaa = aa$, $(a^2-a)a = 0$.

Since N has no non-zero zero divisors, $a^2-a = 0$

$\Rightarrow a^2 = a$.

Consequently N is Boolean.

6) From (5) we get, N is Boolean and we know that every Boolean near-ring is Regular.

Theorem 3.3.10:

Let N be a strong S_1near-ring. Then N is rregular if and only if N is Boolean.

Proof:

For the only if part,

Let $a \in N$.

Since N is regular, there exists $x \in N$ such that $axa = a$.

Therefore $a = xa$. Now $a^2 = aa = a(xa) = a$.

That is $a^2 = a$. Thus N is Boolean.

The proof of if part is obvious.

Theorem 3.3.11:

Let N be a strong S_1near-ring. Then the following are true:

1) Every right identity of N is a left identity of N.

2) xy is a left identity if and only if x and y are left identities for all $x,y \in N$.

3) If $(0 : xy) = \{0\}$ then xy is the identity for all $x,y \in N$.

4) $(0 : xy) = (0 : yx)$ for all $x,y \in N$.

Proof:

Since N is a Strong S_1near-ring, $aba = ba$ for all $a,b \in N$.

1) If e is the right identity of N then $xe = x$ for all $x \in N$.

Now $xe = exe = e(xe) = ex \Rightarrow x = ex$

That is $ex = x$ for all $x \in N$.

Thus e is a left identity of N.

2) Let $x,y \in N$

Assume that xy is a left identity

Let $n \in N$.

Therefore, $xyn = n \Rightarrow y(xyn) = yn \Rightarrow (yxy)n = yn$

$\Rightarrow (xy)n = yn \Rightarrow n = yn$.

That is $yn = n$

Therefore $xn = n$ and $yn = n$ for all $n \in N$.

Now $(xy)n = x(yn) = xn = n$.

That is $(xy)n = n$ for all $n \in N$.

Then xy is a left identity.

3) Let $z \in N$

Now $(z\text{-}zxy)xy = zxy\text{-}z(xy)^2$

$= zxy - zxy$ [$\because xy \in E$ by theorem 3.3.11]

$= 0$

Therefore $z - zxy \in (0 : xy)$.

Since $(0 : xy) = \{0\}$, $z - zxy = 0$

$\Rightarrow z - zxy$ that is $zxy - z$. Thus xy is a right identity of N.

Now (1) implies that, it is a left identity as well and (3) as follows.

4) Let $n \in (0 : xy) \Rightarrow nxy = 0$

Now $nyx = n(yx) = n(xyx)$

$= (nxy)x = 0x = 0.$

That is $nyx = 0 \Rightarrow n \in (0 : yx)$

Therefore $(0 : xy) \subset (0 : yx) \dashrightarrow (1)$

Similarly let $n \in (0 : yx) \Rightarrow nyx = 0.$

Now $nxy = n(xy) = n(yxy)$

$= (nyx)y = 0y = 0.$

That is $nxy = 0 \Rightarrow n \in (0 : xy)$

Therefore $(0 : yx) \subset (0 : xy) \dashrightarrow (2)$

From (1) and (2) we get,

$(0 : xy) = (0 : yx)$

Theorem 3.3.12:

Any homomorphic image of a Strong S_1near-ring is a Strong S_1near-ring.

Proof:

Let N be a strong S_1near-ring and let $f : N \longrightarrow N'$ be a homomorphism.

Since N is strong S_1, by theorem 3.3.8

$xyx = yx$ for all $x,y \in N$

Let $x',y' \in N'$

Then there exists $x,y \in N$ such that $f(x) = x'$ and $f(y) = y'$

Clearly then, $x'y'x' = y'x'$ and the desired result follows.

Theorem 3.3.13:

Every Strong S_1 near-ring is isomorphic to a subdirect product of subdirectly irreducible Strong S_1 near-rings.

Proof:

By defn (Every near-ring is isomorphic to a subdirect product of subdirectly irreducible near-rings.)

N is isomorphic to a subdirect product of subdirectly irreducible near-rings N_i's say and each N_i is a homomorphic image of N under projection map $\prod_i$. The desired result now follows from Theorem 3.3.10.

Theorem 3.3.14:

Let N be a Strong S_1 near-ring with mate function f. Then N is subdirectly irreducible if and only if N is simple.

Proof:

Since N is Strong S_1, by theorem 3.3.5 N is zero symmetric

Suppose N is subdirectly irreducible. First we prove that for any non-zero idempotent e in N, $(0 : e) = \{0\}$. Let $D = \{e \in E - \{0\} / (0 : e) \neq \{0\}\}$

Suppose $D \neq \emptyset$.

Let $B = \bigcap_{e \in D} (0 : e)$

Now, theorem 3.3.9 demands that N has (*IFP)

From theorem 3.3.6 and R(4) we see that $(0 : e)$ is an ideal.

Since N is subdirectly irreducible. R(1) shows that $B \neq \{0\}$

Let $a \in B\text{-}\{0\} \Rightarrow ae = 0$ for all $e \in D \dashrightarrow (1)$

Now $f(a)ae = f(a)0 = 0$ [since $N = N_0$] $\Rightarrow ef(a)a = 0 \Rightarrow e \in (0 : f(a)a)$

$\Rightarrow f(a)a \in D \Rightarrow af(a)a = 0$ [by (1)] $\Rightarrow a = 0$ which is a contradiction to $a \neq 0$.

Consequently, for any non-zero idempotent e in N, $(0:e) = \{0\}$.

Since N is Strong S_1 near-ring from Theorem 3.3.8 we get $exe = xe$

$\Rightarrow (ex-x)e = 0 \Rightarrow ex-x \in (0:e) = \{0\}$

$\Rightarrow ex = x$ for all $x \in N$, (i.e) $x = ex \in Nx \Rightarrow N = Nx$ for all $x \in N$.

Thus N is Simple.

Converse is obvious by proposition 1.38.

We conclude our discussion with the following structure theorem for Strong S_1 near-rings.

Theorem 3.3.15:

Every Strong S_1 near-ring with a mate function is isomorphic to a subdirect product of simple near-rings.

Proof:

Collecting the pieces proved in theorem 3.3.13 and 3.3.14 we get the desired results.

CHAPTER 4 : B_1 NEAR-RING

SECTION 4.1: B_1 NEAR-RING

Definition 4.1.1:

N is a B_1 near-ring if for every $a \in N$, there exists $x \in N^*$ such that $Nax = Nxa$.

Example 4.1.2:

Every constant near-ring is a B_1 near-ring.

Theorem 4.1.3:

Let N be a near-ring. Each of the following statements implies that N is a B_1 near-ring.

(i) N is a zero symmetric nil near-ring.

(ii) N is weak commutative

(iii) N has identity 'I'

(iv) N is a near-field.

Proof:

(i) Let $a \in N$.

If $a = 0$, then for any $x \in N^*$, $Nax = Nxa = N0 = \{0\}$.

If $a \in N^*$, since N is nil, there exists a positive integer k such that $a^k = 0$.

Put $x = a^{k-1} \neq 0$.

Now $Nax = Naa^{k-1} = Na^{k} = Na^{k-1}a \Rightarrow Nxa = N0 = \{0\}$.

Thus N is a B_1 near-ring.

(ii) Let $a \in N$.

For any $x \in N^{*}$, $y \in Nax$

$\Rightarrow y = nax$, where $n \in N$.

Since N is weak commutative, $y = nxa \in Nxa$.

Therefore, $Nax \subset Nxa$.

Similarly, $Nxa \subset Nax$ and hence N is a B_1 near-ring.

(iii) Follows by taking x = 1 in the definition 4.1.1

(iv) Follows from (iii)

Theorem 4.1.4:

Let N be a B_1 near-ring. If N is a Strong S_1 near-ring without non-zero zero divisors then the following are true.

(i) Every non-zero N-subgroup of N is an B_1 near-ring.

(ii) Every non-zero ideal of N is an B_1 near-ring.

Proof:

Since N is a Strong S_1 near-ring,

N is zero symmetric and $aba = ba$ for all $a,b \in N$ ⇢ (1)

(i) Let M be an N-subgroup of N and let $m \in M$.

Then To prove M is a B_1 near-ring.

If $m = 0$ then for any $x \in N^*$, $Nmx = N0 = \{0\}$ [since N is zero symmetric] $= Nxm$.

If $m \neq 0$

Since N is a B_1 near-ring, there exists $y \in N^*$ such that $Nmy = Nym \dashrightarrow (2)$

Let $n = ym$

It follows that $n \in M^*$.

Now, $Mmn = Mn(ym)$

$\subset Nm(ym)$

$= (Nmy)m$

$= N(ym)m$ [by (2)]

$= N(mym)m$ [by (1)]

$= Nm(ym)m \subset M(ym)m = Mnm$.

That is $Mmn \subset Mnm \dashrightarrow (3)$

In a similar fashion we get $Mnm \subset Mmn \dashrightarrow (4)$

From (3) and (4) we get,

$Mmn = Mnm$.

Consequently, M is a B_1 near-ring.

(ii) Since N is zero symmetric,

It demands that every ideal of N is an N-subgroup of N and now (ii) follows from (i).

Theorem 4.1.5:

Let N be a B_1 near-ring. Then for every $a \in N$, there exists $x \in N^*$ such that the following are true.

(i) There exists $n \in N$ such that $axa = nax$.

(ii) $Nax \subset Na \cap Nx$

(iii) If N is Boolean then $Naxa = Nxa$.

(iv) If N is a Strong S_1 near-ring then there exists $n \in N$ such that $xa = nax$.

Proof:

Let $a \in N$.

Since N is a B_1 near-ring, there exists $x \in N^*$ such that $Nax = Nxa \dashrightarrow (1)$.

(i) Since $axa \in Nxa$,

By using (1) we get,

$axa = nax$ for some $n \in N$ and (i) follows.

(ii) From (1) we get,

$Nax = Nxa \subset Na$.

Obviously $Nax \subset Nx$.

Therefore, $Nax \subset Na \cap Nx$

(iii) When N is Boolean,

$Nxa = Nxa^2 = (Nxa)a = (Nax)a$ [by(1)]

and (iii) follows.

(iv) Since N is a Strong S_1 near-ring, the result follows from (i).

SECTION 4.2: STRONG B_1 NEAR-RINGS

Definition 4.2.1:

We say that N is a **Strong B_1 near-ring** if $Nab = Nba$ for all $a,b \in N$.

Example 4.2.2:

Every commutative near-ring is a strong B_1 near-ring.

Theorem 4.2.3:

Every Strong B_1 near-ring is a B_1 near-ring.

Proof:

Straight forward.

Remark 4.2.4:

Converse of theorem 4.2.3 is not valid.

Remark 4.2.5:

It is obvious that the property of N being Strong B_1 is preserved under near-ring homomorphisms.

Theorem 4.2.6:

Every Strong B_1 near-ring is isomorphic to a subdirect product of subdirectly irreducible strong B_1 near-rings.

Proof:

By theorem 1.62, p.26 of Pilz

[N is isomorphic to a subdirect product of subdirectly irreducible near-rings N_i's say, and each N_i is a homomorphic image of N under the usual projection map $\prod$ i]

By remark 4.2.5, Every Strong B_1 near-ring is isomorphic to a subdirect product of subdirectly irreducible Strong B_1 near-ring.

Lemma 4.2.7:

If N is a Strong B_1 near-ring if and only if for all a,b,c $\in$ N, there exists n $\in$ N such that abc = ncb.

Proof:

"Only if" part,

Let a,b,c $\in$ N

Now abc $\in$ Nbc

Since N is Strong B_1 near-ring, Nbc = Ncb.

Therefore, abc $\in$ Ncb and this implies that abc = ncb for some n $\in$ N.

"If" part,

Let a,b,c $\in$ N

Now abc ∈ Nbc.

From our assumption, there exists n ∈ N such that abc = ncb ∈ Ncb.

Therefore, Nbc ⊂ Ncb.

In a similar fashion we get,'

Ncb ⊂ Nbc.

Thus N is a strong B_1 near-ring.

Theorem 4.2.8:

Let N be a Strong B_1 near-ring. If N is regular then we have the following:

(i) For every a ∈ N, there exists x ∈ N such that $a = a^2x$.

(ii) N has no non-zero nilpotent elements.

(iii) Any two principal N-subgroup of N commute with each other.

(iv) N is a P_1 near-ring

(v) N is left bipotent.

Proof:

Since N is regular, For every a ∈ N, there exists x ∈ N such that a = axa ⇢ (1)

(i) Since N is a strong B_1 near-ring, lemma 4.2.7 gurantees that there exists n ∈ N such that axa = nax ⇢ (2).

From (1) and (2) we get

a = nax ⇢ (3).

Now $na = n(axa)$ [by (1)]

$= (nax)a$

$= aa$ [by (3)]

$= a^2$.

That is $a^2 = na \dashrightarrow (4)$.

Using (4) in (3) we get,

$a = a^2x$.

(ii) Let $a \in N$.

Suppose $a^2 = 0$.

Now, (i) demands that there exists $x \in N$ such that $a = a^2x$ and

Therefore $a = 0$.

Now R(1) guarantees that N has no non-zero nilpotent elements.

(iii) First we show that $NaN = Na$ for all $a \in N$.

Let $y \in NaN$.

Then $y = nan'$ for some $n, n' \in N \dashrightarrow (5)$.

Now lemma 4.2.7 demands that $nan' = zn'a$ for some $z \in N \dashrightarrow (6)$.

Combining (5) and (6) we get,

$y = zn'a = (zn')a \in Na$.

Therefore $NaN \subset Na \dashrightarrow (7)$.

Also from (1) we get,

$Na = Naxa = Na(xa) \subset NaN$.

That is $Na \subset NaN \dashrightarrow (8)$.

From (7) and (8) we get,

$NaN = Na \dashrightarrow (9)$.

Let $b,c \in N$.

Now $NbNc = (NbN)c = (Nb)c$ [by (9)]

$= Nbc = Ncb$ [since N is a strong B_1 near-ring]

$= (Nc)b = (NcN)b$ [by (9)] $= NcNb$.

That is $NbNc = NcNb$ and (iii) as follows.

(iv) For any $a \in N$,

Let $y \in aN$.

Then there exists $z \in N$ such that $y = az = (axa)z$ [by (1)]

$= a(xaz)$.

That is $y = a(xaz) \dashrightarrow (10)$.

Now lemma 4.2.7 demands that there exists $n \in N$ such that $xaz = nza \dashrightarrow (11)$.

From (10) and (11) we get,

$y = a(nz)a \in aNa$.

Therefore $aN \subset aNa \dashrightarrow (12)$.

Obviously $aNa \subset aN \dashrightarrow (13)$.

From (12) and (13) we get,

$aNa = aN$.

Thus N is a P_1 near-ring.

(v) From (1), we get

$Na = Naxa = (Nax)a = (Nxa)a$ [Since N is a strong B_1 near-ring]

$= Nxa^2 \subset Na^2$ [since $Nx \subset N$].

Therefore $Na \subset Na^2$

Consequently, $Na = Na^2$.

Thus N is left bipotent.

Corollary 4.2.9:

Let N be a zero symmetric strong B_1 near-ring. If N is regular then N is the subdirect product of integral near-rings.

Proof:

Let N be a strong B_1 near-ring.

Since N is regular. Theorem 4.2.8 (ii) guarantees that, N has non-zero nilpotent elements.

As N is zero symmetric, the desired result now follows from R(3).

Theorem 4.2.10:

Let N be a strong B_1 near-ring. If N is Boolean then the following are true.

(i) NaNb = Nab for all a,b $\in$ N.

(ii) All principal N-subgroups of N commute with one another.

(iii) Every ideal of N is a strong B_1 near-ring

(iv) Every N-subgroup of N is a strong B_1 near-ring.

(v) Every N-subgroup of N is an invariant N-subgroup of N.

Proof:

Since N is a strong B_1 near-ring, Nab = Nba ⇢ (1)

(i) Let a,b $\in$ N.

Since N is Boolean, $a = a^2 \in aN$.

Thus we have $a \in aN$

$\Rightarrow Na \subset NaN \Rightarrow Nab \subset NaNb$.

For the reverse inclusion $y \in NaNb$

$\Rightarrow y = nan'b$ for some $n,n' \in N$ ⇢ (2).

Since N is a strong B_1 near-ring,

By using lemma 4.2.7, we get

$nan' = zn'a$ where $z \in N$.

Therefore from (2) we get,

$y = zn'ab = (zn')ab \in Nab$.

The desired now follows.

(ii) Let $a,b \in N$.

Now $NaNb = Nab$ [by (i)]

$= Nba$ [by (1)] $= NbNa$ [by (i)] and (ii) follows.

(iii) Let I be any ideal of N.

Let $a,b \in I$.

Now $Iab = Ia^2b$ [since N is Boolean]

$= (Ia)ab \subset I(Nab) = I(Nba)$ [by (1)] $\subset Iba$

That is $Iab \subset Iba$.

Similarly we get, $Iba \subset Iab$.

Consequently, I is a strong B_1 near-ring.

(iv) Let M be an N-subgroup of N.

Therefore, $NM \subset M$ ⇢ (3).

Let $x,y \in M$.

Let $z \in Mxy \subset Nxy = Nyx$ [by (1)]

$= Ny^2x$ [since N is Boolean]

$= (Ny)yx \subset (NM)yx \subset Myx$ [by (3)].

Therefore, $Mxy \subset Myx$.

Similarly we get, $Myx \subset Mxy$.

Consequently M is a strong B_1 near-ring.

(v) Let M be a N-subgroup of N.

Let $z \in MN$

$\Rightarrow z = mn = m^2n$ for some $m \in M$ and $n \in N$ ⇢ (4)

Since N is a strong B_1 near-ring, lemma 4.2.7 demands that there exists $n' \in N$ such

that $m^2n = n'nm$ ⇢ (5)

From (4) and (5) we get,

$zn = n'nm \in NM \subset M$ [since M is an N-subgroup of N].

Therefore, $MN \subset M$.

Thus M is an invariant N-subgroup of N.

We conclude our discussion with the following characterization of strongB_1near-rings.

Theorem 4.2.11:

Let N be a Boolean near-ring. Then N is a strong B_1 near-ring if and only if $Na \cap Nb = Nab$ for all $a,b \in N$.

Proof:

Only if part,

Let $y \in Na \cap Nb$.

$\Rightarrow$ $y \in Na$ and $y \in Nb$

Therefore $y = na = n'b$ for some $n, n' \in N$.

Now by lemma 4.2.7, there exists $z \in N$ such that $y^2 = (na)(n'b)$

$= (nan')b = (zn'a)b = (zn'a)b \in Nab$.

$\Rightarrow$ $y^2 \in Nab$

Since N is a Boolean, this yields $y \in Nab$.

Thus $Na \cap Nb \subset Nab \dashrightarrow (1)$

Since N is a strong B_1 near-ring, $Nab = Nba$.

But $Nba \subset Na$ and $Nab \subset Nb$.

Hence $Nab \subset Na \cap Nb \dashrightarrow (2)$

From (1) and (2) we get,

$Na \cap Nb = Nab$.

If part,

Let $a,b \in N$

Now $Nab = Na \cap Nb = Nb \cap Na = Nba$.

Thus N is a strong B_1 near-ring.

A STUDY ON *L*-FUZZY IDEALS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

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in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

P.SUBITHRA

Reg.No: 19SPMT26

Under the guidance of

Dr. P. ANBARASI RODRIGO M.Sc., B.Ed., Ph.D.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON *L*-FUZZY IDEAL"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **P.SUBITHRA (Reg. No: 19SPMT26)**

Signature of the Guide

Signature of the Coordinator

Joyce
Signature of the Director
Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose
Signature of the Principal
Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON *L*-FUZZY IDEALS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Dr. P. Anbarasi Rodrigo M.Sc., B.Ed., Ph.D.,** Assistant Professor, Department of Mathematics (SSC), **St.Mary's College (Autonomous), Thoothukudi.**

P. Subithra

Station: Thoothukudi **Signature of the Student**

Date: 10. 04. 2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER-1

PRELIMINARIES

Definition: 1.1

A set S together with two associative binary operations called addition and multiplication (denoted by + and · respectively) will be called semiring provided

1) Addition is a commutative operation.
2) Multiplication distributes over addition both from the left and from the right.
3) There exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$.

Definition: 1.2

Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

1) $x\alpha(y + z) = x\alpha y + x\alpha z,$
2) $(x + y)\alpha z = x\alpha z + y\alpha z,$
3) $x(\alpha + \beta)y = x\alpha y + x\beta y,$
4) $x\alpha(y\beta z) = (x\alpha y)\beta z,$

then M is called a Γ-semiring.

Definition: 1.3

A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$.

Definition: 1.4

Let M be a Γ-semiring and A be a non-empty subset of M. A is called a Γ-subsemiring of M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

Definition: 1.5

Let M be a Γ-semiring. A subset A of M is called a left ideal of M if A is closed under addition and $M\Gamma A \subseteq A$.

Definition: 1.6

Let M be a Γ-semiring. A subset A of M is called a right ideal of M if A is closed under addition and $A\Gamma M \subseteq A$.

Definition: 1.7

A is called an ideal of M if it is both left and right ideal.

Definition: 1.8

Let M be a non-empty set. A mapping $f: M \to [0,1]$ is called a fuzzy subset of M.

Definition: 1.9

Let f be a fuzzy subset of a non-empty subset M, for $t \in [0,1]$ the set $f_t = \{x \in M / f(x) \geq t\}$ is called level subset of M with respect to f.

Definition: 1.10

Let M be a Γ-semiring. A fuzzy subset μ of M is said to be a fuzzy Γ-subsemiring of M if it satisfies the following conditions

1) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
2) $\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\}, for\ all\ x, y \in M, \alpha \in \Gamma$.

Definition: 1.11

A fuzzy subset μ of a Γ-semiring M is called a fuzzy left ideal of M if for all $x, y \in M, \alpha \in \Gamma$

1) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
2) $\mu(x\alpha y) \geq \mu(y)(\mu(x))$

Definition: 1.12

A fuzzy subset μ of a Γ-semiring M is called a fuzzy ideal of M if for all $x, y \in M, \alpha \in \Gamma$

1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
2) $\mu(x\alpha y) \geq max\{\mu(x), \mu(y)\}$

Definition: 1.13

An ideal I of a Γ-semiring M is called k ideal if for all

$x, y \in M, x + y \in I, y \in I \Rightarrow x \in I$.

Definition: 1.14

A fuzzy subset $\mu: M \to [0,1]$ is non-empty if μ is not the constant function.

Definition: 1.15

For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$.

Definition: 1.16

Let f and g be fuzzy subsets of Γ-semiring M. Then $f \circ g$ is defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z = x\alpha y} \{\min\{f(x), g(y)\}\}, \\ 0, \qquad\qquad otherwise. \end{cases}$$

where $x, y \in M, \alpha \in \Gamma$, for all $z \in M$.

Definition: 1.17

Let f and g be fuzzy subsets of Γ-semiring M. Then $f + g$ is defined by

$$f + g(z) = \begin{cases} \sup\limits_{z = x+y} \{\min\{f(x), g(y)\}\}, \\ 0, \qquad\qquad otherwise. \end{cases}$$

where $x, y \in M, \alpha \in \Gamma$, for all $z \in M$.

Definition: 1.18

Let f and g be fuzzy subsets of Γ-semiring M. Then $f \cup g$ is defined by $f \cup g(z) = max\{f(z), g(z)\}$ where $x, y \in M$, $\alpha \in \Gamma$, for all $z \in M$.

Definition: 1.19

Let f and g be fuzzy subsets of Γ-semiring M. Then $f \cap g$ is defined by $f \cap g(z) = min\{f(z), g(z)\}$ where $x, y \in M, \alpha \in \Gamma$, for all $z \in M$.

Definition: 1.20

A function $f: R \to M$ where R and M are Γ-semirings is said to be a Γ-semiring homomorphism if $f(a + b) = f(a) + f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R, \alpha \in \Gamma$.

Definition: 1.21

Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M is defined by $\chi_A(x) = \begin{cases} 1, if\ x \in A; \\ 0, if\ x \notin A. \end{cases}$

Definition: 1.22

A fuzzy ideal f of a Γ-semiring M with zero 0 is said to be a k-fuzzy ideal of M if $f(x + y) = f(0)$ and $f(y) = f(0) \Rightarrow f(x) = f(0)$, for all $x, y \in M$.

Definition: 1.23

A fuzzy ideal f of a Γ-semiring M is said to be a fuzzy k-ideal of M if $f(x) \geq min\{f(x + y), f(y)\}$, for all $x, y \in M$.

CHAPTER-2

L-FUZZY IDEAL

Definition: 2.1

A L-fuzzy subset μ of a Γ-semiring M is called a L-fuzzy Γ-subsemiring of M if

1) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
2) $\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition: 2.2

A L-fuzzy Γ-subsemiring of a Γ-semiring M is called a L-fuzzy left (right) ideal of M if $\mu(x\alpha y) \geq \mu(y)(\mu(x))$.

Definition: 2.3

If μ is a fuzzy left and a fuzzy right ideal of Γ-semiring M then μ is called a L-fuzzy ideal of M.

Theorem: 2.4

Let μ be a L-fuzzy ideal of Γ-semiring M. Then $\mu(x) \leq \mu(0)$ for all $x \in M$.

Proof:

Let $x \in M, \alpha \in \Gamma$.

Now, $\mu(0) = \mu(0\alpha x) \geq \mu(x)$

Therefore $\mu(0) \geq \mu(x)$

Hence $\mu(x) \leq \mu(0)$, for all $x \in M$.

Theorem: 2.5

Let M be a Γ-semiring. μ is a L-fuzzy left ideal of M if and only if for any $t \in L$ such that $\mu_t \neq \phi$, μ_t is a left ideal of Γ-semiring M.

Proof:

Let μ be a L-fuzzy left ideal of Γ-semiring M and $t \in L$ such that $\mu_t \neq \phi$.

Let $x, y \in \mu_t$

$\Rightarrow \mu(x), \mu(y) \geq t$

$\Rightarrow min\{\mu(x), \mu(y)\} \geq t$

Also $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$

$\Rightarrow \mu(x + y) \geq t$

$\Rightarrow x + y \in \mu_t$

Let $x \in M, y \in \mu_t, \alpha \in \Gamma$.

$\mu(x\alpha y) \geq \mu(y) \geq t$

$\Rightarrow x\alpha y \in \mu_t$

Therefore μ_t is a left ideal of Γ-semiring M.

Conversely suppose that μ_t is a left ideal of Γ-semiring M.

Let $x, y \in M$.

Let $t = min\{\mu(x), \mu(y)\}$

Then $\mu(x), \mu(y) \geq t$

$\Rightarrow x, y \in \mu_t$

$\Rightarrow x + y \in \mu_t$

$\Rightarrow \mu(x + y) \geq t$

$\Rightarrow \mu(x + y) \geq min\{\mu(x), \mu(y)\}$

Let $x, y \in M$

Let $\mu(y) = s$

$\Rightarrow y \in \mu_s$

$\Rightarrow x\alpha y \in \mu_s$

$\Rightarrow \mu(x\alpha y) \geq s$

$\Rightarrow \mu(x\alpha y) \geq \mu(y)$

Therefore μ is a L-fuzzy left ideal.

Theorem: 2.6

Let M be a Γ-semiring and $M_\mu = \{x \in M/\mu(x) \geq \mu(0)\}$. If μ is a L-fuzzy ideal of M then M_μ is an ideal of Γ-semiring.

Proof:

Let μ be a L-fuzzy ideal of Γ-semiring M.

Let $x, y \in M_\mu$

$\Rightarrow \mu(x) \geq \mu(0), \mu(y) \geq \mu(0)$

$\Rightarrow \mu(x + y) \geq min\{\mu(x), \mu(y)\}$

$\Rightarrow \mu(x + y) \geq \mu(0)$

$\Rightarrow x + y \in M_\mu$

Now $\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\}$

$\Rightarrow \mu(x\alpha y) \geq \mu(0)$ (Since $\mu(x), \mu(y) \geq \mu(0)$)

$\Rightarrow x\alpha y \in M_\mu$

Let $x \in M_\mu$, $y \in M, \alpha \in \Gamma$.

$\Rightarrow \mu(x) \geq \mu(0)$

$\Rightarrow \mu(y\alpha x) \geq \mu(x) \geq \mu(0)$

$\Rightarrow \mu(y\alpha x) \geq \mu(0)$

$\Rightarrow y\alpha x \in M_\mu$

Similarly $x\alpha y \in M_\mu$

Hence M_μ is an ideal of Γ-semiring M.

Theorem: 2.7

Let M and S be Γ-semirings and ψ: $M \rightarrow S$ be an onto homomorphism. If μ is a L-fuzzy ideal of S then the pre image of μ under ψ is a L-fuzzy ideal of M.

Proof:

Let μ be a L-fuzzy ideal of S and ϒ be the pre image of μ under ψ.

Let $x, y \in M, \alpha \in \Gamma$

$$\Upsilon(x+y) = \mu(\psi(x+y))$$

$$= \mu(\psi(x) + \psi(y))$$

$$\geq min\{\mu(\psi(x)), \mu(\psi(y))\}$$

$$= min\{\Upsilon(x), \Upsilon(y)\}$$

Therefore $\Upsilon(x+y) \geq min\{\Upsilon(x), \Upsilon(y)\}$

And $\Upsilon(x\alpha y) = \mu(\psi(x\alpha y))$

$$= \mu(\psi(x)\alpha\psi(y))$$

$$\geq min\{\mu(\psi(x)), \mu(\psi(y)\}$$

$$= min\{\Upsilon(x), \Upsilon(y)\}$$

Therefore $\Upsilon(x\alpha y) \geq min\{\Upsilon(x), \Upsilon(y)\}$

Hence ϒ is a L-fuzzy subsemiring of Γ-semiring M.

Let $x, y \in M, \alpha \in \Gamma$.

$$\Upsilon(x\alpha y) = \mu(\psi(x\alpha y))$$

$$= \mu(\psi(x)\alpha\psi(y))$$

$$\geq \mu(\psi(x))$$

$$= \Upsilon(x)$$

$\Rightarrow \curlyvee(x\alpha y) \geq \curlyvee(x)$

Therefore $\curlyvee$ is a L-fuzzy left ideal of Γ-semiring M.

Similarly $\curlyvee$ is a L-fuzzy right ideal of Γ-semiring M.

Hence $\curlyvee$ is a L-fuzzy ideal of M.

Theorem: 2.8

Let M be a Γ-semiring. If A is an ideal of Γ-semiring M then there exist a L-fuzzy ideal μ of M such that $\mu_t = A$, for some $t \in L$.

Proof:

Suppose A is an ideal of Γ-semiring M and $t \in L$.

We define L-fuzzy subset of M by $\mu(x) = \begin{cases} t, & if\ x \in A \\ 0, & otherwise \end{cases}$

$\Rightarrow \mu(t) = A$.

Let $s \in L$.

We have $\mu_s = \begin{cases} M, & if\ s = 0 \\ A, & if\ 0 < s \leq t \\ \phi, & otherwise \end{cases}$

Hence every non-empty subset μ_s of μ is an ideal of Γ-semiring M.

By theorem: 2.5,

μ is a L-fuzzy ideal of Γ-semiring M.

Theorem: 2.9

Let μ and ϒ be two L-fuzzy ideals of Γ-semiring M. Then $\mu \cap \Upsilon$ is a L-fuzzy ideal of Γ-semiring M.

Proof:

Let $a, b \in M, \alpha \in \Gamma$.

Now,

$$\mu \cap \Upsilon(a + b) = min\{\mu(a + b), \Upsilon(a + b)\}$$

$$\geq min\{min\{\mu(a), \mu(b)\}, min\{\Upsilon(a), \Upsilon(b)\}\}$$

$$= min\{min\{\mu(a), \Upsilon(a)\}, min\{\mu(b), \Upsilon(b)\}\}$$

$$= min\{\mu \cap \Upsilon(a), \mu \cap \Upsilon(b)\}$$

$$\Rightarrow \mu \cap \Upsilon(a + b) \geq min\{\mu \cap \Upsilon(a), \mu \cap \Upsilon(b)\}$$

Also,

$$\mu \cap \Upsilon(a\alpha b) = min\{\mu(a\alpha b), \Upsilon(a\alpha b)\}$$

$$\geq min\{max\{\mu(a), \mu(b)\}, max\{\Upsilon(a), \Upsilon(b)\}\}$$

$$= max\{min\{\mu(a), \Upsilon(a)\}, min\{\mu(b), \Upsilon(b)\}\}$$

$$= max\{\mu \cap \Upsilon(a), \mu \cap \Upsilon(b)\}$$

$$\Rightarrow \mu \cap \Upsilon(a\alpha b) \geq max\{\mu \cap \Upsilon(a), \mu \cap \Upsilon(b)\}$$

Hence $\mu \cap \Upsilon$ is a L-fuzzy ideal of Γ-semiring M.

Definition: 2.10

A mapping $\mu: M \times M \to L$ is called a L-fuzzy subset of M^2.

Definition: 2.11

A L-fuzzy subset μ of $M \times M$ is called a L-fuzzy Γ-subsemiring of M^2 if the following conditions are satisfied

1) $\mu(x + y, z + m) \geq min\{\mu(x, z), \mu(y, m\}$,
2) $\mu(x\alpha y, z\alpha m) \geq min\{\mu(x, z), \mu(y, m)\}$,

for all $(x, z), (y, m) \in M \times M, \alpha \in \Gamma$.

Definition: 2.12

A L-fuzzy Γ-subsemiring of $M \times M$, μ is called a L-fuzzy left(right) ideal of M^2 if $\mu(x\alpha y, z\alpha m) \geq \mu(y, m)(\mu(x, z))$.

Definition: 2.13

If μ is a L-fuzzy left and L-fuzzy right ideal of $M \times M$, μ is called a L-fuzzy ideal of M^2.

Theorem: 2.14

Let μ be a L-fuzzy ideal of $M \times M$. Then $\mu(x, y) \leq \mu(0,0)$ for all

$(x, y) \in M \times M$.

Proof:

Let $(x, y) \in M \times M, \alpha \in \Gamma$.

Now,

$\mu(0,0) = \mu(0\alpha x, 0\alpha y)$

$\geq \mu(x, y)$

$\Rightarrow \mu(0,0) \geq \mu(x, y)$

Hence $\mu(x, y) \leq \mu(0,0)$, for all $(x, y) \in M \times M$

Theorem: 2.15

μ is a L-fuzzy left ideal of M^2 iff for $t \in L$ such that $\mu_t \neq \phi$, μ_t is a left ideal of $M \times M$.

Proof:

Let μ be a L-fuzzy left ideal of $M \times M$ and $t \in L$ such that $\mu_t \neq \phi$.

Let $(x, z), (y, m) \in \mu_t$

$\Rightarrow \mu(x, z), \mu(y, m) \geq t$

$\Rightarrow \mu(x + y, z + m) \geq min\{\mu(x, z), \mu(y, m)\} \geq t$

$\Rightarrow \mu(x + y, z + m) \geq t$

Let $(x, z) \in M \times M, (y, m) \in \mu_t$ and $\alpha \in \Gamma$.

Then $\mu(x\alpha y, z\alpha m) \geq \mu(y, m) \geq t$ (Since $(y, m) \in \mu_t, \mu(y, m) \geq t$)

$\Rightarrow (x\alpha y, z\alpha m) \in \mu_t$

Therefore μ_t is a left ideal of $M \times M$.

Suppose that μ_t is a left ideal of $M \times M$.

Let $(x,z),(y,m) \in M \times M$ and $t = min\{\mu(x,z),\mu(y,m)\}$

Then $\mu(x,z),\mu(y,m) \geq t$

$\Rightarrow (x,z),(y,m) \in \mu_t$

$\Rightarrow \mu(x+y,z+m) \geq t$

$\Rightarrow \mu(x+y,z+m) \geq min\{\mu(x,z),\mu(y,m)\}$

Now, let $(x,z),(y,m) \in M^2$

Let $\mu(y,m) = s$

$\Rightarrow (y,m) \in \mu_s$

$\Rightarrow (x\alpha y, z\alpha m) \in \mu_s$

$\Rightarrow \mu(x\alpha y, z\alpha m) \geq s = \mu(y,m)$

$\Rightarrow \mu(x\alpha y, z\alpha m) \geq \mu(y,m)$

Hence μ is a L-fuzzy left ideal.

Theorem:2.16

Define $M_\mu = \{(x,y) \in M \times M / \mu(x,y) \geq \mu(0,0)\}$. If μ is a L-fuzzy ideal of $M \times M$, then M_μ is an ideal of M^2.

Proof:

Let μ be a L-fuzzy ideal of $M \times M$.

Let $(x,z),(y,m) \in M_\mu$

$\Rightarrow \mu(x,z) \geq \mu(0,0),\ \ \mu(y,m) \geq \mu(0,0)$

Now,

$\mu(x+y,z+m) \geq min\{\mu(x,z),\mu(y,m)\}$

$\Rightarrow \mu(x+y,z+m) \geq \mu(0,0)$

$\Rightarrow (x+y,z+m) \in M_\mu$

Also,

$\mu(x\alpha y,z\alpha m) \geq min\{\mu(x,z),\mu(y,m)\}$

$\Rightarrow \mu(x\alpha y,z\alpha m) \geq \mu(0,0)$

$\Rightarrow (x\alpha y,z\alpha m) \in M_\mu$

Now, let $(x,z) \in M_\mu$, $(y,m) \in M$ and $\alpha \in \Gamma$.

$\Rightarrow \mu(x,z) \geq \mu(0,0)$

$\Rightarrow \mu(y\alpha x,m\alpha z) \geq \mu(x,z)$

$\Rightarrow \mu(y\alpha x,m\alpha z) \geq \mu(0,0)$ (Since $(x,z) \in M_\mu$, $\mu(x,z) \geq \mu(0,0)$)

$\Rightarrow (y\alpha x,m\alpha z) \in M_\mu$

Similarly $(x\alpha y,z\alpha m) \in M_\mu$

Hence M_μ is an ideal of $M \times M$.

Theorem:2.17

Let μ and ϒ be two L-fuzzy ideals of $M \times M$, then $\mu \cap \Upsilon$ is a L-fuzzy ideal of $M \times M$.

Proof:

Let $(x, z), (y, m) \in M \times M$ and $\alpha \in \Gamma$.

Now,

$$(\mu \cap \Upsilon)(x + y, z + m) = min\{\mu(x + y, z + m), \Upsilon(x + y, z + m)\}$$

$$\geq min\{min\{\mu(x, z), \mu(y, m)\}, min\{\Upsilon(x, z), \Upsilon(y, m)\}\}$$

$$\geq min\{min\{\mu(x, z), \Upsilon(x, z)\}, min\{\mu(y, m), \Upsilon(y, m)\}\}$$

$$= min\{(\mu \cap \Upsilon)(x, z), (\mu \cap \Upsilon)(y, m)\}$$

$$\Rightarrow (\mu \cap \Upsilon)(x + y, z + m) \geq min\{(\mu \cap \Upsilon)(x, z), (\mu \cap \Upsilon)(y, m)\}$$

Also,

$$(\mu \cap \Upsilon)(x\alpha y, z\alpha m) = min\{\mu(x\alpha y, z\alpha m), \Upsilon(x\alpha y, z\alpha m)\}$$

$$\geq min\{max\{\mu(x, z), \mu(y, m)\}, max\{\Upsilon(x, z), \Upsilon(y, m)\}\}$$

$$\geq max\{min\{\mu(x, z), \Upsilon(x, z)\}, max\{\mu(y.m), \Upsilon(y, m)\}\}$$

$$= max\{(\mu \cap \Upsilon)(x, z), (\mu \cap \Upsilon)(y, m)\}$$

$$\Rightarrow (\mu \cap \Upsilon)(x\alpha y, z\alpha m) \geq max\{(\mu \cap \Upsilon)(x, z), (\mu \cap \Upsilon)(y, m)\}$$

Hence $\mu \cap \Upsilon$ is a L-fuzzy ideal of $M \times M$.

CHAPTER-3

L-FUZZY k IDEAL AND $L - k$ FUZZY IDEAL

Definition: 3.1

A L-fuzzy ideal μ of Γ-semiring M is called a L-fuzzy k ideal of M if

$\mu(x) \geq min\{\mu(x+y), \mu(y)\}$, for all $x, y \in M$.

Definition: 3.2

A L-fuzzy ideal μ of Γ-semiring M is called a $L - k$ fuzzy ideal of M if

$\mu(x+y) = 0, \mu(y) = 0 \Rightarrow \mu(x) = 0$, for all $x, y \in M$.

Theorem: 3.3

Let f and g be a L-fuzzy k ideals of M. Then $f \cap g$ is a L-fuzzy k ideal of Γ-semiring M.

Proof:

Let f and g be a L-fuzzy k ideals of M.

By theorem: 2.9,

Since f and g are L-fuzzy ideals of Γ-semiring M,

$f \cap g$ is a L-fuzzy ideal of Γ-semiring M.

Let $x, y \in M$.

$$f \cap g(a) = min\{f(x), g(x)\}$$

$$\geq min\{min\{f(x+y), f(y)\}, min\{g(x+y), g(y)\}\}$$

$$= min\{min\{f(x+y), g(x+y)\}, min\{f(y), g(y)\}\}$$

$$= min\{f \cap g(x+y), f \cap g(y)\}$$

$$\Rightarrow f \cap g(a) \geq min\{f \cap g(x+y), f \cap g(y)\}$$

Hence $f \cap g$ is a L-fuzzy k ideal of M.

Theorem: 3.4

A L-fuzzy subset μ of M is a L-fuzzy k ideal of Γ-semiring M if and only if μ_t is a k ideal of Γ-semiring M for any $t \in L$, $\mu_t \neq \phi$.

Proof:

Let μ be a L-fuzzy k ideal of Γ-semiring M.

If $\mu_t \neq \phi$ then μ_t is an ideal of Γ-semiring M for any $t \in L$.

Suppose $a, a + x \in \mu_t$

$$\Rightarrow \mu(a) \geq t,\ \mu(a + x) \geq t$$

Since μ is a L-fuzzy k ideal of Γ-semiring M, we have

$$\mu(x) \geq min\{\mu(a + x), \mu(a)\}$$

$$\Rightarrow \mu(x) \geq t$$

$$\Rightarrow x \in \mu_t$$

Hence μ_t is a k ideal of Γ-semiring M.

Conversely,

Assume that μ_t is a k ideal of Γ-semiring M with $\mu_t \neq \phi$.

Let $\mu(a) = t_1, \mu(x + a) = t_2$

Let $t = min\{t_1, t_2\}$

Then $a \in \mu_t,\ x + a \in \mu_t$ for some $x \in M$.

$\Rightarrow x \in \mu_t$

$\Rightarrow \mu(x) \geq t$

But $t = min\{t_1, t_2\} = min\{\mu(x + a), \mu(a)\}$

Therefore $\mu(x) \geq\ min\{\mu(x + a), \mu(a)\}$

Hence μ is a L-fuzzy k ideal of Γ-semiring M.

Theorem: 3.5

Let M be a Γ-semiring. If μ is a L-fuzzy k ideal of M then μ is a $L - k$ fuzzy ideal of M.

Proof:

Let μ be a L-fuzzy k ideal of M.

Let $x, y \in M$ and $\mu(0) = t \in L$

$\mu(x + y) = \mu(0)$ and $\mu(y) = \mu(0)$

Now $\mu(0) = t$

$\Rightarrow \mu(x+y) = t, \mu(y) = t$

$\Rightarrow x + y \in \mu_t, y \in \mu_t$.

By theorem: 3.4,

μ_t is a k ideal of M.

$\Rightarrow x \in \mu_t$

$\Rightarrow \mu(x) \geq t = \mu(0)$

$\Rightarrow \mu(x) \geq \mu(0)$

Also $\mu(x) \leq \mu(0)$, for all $x \in M$.

Hence $\mu(x) = \mu(0)$.

Therefore μ is a $L - k$ fuzzy ideal of Γ-semiring M.

Definition: 3.6

Let μ be a L-fuzzy subset of X and $a, b \in L$. The mapping $\mu_a^T: X \to L$ is called fuzzy translation of μ, if $\mu_a^T(x) = \mu(x) \vee a$ for all $x \in X$.

Definition: 3.7

Let μ be a L-fuzzy subset of X and $a, b \in L$. The mapping $\mu_b^M: X \to L$ is called fuzzy multiplication of μ, if $\mu_b^M(x) = b \wedge \mu(x)$ for all $x \in X$.

Definition: 3.8

Let μ be a L-fuzzy subset of X and $a, b \in L$. The mapping $\mu_{b,a}^{MT}: X \to L$ is called fuzzy magnified translation of μ, if $\mu_{b,a}^{MT}(x) = (b \wedge \mu(x)) \vee a$ for all $x \in X$.

Theorem:3.9

Let μ be a L-fuzzy subset of Γ-semiring M. Then $a \in L$, μ is a L-fuzzy ideal of Γ-semiring M if any only if μ_a^T, the fuzzy translation is a L-fuzzy ideal of Γ-semiring M.

Proof:

Suppose μ is a L-fuzzy ideal of Γ-semiring M.

Let $x, y \in M$, $\alpha \in \Gamma$.

$$\mu_a^T(x+y) = \mu(x+y) \vee a$$

$$\geq min\{\mu(x), \mu(y)\} \vee a$$

$$= min\{\mu(x) \vee a, \mu(y) \vee a\}$$

$$= min\{\mu_a^T(x), \mu_a^T(y)\}$$

$$\therefore \mu_a^T(x+y) \geq min\{\mu_a^T(x), \mu_a^T(y)\}$$

$$\mu_a^T(x\alpha y) = \mu(x\alpha y) \vee a$$

$$\geq min\{\mu(x), \mu(y)\} \vee a$$

$$= min\{\mu(x) \vee a, \mu(y) \vee a\}$$

$$= min\{\mu_a^T(x), \mu_a^T(y)\}$$

$$\therefore \mu_a^T(x\alpha y) \geq min\{\mu_a^T(x), \mu_a^T(y)\}$$

Hence μ_a^T, the fuzzy translation is a L-fuzzy ideal of Γ-semiring M.

Conversely,

Suppose that $a \in L$, μ_a^T, the fuzzy translation is a L-fuzzy ideal of

Γ-semiring M.

Let $x, y \in M, \alpha \in \Gamma$.

Now,

$\mu_a^T(x + y) \geq min\{\mu_a^T(x), \mu_a^T(y)\}$

$= min\{\mu(x) \vee a, \mu(y) \vee a\}$

$= \mu(x + y) \vee a$

$\therefore \mu_a^T(x + y) \geq \mu(x + y) \vee a$

$\Rightarrow \mu_a^T(x + y) \vee a \geq min\{\mu(x) \vee a, \mu(y) \vee a\} \vee a$

$\Rightarrow \mu_a^T(x + y) \geq min\{\mu(x), \mu(y)\}$

Also,

$\mu_a^T(x\alpha y) \geq max\{\mu_a^T(x), \mu_a^T(y)\}$

$\Rightarrow \mu(x\alpha y) \vee a \geq max\{\mu(x) \vee a, \mu(y) \vee a\}$

$\Rightarrow \mu(x\alpha y) \vee a \geq max\{\mu(x), \mu(y)\} \vee a$

$\Rightarrow \mu(x\alpha y) \geq max\{\mu(x), \mu(y)\}$

Hence μ is a L-fuzzy ideal of Γ-semiring M.

Theorem: 3.10

Let μ be a L-fuzzy subset of Γ-semiring M and $a \in L$. Then μ is a L-fuzzy k ideal of Γ-semiring M if and only if μ_a^T, the fuzzy translation is a L-fuzzy k ideal of Γ-semiring M.

Proof:

Suppose μ is a L-fuzzy k ideal of Γ-semiring M.

By theorem: 3.9,

μ_a^T is a L-fuzzy ideal of Γ-semiring M.

$\mu_a^T(x) = \mu(x) \vee a$

$\geq min\{\mu(x+y), \mu(y)\} \vee a$

$= min\{\mu(x+y) \vee a, \mu(y) \vee a\}$

$= min\{\mu_a^T(x+y), \mu_a^T(y)\}$

$\Rightarrow \mu_a^T(x) \geq min\{\mu_a^T(x+y), \mu_a^T(y)\}$, for all $x, y \in M$

Hence μ_a^T is a L-fuzzy k ideal of Γ-semiring M.

Conversely,

Suppose that $a \in L$, μ_a^T is a L-fuzzy k ideal of Γ-semiring M.

$\mu(x) \vee a = \mu_a^T(x)$

$\geq min\{\mu_a^T(x+y), \mu_a^T(y)\}$

$= min\{\mu(x+y)\vee a, \mu(y)\vee a\}$

$= min\{\mu(x+y), \mu(y)\}\vee a$

$\Rightarrow \mu(x)\vee a \geq min\{\mu(x+y), \mu(y)\}\vee a$

$\Rightarrow \mu(x) \geq min\{\mu(x+y), \mu(y)\}$, for all $x, y \in M$.

Therefore μ is a L-fuzzy k ideal of Γ-semiring M.

Theorem: 3.11

Let μ be a L-fuzzy subset of Γ-semiring M. Then μ is a L-fuzzy ideal of Γ-semiring M if and only if $b \in L$, μ_b^M, fuzzy multiplication is a L-fuzzy ideal of Γ-semiring M.

Proof:

Suppose μ is a L-fuzzy ideal of Γ-semiring M.

Let $x, y \in M, \alpha \in \Gamma, b \in L$.

$\mu_b^M(x+y) = \mu(x+y)\wedge b$

$\geq min\{\mu(x), \mu(y)\}\wedge b$

$= min\{\mu(x)\wedge b, \mu(y)\wedge b\}$

$= min\{\mu_b^M(x), \mu_b^M(y)\}$

$\Rightarrow \mu_b^M(x+y) \geq min\{\mu_b^M(x), \mu_b^M(y)\}$

Now,

$\mu_b^M(x\alpha y) = \mu(x\alpha y)\wedge b$

$\geq max\{\mu(x), \mu(y)\} \wedge b$

$= max\{\mu(x) \wedge b, \mu(y) \wedge b\}$

$= max\{\mu_b^M(x), \mu_b^M(y)\}$

$\Rightarrow \mu_b^M(x\alpha y) \geq max\{\mu_b^M(x), \mu_b^M(y)\}$

Hence μ_b^M, fuzzy multiplication is a L-fuzzy ideal of Γ-semiring M.

Conversely,

Suppose μ_b^M, fuzzy multiplication is a L-fuzzy ideal of Γ-semiring M.

Let $x, y \in M, \alpha \in \Gamma$.

$\mu_b^M(x + y) \geq min\{\mu_b^M(x), \mu_b^M(y)\}$

$\Rightarrow \mu(x + y) \wedge b \geq min\{\mu(x) \wedge b, \mu(y) \wedge b\}$

$\Rightarrow \mu(x + y) \wedge b \geq min\{\mu(x), \mu(y)\} \wedge b$

$\Rightarrow \mu(x + y) \geq min\{\mu(x), \mu(y)\}$

Also,

$\mu_b^M(x\alpha y) \geq max\{\mu_b^M(x), \mu_b^M(y)\}$

$\Rightarrow \mu(x\alpha y) \wedge b \geq max\{\mu(x) \wedge b, \mu(y) \wedge b\}$

$\Rightarrow \mu(x\alpha y) \wedge b \geq max\{\mu(x), \mu(y)\} \wedge b$

$\Rightarrow \mu(x\alpha y) \geq max\{\mu(x), \mu(y)\}$

Hence μ is a L-fuzzy ideal of Γ-semiring M.

Theorem: 3.12

Let μ be a L-fuzzy subset of Γ-semiring M. Then μ is a L-fuzzy k ideal of Γ-semiring M if and only if $b \in L$, μ_b^M, fuzzy multiplication is a L-fuzzy k ideal of Γ-semiring M.

Proof:

Suppose μ is a L-fuzzy k ideal of Γ-semiring M.

By theorem: 3.11,

μ_b^M is a L-fuzzy ideal of Γ-semiring M.

Let $x, y \in M, \alpha \in \Gamma, b \in L$.

$$\mu_b^M(x) = \mu(x) \wedge b$$

$$\geq min\{\mu(x+y), \mu(y)\} \wedge b$$

$$= min\{\mu(x+y) \wedge b, \mu(y) \wedge b\}$$

$$= min\{\mu_b^M(x+y), \mu_b^M(y)\}$$

$\Rightarrow \mu_b^M(x) \geq min\{\mu_b^M(x+y), \mu_b^M(y)\}$, for all $x, y \in M$.

Hence μ_b^M is a L-fuzzy k ideal of Γ-semiring M.

Conversely,

Suppose that $a \in L$, μ_b^M is a L-fuzzy k ideal of Γ-semiring M.

$\mu(x) \wedge b = \mu_b^M(x)$

$\geq min\{\mu_b^M(x+y), \mu_b^M(y)\}$

$= min\{\mu(x+y) \wedge b, \mu(y) \wedge b\}$

$= min\{\mu(x+y), \mu(y)\} \wedge b$

$\Rightarrow \mu(x) \wedge b \geq min\{\mu(x+y), \mu(y)\} \wedge b$

$\Rightarrow \mu(x) \geq min\{\mu(x+y), \mu(y)\}$, for all $x, y \in M$.

Therefore μ is a L-fuzzy k ideal of Γ-semiring M.

Theorem: 3.13

Let μ be a L-fuzzy k subset of Γ-semiring M. Then μ is a L-fuzzy k ideal of Γ-semiring M if and only if $\mu_{b,a}^{MT}$ is a L-fuzzy k ideal of Γ-semiring M.

Proof:

Let μ be a L-fuzzy k ideal of Γ-semiring M.

$\Leftrightarrow$ μ_b^M is a L-fuzzy k ideal of Γ-semiring M. (By theorem: 3.12)

$\Leftrightarrow \mu_{b,a}^{MT}$ is a L-fuzzy k ideal of Γ-semiring M. (By theorem: 3.10)

Definition: 3.14

Let μ be a L-fuzzy subset of Γ-semiring M. Then the set M_μ is defined by $M_\mu = \{x \in M \,/\, \mu(x) = \mu(0)\}$.

Theorem: 3.15

If μ is a L-fuzzy k ideal of Γ-semiring M then M_μ is a k ideal of Γ-semiring M.

Proof:

Let μ be a L-fuzzy k ideal of Γ-semiring M.

Let $x, y \in M_\mu$.

Then $\mu(x) = \mu(0) = \mu(y)$

Now $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$

$= min\{\mu(0), \mu(0)\}$

$= \mu(0)$

$\Rightarrow \mu(x + y) \geq \mu(0)$

Also $\mu(x + y) \leq \mu(0)$

Therefore $\mu(x + y) = \mu(0)$

Hence $x + y \in M_\mu$.

Let $x \in M_\mu, y \in M, \alpha \in \Gamma$.

Then $\mu(x) = 0$

$\mu(x\alpha y) \geq max\{\mu(x), \mu(y)\}$

$= max\{\mu(0), \mu(y)\}$

$= \mu(0)$

$\Rightarrow \mu(x\alpha y) \geq \mu(0)$

Also $\mu(x\alpha y) \leq \mu(0)$

Therefore $\mu(x\alpha y) = \mu(0)$

Hence $x\alpha y \in M_\mu$.

$\therefore M_\mu$ is an ideal of Γ-semiring M.

Let $x + y, x \in M_\mu$.

Then $\mu(x + y) = \mu(0) = \mu(x)$.

Since μ is a L-fuzzy k ideal of Γ-semiring M,

$\mu(y) \geq min\{\mu(x + y), \mu(x)\}$

$= min\{\mu(0), \mu(0)\}$

$= \mu(0)$

$\Rightarrow \mu(y) \geq \mu(0)$

Also $\mu(y) \leq \mu(0)$

Therefore $\mu(y) = \mu(0)$

Hence $y \in M_\mu$

Thus M_μ is a k ideal of Γ-semiring M.

Theorem: 3.16

If $t \in L$ such that $\mu_t \neq \phi$, μ_t is a k ideal of Γ-semiring M then μ is a $L -$ k fuzzy ideal of M.

Proof:

Let μ_t be a k ideal of Γ-semiring M.

Let $t \in L$.

Let $\mu(x + y) = \mu(0)$ and $\mu(y) = \mu(0)$

$\Rightarrow x + y \in M_\mu$

Since M_μ is a k ideal of Γ-semiring M, $x \in M_\mu$.

Therefore $\mu(x) \geq \mu(0)$

Also, $\mu(x) \leq \mu(0)$ (By theorem: 2.4)

Hence $\mu(x) = \mu(0)$.

Therefore μ is a $L - k$ fuzzy ideal of M.

Definition: 3.17

Let M be a Γ-semiring and μ be a L-fuzzy ideal of $M \times M$. μ is called L-fuzzy k ideal of M^2 if $\mu(x, z) \geq min\{\mu(x + y, z + m), \mu(y, m)\}$ for all $x, y, z, m \in M$.

Definition:3.18

Let M be a Γ-semiring and μ be a L-fuzzy ideal of $M \times M$. μ is called $L - k$ fuzzy ideal of $M \times M$ if $\mu(x + y, z + m) = 0, \mu(y, m) = 0 \Rightarrow \mu(x, z) = 0$ for all $x, y, z, m \in M$.

Theorem: 3.19

Let M be a Γ-semiring and let f and g be L-fuzzy k ideals of M^2. Then $f \cap g$ is a L-fuzzy k ideal of M^2.

Proof:

Let M be a Γ-semiring.

Let f and g be L-fuzzy k ideals of M^2.

By theorem: 2.17 ,

$f \cap g$ is a L-fuzzy ideal of M^2.

Let $x, y, z, m \in M$.

$$(f \cap g)(x, z) = min\{f(x, z), g(x, z)\}$$

$$\geq min\{min\{f(x + y, z + m), f(y, m)\},$$

$$min\{g(x + y, z + m), g(y, m)\}\}$$

$$\geq min\{min\{f(x + y, z + m), g(x + y, z + m)\},$$

$$min\{f(y,m), g(y,m)\}\}$$

$$= min\{(f \cap g)(x+y, z+m), (f \cap g)(y,m)\}$$

$$\Rightarrow (f \cap g)(x,z) \geq \ min\{(f \cap g)(x+y, z+m), (f \cap g)(y,m)\}$$

Hence $f \cap g$ is a L-fuzzy k ideal of $M \times M$.

Definition: 3.20

Let X be a set and μ be a L-fuzzy subset of $X \times X$ and $a, b \in L$. The mapping $\mu_a^T: X \times X \to L$, $\mu_b^M: X \times X \to L$, $\mu_{b,a}^{MT}: X \times X \to L$ are called fuzzy type translation, fuzzy type multiplication and fuzzy type magnified translation of μ respectively, if for all $x, z \in M$,

$$\mu_a^T(x,z) = \mu(x,z) \vee a,\ \mu_b^M(x,z) = b \wedge \mu(x,z),\ \mu_{b,a}^{MT}(x,z) = (b \wedge \mu(x,z)) \vee a.$$

Theorem: 3.21

Let M be a Γ-semiring and let μ be a L-fuzzy subset of $M \times M$ and let $a \in L$. μ is a L-fuzzy ideal of $M \times M$ iff μ_a^T is a L-fuzzy ideal of $M \times M$.

Proof:

Suppose μ is a L-fuzzy ideal of $M \times M$.

Let $x, y, z, m \in M$ and $\alpha \in \Gamma$.

Now,

$\mu_a^T(x+y, z+m) = \mu(x+y, z+m) \vee a$

$$\geq min\{\mu(x,z),\mu(y,m)\}\vee a$$

$$= min\{\mu(x,z)\vee a,\mu(y,m)\vee a\}$$

$$= min\{\mu_a^T(x,z),\mu_a^T(y,m)\}$$

$$\Rightarrow \mu_a^T(x+y,z+m) \geq \ min\{\mu_a^T(x,z),\mu_a^T(y,m)\}$$

Also,

$$\mu_a^T(x\alpha y,z\alpha m) = \mu(x\alpha y,z\alpha m)\vee a$$

$$\geq min\{\mu(x,z),\mu(y,m)\}\vee a$$

$$= min\{\mu(x,z)\vee a,\mu(y,m)\vee a\}$$

$$= min\{\mu_a^T(x,z),\mu_a^T(y,m)\}$$

$$\Rightarrow \mu_a^T(x\alpha y,z\alpha m) \geq \ min\{\mu_a^T(x,z),\mu_a^T(y,m)\}$$

Hence μ_a^T is a L-fuzzy ideal of $M \times M$.

Suppose that $a \in L$, μ_a^T is a L-fuzzy ideal of $M \times M$.

Let $x,y,z,m \in M$ and $\alpha \in \Gamma$.

Now,

$$\mu_a^T(x+y,z+m) \geq \ min\{\mu_a^T(x,z),\mu_a^T(y,m)\}$$

$$\Rightarrow \mu(x+y,z+m)\vee a \geq min\{\mu(x,z)\vee a,\mu(y,m)\vee a\}$$

$$\Rightarrow \mu(x+y,z+m)\vee a \geq min\{\mu(x,z),\mu(y,m)\}\vee a$$

$$\Rightarrow \mu(x+y, z+m) \geq min\{\mu(x,z), \mu(y,m)\}$$

Also,

$$\mu_a^T(x\alpha y, z\alpha m) \geq max\{\mu_a^T(x,z), \mu_a^T(y,m)\}$$

$$\Rightarrow \mu(x\alpha y, z\alpha m)\vee a \geq max\{\mu(x,z)\vee a, \mu(y,m)\vee a\}$$

$$\Rightarrow \mu(x\alpha y, z\alpha m)\vee a \geq max\{\mu(x,z), \mu(y,m)\}\vee a$$

$$\Rightarrow \mu(x\alpha y, z\alpha m) \geq max\{\mu(x,z), \mu(y,m)\}$$

Hence μ is a L-fuzzy ideal of $M \times M$.

CHAPTER-4

NORMAL L-FUZZY IDEAL

Definition: 4.1

A L-fuzzy subset μ of a Γ-semiring M is said to be normal if $\mu(0) = 1$.

Definition: 4.2

Let μ be a L-fuzzy subset of Γ-semiring M. We define μ^+on S by

$\mu^+(x) = \mu(x) \vee (\mu(0))'$, where $(\mu(0))'$ is the complement of $\mu(0)$.

Definition: 4.3

Let A be an ideal of Γ-semiring M. If we define L-fuzzy subset on M by

$\chi_A(x) = \begin{cases} 1, & if\ x \in A \\ 0, & if\ x \notin A \end{cases}$,. for all $x \in M$. Then χ_A is a normal L-fuzzy ideal of M and

$M_{\chi_A} = A$.

Result: 4.4

If μ and λ are normal L-fuzzy ideals of Γ-semiring M then $\mu \cap \lambda$ is a normal L-fuzzy ideal of Γ-semiring.

Theorem: 4.5

If μ is a normal L-fuzzy ideal of Γ-semiring M then μ_a^T, fuzzy translation is a normal L-fuzzy ideal.

Proof:

Let μ be a normal L-fuzzy ideal of Γ-semiring M.

By theorem: 3.9,

μ_a^T, the fuzzy translation is a L-fuzzy ideal of Γ-semiring M.

$\mu_a^T(x) = \mu(x) \vee a$, for all $x \in M$.

$$\Rightarrow \mu_a^T(0) = \mu(0) \vee a$$

$$= 1 \vee a$$

$$= 1$$

$\therefore \mu_a^T(0) = 1$

Hence μ_a^T, fuzzy translation is a normal L-fuzzy ideal.

Theorem: 4.6

Let μ be a L-fuzzy subset of Γ-semiring M. Then

i) μ^+ is a normal L-fuzzy subset of M containing μ.

ii) $(\mu^+)^+ = \mu$

iii) μ is a normal if and only if $\mu^+ = \mu$

iv) If there exists a L-fuzzy subset Υ of M satisfying $\Upsilon^+ \subseteq \mu$ then μ is a normal.

Proof:

i) Let μ be a L-fuzzy subset of Γ-semiring.

Then $\mu^+(x) = \mu(x) \vee (\mu(0))'$, for all $x \in M$.

$$\Rightarrow \mu^+(0) = \mu(0) \vee (\mu(0))' = 1,$$

$\mu(x) \leq \mu(x) \vee (\mu(0))' = \mu^+(x)$, for all $x \in M$.

$\therefore \mu(x) \leq \mu^+(x)$, for all $x \in M$.

Hence μ^+ is a normal L-fuzzy subset of M containing μ.

ii) $(\mu^+)^+(x) = \mu^+(x) \vee (\mu^+(0))'$

$$= \mu^+(x) \vee 1'$$

$$= \mu^+(x) \vee 0$$

$\Rightarrow (\mu^+)^+(x) = \mu^+(x)$, for all $x \in M$.

Therefore $(\mu^+)^+ = \mu$

iii) Suppose $\mu = \mu^+$.

Then $\mu^+(x) = \mu(x) \vee (\mu(0))'$, for all $x \in M$.

$\Rightarrow \mu(x) = \mu(x) \vee \big(\mu(0)\big)'$

$\Rightarrow \mu(0) = \mu(0) \vee \big(\mu(0)\big)'$

$\Rightarrow \mu(0) = 1$

Hence μ is normal.

Conversely,

Suppose that μ is normal.

Then $\mu^+(x) = \mu(x) \vee \big(\mu(0)\big)'$

$$= \mu(x) \vee 1'$$

$\Rightarrow \mu^+(x) = \mu(x)$

Therefore $\mu^+ = \mu$

iv) Now $\Upsilon^+(x) = \Upsilon(x) \vee (\Upsilon(0))'$

$\Rightarrow \Upsilon^+(0) = \Upsilon(0) \vee (\Upsilon(0))'$

$\Rightarrow \Upsilon^+(0) = 1$

Also $\Upsilon^+ \subseteq \mu$

$\Rightarrow \Upsilon^+(0) \leq \mu(0)$

$\Rightarrow 1 \leq \mu(0)$

$\Rightarrow \mu(0) = 1.$

Hence μ is normal.

Theorem: 4.7

Let μ be a L-fuzzy subset of Γ-semiring M. If μ is a L-fuzzy ideal of M then μ^+ is a normal L-fuzzy ideal of M containing μ.

Proof:

Let $x, y \in M, \alpha \in \Gamma$.

Then $\mu^+(x + y) = \mu(x + y) \vee (\mu(0))'$

$\geq min\{\mu(x), \mu(y)\} \vee (\mu(0))'$

$= min\{\mu(x) \vee (\mu(0))', \mu(y) \vee (\mu(0))'\}$

$= min\{\mu^+(x), \mu^+(y)\}$

$\Rightarrow \mu^+(x + y) \geq min\{\mu^+(x), \mu^+(y)\}$

Also, $\mu^+(x\alpha y) = \mu(x\alpha y) \vee (\mu(0))'$

$$\geq \mu(y) \vee (\mu(0))'$$

$$= \mu^+(y)$$

$\Rightarrow \mu^+(x\alpha y) \geq \mu^+(y)$

Similarly, $\mu^+(x\alpha y) \geq \mu^+(x)$.

By theorem: 4.6,

μ^+ is a normal L-fuzzy ideal containing μ.

Corollary: 4.8

Let μ be a L-fuzzy subset of Γ-semiring and $x \in M$. If $\mu^+(x) = 0$ then $\mu(x) = 0$.

Proof:

By theorem: 4.6 i),

$\mu(x) \leq \mu^+(x)$

$$\Rightarrow \mu(x) \leq \mu^+(x) = 0$$

$$\Rightarrow \mu(x) \leq 0$$

Therefore $\mu(x) = 0$.

Theorem: 4.9

Let M be a Γ-semiring, $\psi: M \to M$ be an onto homomorphism and μ be a L-fuzzy subset of M. Define $\mu^{\psi}: M \to L$ by $\mu^{\psi}(x) = \mu(\psi(x))$, for all $x \in M$. If μ is a L-fuzzy ideal of M then μ^{ψ} is a L-fuzzy ideal of M.

Proof:

Let M be a Γ-semiring.

Let $x, y \in M, \alpha \in \Gamma$.

Now,

$\mu^{\psi}(x+y) = \mu(\psi(x+y))$

$= \mu(\psi(x) + \psi(y))$

$\geq min\{\mu(\psi(x)), \mu(\psi(y))\}$

$= min\{\mu^{\psi}(x), \mu^{\psi}(y)\}$

$\Rightarrow \mu^{\psi}(x+y) \geq min\{\mu^{\psi}(x), \mu^{\psi}(y)\}$

Also,

$\mu^{\psi}(x\alpha y) = \mu(\psi(x\alpha y))$

$= \mu(\psi(x)\alpha\psi(y))$

$\geq min\{\mu(\psi(x)), \mu(\psi(y))\}$

$= min\{\mu^{\psi}(x), \mu^{\psi}(y)\}$

$\Rightarrow \mu^{\psi}(x+y) \geq min\{\mu^{\psi}(\mathrm{x}), \mu^{\psi}(\mathrm{y})\}$

Therefore μ^{ψ} is a L-fuzzy Γ-subsemiring of M.

Now,

$\mu^{\psi}(x\alpha y) = \mu(\psi(x\alpha y))$

$= \mu(\psi(x)\alpha\psi(y))$

$\geq \mu(\psi(y))$

$= \mu^{\psi}(y)$

$\Rightarrow \mu^{\psi}(x\alpha y) \geq \mu^{\psi}(y)$

Similarly, $\mu^{\psi}(x\alpha y) \geq \mu^{\psi}(x)$.

Hence μ^{ψ} is an ideal of Γ -semiring M.

Theorem: 4.10

Let μ and Υ be L-fuzzy ideals of Γ-semiring M. If $\mu \subseteq \Upsilon$ and $\mu(0) = \Upsilon(0)$ then $M_{\mu} \subseteq M_{\Upsilon}$.

Proof:

Let μ and Υ be two L-fuzzy ideals of Γ-semiring M.

Suppose that $\mu \subseteq \Upsilon$ and $\mu(0) = \Upsilon(0)$.

Let $x \in M_{\mu}$.

$\Rightarrow \mu(x) = \mu(0) = \Upsilon(0)$

$\Rightarrow \Upsilon(0) = \mu(x) \leq \Upsilon(x)$, for all $x \in M$.

Also $\Upsilon(x) \leq \Upsilon(0)$, for all $x \in M$.

$\Rightarrow \Upsilon(x) = \Upsilon(0)$.

Therefore $x \in M_{\Upsilon}$

Hence $M_{\mu} \subseteq M_{\Upsilon}$.

Corollary: 4.11

Let μ and Υ be normal L-fuzzy ideals of Γ-semiring M. If $\mu \subseteq \Upsilon$ then $M_{\mu} \subseteq M_{\Upsilon}$.

Theorem: 4.12

If μ and Υ be normal L-fuzzy ideals of Γ-semiring M then $M_{\mu \cap \Upsilon} = M_{\mu} \cap M_{\Upsilon}$.

Proof:

Let μ and Υ be normal L-fuzzy ideals of Γ-semiring M.

Suppose $x \in M_{\mu \cap \Upsilon}$

$\Leftrightarrow \mu \cap \Upsilon(x) = \mu \cap \Upsilon(0)$

$\Leftrightarrow min\{\mu(x), \Upsilon(x)\} = min\{\mu(0), \Upsilon(0)\} = 1$

$\Leftrightarrow \mu(x) = 1 \; and \; \Upsilon(x) = 1$

$\Leftrightarrow \mu(x) = \mu(0) \; and \; \Upsilon(x) = \Upsilon(0)$

$\Leftrightarrow x \in M_{\mu} \cap M_{\Upsilon}$

Hence $M_{\mu \cap \Upsilon} = M_{\mu} \cap M_{\Upsilon}$.

Definition: 4.13

A non-constant L-fuzzy ideal μ of Γ-semiring M is said to be a maximal L-fuzzy ideal if μ^+ is a maximal element of $(N(N), \subseteq)$. $(N(M), \subseteq)$ denotes the partially ordered set of normal L-fuzzy ideals of Γ-semiring M under set inclusion.

Theorem: 4.14

If μ be a non-constant maximal normal L-fuzzy ideal of Γ-semiring M then μ takes the values only 0 and 1.

Proof:

Let $y \in M$.

Let μ be a maximal normal L-fuzzy ideal of Γ-semiring M.

$0 < \mu(y) < 1$ and $\mu(y) = a$

Define L-fuzzy subset Υ of M by $\Upsilon(x) = \mu(x) \vee a$ for all $x \in M$.

Then $\Upsilon(x) = \mu_a^T(x)$ and $\Upsilon(x) \geq \mu(x)$, for all $x \in M$.

By theorem: 4.5,

Υ is a normal L-fuzzy ideal of Γ-semiring M.

If $x \neq 0$, $\mu(x) < \Upsilon(x)$.

Therefore μ is not a maximal, which is a contradiction.

Hence the theorem.

Theorem: 4.15

If μ is a maximal L-fuzzy ideal of Γ-semiring M then M_μ is a maximal ideal of Γ-semiring M.

Proof:

Let μ be a maximal L-fuzzy ideal of Γ-semiring M.

Then μ^+ is a maximal element of $(N(N), \subseteq)$

By theorem: 4.14 ,

μ^+ takes only the values 0 and 1.

If $\mu^+ = 1$

$\Rightarrow \mu(x) \vee (\mu(0))' = 1$

$\Rightarrow \mu(0) = 1$ (Since $\mu(0) \geq \mu(x)$, for all $x \in M$)

Also $\mu(x) \leq \mu^+(x)$, for all $x \in M$.

If $\mu^+(x) = 0$

$\Rightarrow \mu(x) \vee (\mu(0))' = 0$

$\Rightarrow \mu(x) = 0$ and $(\mu(0))' = 0$

$\Rightarrow \mu(0) = 1.$

Therefore μ is a normal L-fuzzy ideal of Γ-semiring M.

Now, M_μ is a proper ideal of Γ-semiring M, since μ is a non-constant.

Let A be an ideal of Γ-semiring M such that $M_\mu \subseteq A$.

$\Rightarrow \chi_{M_\mu} \subseteq \chi_A$

$\Rightarrow \mu = \chi_{M_\mu} \subseteq \chi_A$

Since μ and χ_A are normal L-fuzzy ideals of M and $\mu = \mu^+$

$\Rightarrow \mu$ is a maximal element of $N(M)$.

$\Rightarrow \mu = \chi_A$ or $\chi_A = 1$

where $1: M \to L$, $1(x) = 1$, for all $x \in M$ is a L-fuzzy ideal.

If $\chi_A = 1$, then $A = M$.

If $\mu = \chi_A$, then $M_\mu = M_{\chi_A} = A$

Hence M_μ is a maximal ideal of Γ-semiring M.

A STUDY ON TENSOR ALGEBRA

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

R.VISVAPRIYA

Reg.No:19SPMT27

Under the guidance of

Ms. A. FERDINA M.Sc., M.Phil., SET.,

FIDE VIVANT

ST. MARY'S COLLEGE THOOTHUKUDI

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON TENSOR ALGEBRA"** is submitted to **St. Mary's college (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by

R.VISVAPRIYA (REG.NO: 19SPMT27)

Ferdina

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON TENSOR ALGEBRA"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. A. Ferdina M.Sc., M.Phil., SET.,** Assistant Professor, Department of Mathematics (SSC), **St. Mary's College (Autonomous), Thoothukudi .**

R. Visva Priya

Signature of student

Station : Thoothukudi

Date: 10.04.2021

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CONTENT

CHAPTER - 1

PRELIMINARIES

This chapter provides the basic definitions and results of algebra which are needed to the subsequent chapters.

Definition : 1.1

In three dimensional rectangular space, the coordinates of a point are (x, y, z). It is convenient to write (x^1,x^2,x^3) for (x,y,z). *The coordinates of point in four* dimensional space are given by (x^1,x^2,x^3,x^4). In general, the coordinates of a point in **n-dimensional space** are given by $(x^1,x^2,\ldots,x^n)$ such n-dimensional space is denoted by $V_n(R)$.

Definition :1.2

In the symbol A^{ij}_{kl}, the indices i , j written in the upper position are called **superscripts** and k,l written in the lower position are called **subscripts**.

Note : 1.3

A superscript is always used to indicate **contravariant component** and a subscript is always used to indicate **covariant component**.

Definition: 1.4

i The symbol δ^i_j, called **Krönecker Delta** and it is defined by,

$$\delta^i_j = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$

similarly, δ^{ij} and δ_{ij} are defined as,

$$\delta^{ij} = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases} \quad \text{and}$$

$$\delta_{ij} = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$

Definition: 1.5

A function $\phi(x^1,x^2,...,x^n)$ is called **Scalar or an invariant** if its original value not change upon transformation action of coordinates from $x^1,x^2,...,x^n$ to $\overline{x}^1,\overline{x}^2,...,\overline{x}^n$.

$$(i.e)\phi(x^1,x^2,...,x^n) = \overline{\phi}\,(\overline{x}^1,\overline{x}^2,...,\overline{x}^n)$$

Scalar is also called **tensor of rank zero**.

Example : 1.6

A^iB_i is scalar

Definition : 1 . 7

In three dimensional rectangular space, the coordinate of a point are (x,y,z) where x, y, z are real numbers. It is convenient to write (x^1,x^2,x^3) for (x,y,z) or simply x^i, where $i=1,2,3$. Similarly in n-dimensional space, the coordinate of a point are n-independent variables $(x^1,x^2,...,x^n)$ in X-coordinate system. Let $(\overline{x}^1,\overline{x}^2,...,\overline{x}^n)$ be coordinate of the same point in Y-coordinate system.

Let $\overline{x}^1,\overline{x}^2,...,\overline{x}^n$ be independent single valued function of $x^1,x^2,...,x^n$, so that,

$$\overline{x}^1 = \overline{x}^1\,(x^1,x^2,...,x^n)$$

$$\overline{x}^2 = \overline{x}^2\,(x^1,x^2,...,x^n)$$

$\bar{x}^3 = \bar{x}^3(x^1, x^2, ..., x^n)$

………………….

$\bar{x}^n = \bar{x}^n(x^1, x^2, ..., x^n)$ (*or*)

$\bar{x}^i = \bar{x}^i(x^1, x^2, ..., x^n)$; $i = 1, 2, 3, ..., n$ …..(1)

Solving these equations and expressing x^i as functions of $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$, so that,

$x^i = x^i(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$; $i = 1, 2, ..., n$ …..(2)

The equations (1) and (2) are said to be a **transformation of the coordinates** from one coordinate system to another.

Definition : 1.8

Consider the sum of the series $S = a_1x^1 + a_2x^2 + ... + a_n x^n = \sum_{i=1}^{n} a_i \ x^i$.

By using summation convention, drop the sigma sign and write convention as

$$\sum_{i=1}^{n} a_i \ x^i = a_i \ x^i$$

This convention is called **Einstein's summation convention**

Definition : 1.9

Any index which is repeated in a given term is called a **dummy index or dummy suffix**. This is also called **Umbral or Dextral Index**.

For example, Consider the expression $a_i \, x^i$ where i is dummy index; then,

$a_i x^i = a_1x^1 + a_2x^2 + ... + a_nx^n$

and

$$a_jx^j = a_1x^1 + a_2x^2 + \ldots + a_nx^n$$

These two equations prove that

$$a_i x^i = a_jx^j$$

So, any dummy index can be replaced by any other index ranging the same numbers.

Definition : 1.10

Any index occurring only once in a given term is called a **Free Index**.

e.g. Consider the expression $a_i{}^jx^i$ where j is free index.

Definition : 1.11

If $V_1, V_2, \ldots, V_k$ and W are vector spaces. A function

$f : V_1 \times V_2 \times, \ldots, \times V_K \to W$ is called **Multilinear** if it is linear in each of its variables.

$f(v_1, \ldots, v_{i-1}, av_i + v_i', v_{i+1}, \ldots, v_k)$

$$= af(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k) + bf(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_k)$$

for all **a,b** $\in$**R**, $v_j \in V_j$ for j=1,…..k and $v_i' \in V_i$ for i=1,…..,k

Definition : 1.12

Multilinear algebra is a generalization of linear algebra since a linear function is also multilinear in one variable. Multilinear algebra are tensor of **rank one**.

If $V_{1,}V_2,\ldots\ldots, V_k$ and W are vector spaces then multilinear maps

g : $V_1 \times V_2 \times_{,\ldots\ldots,} \times V_K \rightarrow W$

Definition : 1.13

The definition of **tensors** is based on multilinear algebra with a multilinear map. Consider the real vector spaces $\mathbf{U}_1,\text{-}..,\mathbf{U}_n$ and their respective dual vector spaces $\mathbf{V}_1,\ldots,\mathbf{V}_m$. Each of their vector spaces belongs to the finite **N**-dimensional space $\mathbf{R}^{\mathbf{N}}$, the image vector space **W**, to the real space **R**.

A mixed tensor of type (m, n) can be defined as a multilinear functional **T** that maps an (m+ n) tuple of vectors of the vector spaces U and V into W

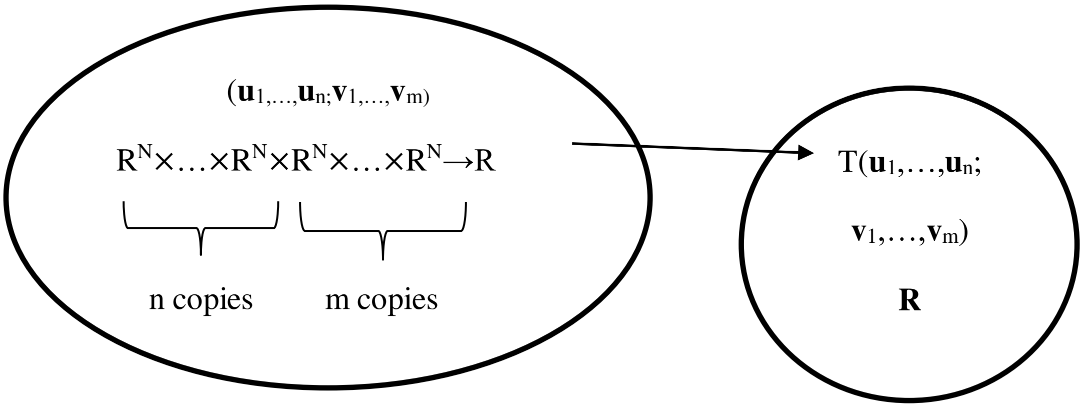
$$T:(\mathbf{U}_1\times\ldots\times\mathbf{U}_{n,})\times(\mathbf{V}_1\times\ldots\times\mathbf{V}_m)\rightarrow W$$

$$\underbrace{R^N\times\ldots\times R^N}_{\text{n copies}}\times\underbrace{R^N\times\ldots\times R^N}_{\text{m copies}}\rightarrow R \quad \ldots\ldots (1)$$

$$(\mathbf{u}_1,\ldots,\mathbf{u}_n;\mathbf{v}_1,\ldots,\mathbf{v}_m)\rightarrow T(\mathbf{u}_1,\ldots,\mathbf{u}_n;\mathbf{v}_1,\ldots,\mathbf{v}_m)\in \mathbf{R}.$$

Mapping the multilinear functional T of the tensor type (m, n) to the contra variant basis $\{g^{im}\}$ of **U** and covariant basis $\{g_{jn}\}$ of **V**, one obtains its images in $\mathbf{W}\subset \mathbf{R}$. These images are called the components of the (m + n) order mixed tensor **T** with respect to the relating bases:

$$T^{i1\ldots im}_{j1\ldots jn} \equiv \boldsymbol{T}(\boldsymbol{g}i1,\ldots,\boldsymbol{g}in;\boldsymbol{g}^{i1},\ldots,\boldsymbol{g}^{im}) \in \mathbf{R} \qquad \ldots..(2)$$



Multilinear functional **T**

Thus, the (m + n)-order mixed tensor **T** can be expressed in the covariant and contravariant bases of the respective vector spaces **V** and **U**. In total, the (m + n) order tensor**T** has $N^{(m+n)}$ components, as shown in Equation (3).

$$\mathbf{T} = T^{i1\ldots im}_{j1\ldots jm}\mathbf{g}i1\ldots\mathbf{g}im\boldsymbol{g}^{j1}\ldots\boldsymbol{g}^{jn};$$

$$\mathbf{T}\in \underbrace{R^N \text{ x}\ldots\text{x } R^N}_{\text{n copies}} \text{ x } \underbrace{R^N \text{ x}\ldots\text{x } R^N}_{\text{m copies}} \qquad \ldots.(3)$$

In the case of covariant and contravariant tensors **T,** the dual vector spaces **V**and real vector spaces **U**are omitted in Eqution (1), respectively.

i) n-order covariant tensors:

$$\mathbf{T}= \mathbf{T}_{j1\ldots jn}\mathbf{g}^{j1}\ldots.\mathbf{g}^{jn} \in(\mathbf{U}_1 \text{ x...x } \mathbf{U}_n)$$

(ii) m-order contravariant tensors:

$$\mathbf{T} =\mathbf{T}^{i1\ldots im}\mathbf{g}_{i1}\ldots\mathbf{g}_{im}\in(\mathbf{V}_1 \text{ x...x } \mathbf{V}_m)$$

CHAPTER - 2

RANKS OF TENSORS

In this section, I have discussing about contravariant vector, covariant vector, contravariant tensor of rank two, covariant tensor of rank two, mixed tensor of rank two, contravariant tensor of rank r and covariant tensor of rank s.

2.1.Tensor of rank one :

Definition : 2.1.1

Let $(x^1, x^2, \dots, x^n)$ or x^i be coordinate of a point in X- coordinate system and $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ or $\bar{x}^i$be coordinates of the same point in the Y - coordinate system.

Let A^i, $i = 1, 2, \dots, n$ (or $A^1, A^2, \dots, A^n$)be n functions of coordinates $x^1, x^2, \dots, x^n$, in X - coordinate system. If the quantities A^i are transformed to $\bar{A}^i$in Y - coordinate system then according to the law of transformation.

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \qquad \text{or} \qquad A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i$$

Then A^i called components of **contra variant vector.**

Example : 2.1.2

If x^i be the coordinate of a point in n-dimensional space show that dx^i are component of a contravariant vector.

Solution :

Let $x^1, x^2, \dots, x^n$ or x^i are coordinates in X - coordinate system and $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ or $\bar{x}^i$ are coordinates in Y-coordinate system.

If

$$\bar{x}^i = \bar{x}^i\,(x^1, x^2, \dots, x^n)$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1} dx^1 + \frac{\partial \bar{x}^i}{\partial x^2} dx^2 + \cdots + \frac{\partial \bar{x}^i}{\partial x^n} dx^n$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j$$

It is law of transformation of contra variant vector. So, dx^i are components of a contravariant vector.

Definition : 2.1.3

Let, $A_i,\ i = 1, 2, \dots, n$ (or $A_1, A_{2,} \dots, A_n$) be n functions of the coordinates $x^1, x^2, \dots, x^n$, in X - coordinate system. If the quantities A_i are transformed to $\bar{A}_i$in Y - coordinate system then according to the law of transformation.

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \qquad \text{or} \qquad A_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{A}_i$$

Then A_i are called components of **covariant vector.**

Note : 2.1.4

The contravariant (or covariant) vector is also called contravariant or covariant) tensor of rank one.

Example : 2.1.5

$\frac{\partial \phi}{\partial x^i}$ is a covariant vector where ϕ is a scalar function.

Solution :

Let $x^1, x^2, \dots, x^n$ or x^i are coordinates in X - coordinate system and $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ or $\bar{x}^i$ are coordinates in Y - coordinates system.

Consider $\phi(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n) = \phi(x^1, x^2, \dots, x^n)$

$$\partial \phi = \frac{\partial \phi}{\partial x^1}\partial x^1 + \frac{\partial \phi}{\partial x^2}\partial x^2 + \cdots + \frac{\partial \phi}{\partial x^n}\partial x^n$$

$$\frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial \bar{x}^i} + \cdots + \frac{\partial \phi}{\partial x^n}\frac{\partial x^n}{\partial \bar{x}^i}$$

$$\frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i}$$

(or) $$\frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial \phi}{\partial x^j}$$

It is law of transformation of component of covariant vector. So, $\frac{\partial \phi}{\partial x^i}$ is component of covariant vector.

2.2 Tensor of rank two :

Definition : 2.2.1

Let A^{ij} $(i, j = 1, 2, \dots, n)$ be $n^2,$ functions of coordinates $x^1, x^2, \dots, x^n$ in X - coordinates system. If the quantities A^{ij} are transformed to $\bar{A}^{ij}$ in Y - coordinate system having coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$. Then according to the law of transformation.

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^l}A^{kl}$$

Then A^{ij} are called components of **contravariant Tensor of rank two.**

Definition: 2.2.2

Let A_{ij} $(i,j = 1,2,\dots,n)$ be n^2 functions of coordinates $x^1, x^2, \dots, x^n$ in X - coordinates system. If the quantities A_{ij} are transformed to $\bar{A}_{ij}$ in Y - coordinate system having coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ then according to the law of transformation.

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}A_{kl}$$

Then A_{ij} called components of **covariant tensor of rank two.**

Definition: 2.2.3

Let $A^i_j(i,j = 1,2,\dots,n)$ be n^2 functions of coordinates $x^1, x^2, \dots, x^n$ in X - coordinates system. If the quantities A^i_j are transformed to $\bar{A}^i_j$ in Y - coordinates system having coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$then according to the law of transformation.

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}A^k_l$$

Then A^i_j are called components of **mixed tensor of rank two**.

Theorem : 2.2.4

The kronecker delta is a mixed tensor of rank two.

Proof :

Let X and Y be two coordinate systems. Let the component of Kronecker delta in X - coordinate system δ_j^i and component of Kronecker delta in Y - coordinate be $\bar{\delta}_j^i$, then according to the law of transformation.

$$= \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^k}{\partial x^l}$$

$$\bar{\delta}_j^i = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}\delta_l^k$$

This shows that Kronecker δ_j^i is mixed tensor of rank two.

Theorem : 2.2.5

δ_j^i is an invariant i.e., it has same components in every coordinate system.

Proof :

Since δ_j^i is a mixed tensor of rank two, then

$$\bar{\delta}_j^i = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}\delta_l^k$$

$$= \frac{\partial \bar{x}^i}{\partial x^k}\left(\frac{\partial x^l}{\partial \bar{x}^j}\delta_l^k\right)$$

$$= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j} \text{ as } \frac{\partial x^l}{\partial \bar{x}^j}\delta_l^k = \frac{\partial x^k}{\partial \bar{x}^j}$$

$$\bar{\delta}_j^i = \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta_j^i \text{ , as } \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta_j^i$$

So, δ_j^i is an invariant

Theorem : 2.2.6

The transformation of a contravariant vector is transitive

(or)

The transformation of a contravariant vector form a group

Proof :

Let A^i be a contravariant vector in a coordinate system x^i $(i = 1, 2, \dots, n)$.

Let the coordinates x^i be transformed to the coordinate system $\bar{x}^i$ and $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$.

When coordinate x^i be transformed to $\bar{x}^i$, the law transformation of a contravariant vector is

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q \qquad \dots. (1)$$

When coordinate $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$, the law of transformation of contravariant vector is

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^p} \bar{A}^p$$

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^p} \frac{\partial \bar{x}^p}{\partial x^q} A^q \text{ from (1)}$$

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial x^q} A^q$$

This shows that the direct transformation from x^i to $\bar{\bar{x}}^i$, gives the same law of transformation. This property is called that transformation of contravariant vectors is transitive or form a group.

Theorem : 2.2.7

The transformation of a covariant vector is transitive.

(or)

The transformation of a covariant vector form a group

Proof :

Let, A_i be a covariant vector in a coordinate system x^i $(i = 1, 2, \dots, n)$. Let the coordinates x^i be transformed to the coordinate system $\bar{x}^i$ and $\bar{x}^i$, be transformed to $\bar{\bar{x}}^i$.

When coordinate x^i be transformed to $\bar{x}^i$, the law of transformation of a covariant vector is

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q \qquad \text{.... (1)}$$

When coordinate $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$, the law of transformation of a convariant vector is

$$\bar{\bar{A}}_i = \frac{\partial \bar{x}^p}{\partial \bar{\bar{x}}^i} \bar{A}_p$$

$$\bar{\bar{A}}_i = \frac{\partial \bar{x}^p}{\partial \bar{\bar{x}}^i} \frac{\partial x^q}{\partial \bar{x}^p} A_q$$

$$\bar{\bar{A}}_i = \frac{\partial x^q}{\partial \bar{\bar{x}}^i} A_q$$

This shows that the direct transformation from x^i to $\bar{\bar{x}}^i$, gives the same law of transformation. This property is called that transformation of covariant vectors is transitive or form a group.

Theorem : 2.2.8

The transformation of tensors form a group

(or)

The equation of transformation a tensor (Mixed tensor) posses the group property.

Proof :

Let A_j^i be a mixed tensor of rank two in a coordinate system x^i $(i = 1, 2, \dots, n)$. Let the coordinates x^ibe transformed to the coordinate system $\bar{x}^i$and $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$

When coordinate x^i be transformed to $\bar{x}^i$, the transformation of mixed tensor of rank two is

$$\bar{A}_q^p = \frac{\partial \bar{x}^p}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^q}A_s^r \qquad \text{..... (1)}$$

When coordinate $\bar{x}^i$be transformed to $\bar{\bar{x}}^i$,the law of transformation of a mixed tensor of rank two is

$$\bar{\bar{A}}_j^i = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^p}\frac{\partial \bar{x}^q}{\partial \bar{\bar{x}}^j}\bar{A}_q^p$$

$$= \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^p}\frac{\partial \bar{x}^q}{\partial \bar{\bar{x}}^j}\frac{\partial \bar{x}^p}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^q}A_s^r \qquad \text{from (1)}$$

$$\bar{\bar{A}}_j^i = \frac{\partial \bar{\bar{x}}^i}{\partial x^r}\frac{\partial x^s}{\partial \bar{\bar{x}}^j} A_s^r$$

This shows that the direct transformation from x^i to $\bar{\bar{x}}^i$, gives the same law of transformation. This property is called that transformation of tensors form a group.

2.3 Tensors of higher order

Definition : 2.3.1

Let $A^{i_1,i_2,\dots,i_r}$be n^r function of coordinates $x^1, x^2, \dots, x^n$ in X - coordinate system. If the quantities $A^{i_1,i_2,\dots,i_r}$, are transformed to $\bar{A}^{i_1,i_2,\dots,i_r}$in Y - coordinate system having coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$. Then according to the law of transformation

$$\bar{A}^{i_1,i_2,\dots,i_r} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}}\frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \cdots \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} A^{p_1 p_2 \dots p_r}$$

Then $A^{i_1,i_2,\dots,i_r}$ are called components of **contravariant tensor of rank r**

Definition : 2.3.2

Let $A_{j_1,j_2,\dots,j_s}$ be n^s functions of coordinates $x^1, x^2, \dots, x^n$ in X - coordinate system. If the quantities $A_{j_1,j_2,\dots,j_s}$ are transformed to $\bar{A}_{j_1,j_2,\dots,j_s}$in Y - coordinate system having coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$. Then according to the law of transformation.

$$\bar{A}_{j_1,j_2,\dots,j_s} = \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}}\frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} A_{q_1,q_2,\dots,q_s}$$

Then $A_{j_1,j_2,\dots,j_s}$are called the components of **covariant tensor of rank s.**

Definition : 2.3.3

Let$A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ be n^{r+s} functions of coordinates $x^1, x^2, \dots, x^n$ in X - coordinate system. If the quantities $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ are transformed to $\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ in Y – coordinate system having coordinate $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$. Then according to the law of transformation.

$$\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} A^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$$

Then $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ are called component of mixed tensor of rank r + s

A tensor of type $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$is known as tensor of type (r, s). In (r, s) the first component r indicates the rank of contravariant tensor and the second components indicates the rank of covariant tensor.

Thus the tensors A_{ij} and A^{ij} are type (0, 2) and (2, 0) respectively while tensor A^i_j is type (1, 1)

CHAPTER – 3

OPERATIONS IN TENSOR

In this chapter, I have discussing about the Addition and subtraction of tensors, multiplication of tensors, contraction of a tensor and inner product of the given tensors.

3.1 Addition and subtraction of tensors

Theorem : 3.1.1

The sum (or difference) of two tensors which have same number of covariant and the same contravariant indices is again a tensor of the same rank and type as the given tensors.

Proof :

Consider two tensors$A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ and $B^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ of the same rank and type

(i.e., covariant tensor of rank s and contravariant tensor of rank r). Then according to the law of transformation

$$\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \dots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} A^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$$

and

$$\bar{B}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \dots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} B^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$$

Then

$$\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \pm \bar{B}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \dots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} (A^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} \pm B^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s})$$

If,

$$\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \pm \bar{B}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \bar{C}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$$

and

$$A^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} \pm B^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} = C^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$$

So,

$$\bar{C}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \dots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} C^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$$

This is law of transformation of a mixed tensor r+s. So, $\bar{C}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ is a mixed tensor of rank r+s or of type (r, s).

Example : 3.1.2

If A^{ij}_k and B^{ij}_k are tensors then their sum and difference are tensors of the same rank and type.

Solution :

As given A^{ij}_k and B^{ij}_k are tensors. Then according to the law of transformation.

$$\bar{A}^{ij}_k = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^k} A^{pq}_r$$

and

$$\bar{B}^{ij}_k = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^k} B^{pq}_r$$

then

$$\bar{A}_k^{ij} \pm \bar{B}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\left(A_r^{pq} \pm B_r^{pq}\right)$$

If

$$\bar{A}_k^{ij} \pm \bar{B}_k^{ij} = \bar{C}_k^{ij} \text{ and } A_r^{pq} \pm B_r^{pq} = C_r^{pq}$$

So,

$$\bar{C}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k} C_r^{pq}$$

This shows that $\bar{C}_k^{ij}$ is a tensor of same rank and type as A_k^{ij} and B_k^{ij}

3.2 Multiplication of tensors

Theorem : 3.2.1

The multiplication of two tensors is a tensor whose rank is the sum of the ranks of two tensors.

Proof :

Consider two tensors $A_{j_1 j_2 \dots j_s}^{i_1 i_2 \dots i_r}$ (which is covariant tensor of rank s and contravariant tensor of rank r) and $B_{l_1 l_2 \dots l_n}^{k_1 k_2 \dots k_m}$ (which is covariant tensor of rank m and contravariant tensor of rank n). Then according to the law of transformation.

$$\bar{A}_{j_1 j_2 \dots j_s}^{i_1 i_2 \dots i_r} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}}\frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}}\frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \dots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} A_{q_1 q_2 \dots q_s}^{p_1 p_2 \dots p_r}$$

and

$$\bar{B}^{k_1k_2\ldots k_m}_{l_1l_2\ldots l_n} = \frac{\partial\bar{x}^{k_1}}{\partial x^{\alpha_1}}\frac{\partial\bar{x}^{k_2}}{\partial x^{\alpha_2}}\cdots\frac{\partial\bar{x}^{k_m}}{\partial x^{\alpha_m}}\frac{\partial x^{\beta_1}}{\partial\bar{x}^{l_1}}\frac{\partial x^{\beta_2}}{\partial\bar{x}^{l_2}}\cdots\frac{\partial x^{\beta_n}}{\partial\bar{x}^{l_n}}B^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}$$

Then their product is

$$\bar{A}^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s}\bar{B}^{k_1k_2\ldots k_m}_{l_1l_2\ldots l_n} = \frac{\partial\bar{x}^{i_1}}{\partial x^{p_1}}\cdots\frac{\partial\bar{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial\bar{x}^{j_1}}\cdots\frac{\partial x^{q_s}}{\partial\bar{x}^{j_s}}\frac{\partial\bar{x}^{k_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial\bar{x}^{k_m}}{\partial x^{\alpha_m}}\frac{\partial x^{\beta_1}}{\partial\bar{x}^{l_1}}\cdots\frac{\partial x^{\beta_n}}{\partial\bar{x}^{l_n}}A^{p_1p_2\ldots p_r}_{q_1q_2\ldots q_s}B^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}$$

If

$$\bar{C}^{i_1i_2\ldots i_rk_1k_2\ldots k_m}_{j_1j_2\ldots j_sl_1l_2\ldots l_n} = \bar{A}^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s}\bar{B}^{k_1k_2\ldots k_m}_{l_1l_2\ldots l_n}$$

and

$$C^{P_1P_2\ldots p_r\alpha_1\alpha_2\ldots\alpha_m}_{q_1q_2\ldots q_s\beta_1\beta_2\ldots\beta_n} = A^{p_1p_2\ldots p_r}_{q_1q_2\ldots q_s}B^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}$$

So,

$$\bar{C}^{i_1i_2\ldots i_rk_1k_2\ldots k_m}_{j_1j_2\ldots j_sl_1l_2\ldots l_n} = \frac{\partial\bar{x}^{i_1}}{\partial x^{p_1}}\cdots\frac{\partial\bar{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial\bar{x}^{j_1}}\cdots\frac{\partial x^{q_s}}{\partial\bar{x}^{j_s}}\frac{\partial\bar{x}^{k_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial\bar{x}^{k_m}}{\partial x^{\alpha_m}}\frac{\partial x^{\beta_1}}{\partial\bar{x}^{l_1}}\cdots\frac{\partial x^{\beta_n}}{\partial\bar{x}^{l_n}}$$

$$C^{P_1P_2\ldots p_r\alpha_1\alpha_2\ldots\alpha_m}_{q_1q_2\ldots q_s\beta_1\beta_2\ldots\beta_n}$$

This is law of transformation of a mixed tensor of rank *r*+*m*+*s*+*n*. So, $\bar{C}^{i_1i_2\ldots i_rk_1k_2\ldots k_m}_{j_1j_2\ldots j_sl_1l_2\ldots l_n}$ is a mixed tensor of rank *r*+*m*+*s*+*n*. or of type (*r*+*m, s*+*n)*. Such product is called outer product or open product of two tensors.

Theorem : 3.2.2

If A^i and B_j are the components of a contravariant and covariant tensors of rank one then prove that A^iB_j are components of a mixed tensor of rank two.

Proof :

As A^i is contravariant tensor of rank one and B_j is convariant tensor of rank one. Then according to the law of transformation

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^k} A^k \qquad \dots (1)$$

and

$$\bar{B}_j = \frac{\partial x^l}{\partial \bar{x}^j} B_l \qquad \dots (2)$$

Multiply (1) and (2), we get

$$\bar{A}^i \bar{B}_j = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^j} A^k B_l$$

This is law of transformation of tensor of rank two. So, $A^i B_j$ are mixed tensor of rank two. Such product is called outer product of two tensors.

Example: 3.2.3

The product of two tensors A_j^i and B_m^{kl} is a tensor of rank five.

Solution :

As A_j^i and B_m^{kl} are tensors. Then by law of transformation.

$$\bar{A}_j^i = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} A_q^p \text{ and } \bar{B}_m^{kl} = \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial \bar{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \bar{x}^m} \quad B_t^{rs}$$

Multiplying these, we get

$$\bar{A}_j^i \bar{B}_m^{kl} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial \bar{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \bar{x}^m} A_q^p \, B_t^{rs}$$

This is law of transformation of tensor of rank five. So, $A_j^i B_m^{kl}$ is a tensor of rank five.

Definition: 3.2.4

The process of getting a tensor of lower order (reduced by 2) by putting a covariant index equal to a contravariant index and performing the summation indicated is known contraction.

In other words, if in a tensor we put one contravariant and one covariant indices equal, the process is called **contraction of a tensor**.

Example : 3.2.5

consider a mixed tensor A_{lm}^{ijk} of order five. Then by law of transformation,

$$\bar{A}_{lm}^{ijk} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^l}\frac{\partial x^t}{\partial \bar{x}^m} A_{st}^{pqr}$$

Put the covariant index l = contravariant index i, so that

$$\bar{A}_{im}^{ijk} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial x^t}{\partial \bar{x}^m} A_{st}^{pqr}$$

$$= \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^s}{\partial x^p}\frac{\partial x^t}{\partial \bar{x}^m} A_{st}^{pqr}$$

$$= \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^t}{\partial \bar{x}^m} \delta_p^s A_{st}^{pqr} \qquad \text{Since } \frac{\partial x^s}{\partial x^p} = \delta_p^s$$

$$\bar{A}_{im}^{ijk} = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^t}{\partial \bar{x}^m} A_{st}^{pqr}$$

This is law of transformation of tensor of rank 3. So, A_{lm}^{ijk} is a tensor of rank 3 and type (1, 2) while A_{lm}^{ijk} is a tensor of rank 5 and type (2,3). It means that contraction reduces rank of tensor by two.

Definition: 3.2.6

Consider the tensors A_k^{ij} and B_{mn}^{l} if we first form their outer product $A_k^{ij}B_{mn}^{l}$ and contract this by putting $l = k$ then the result is $A_k^{ij}B_{mn}^{k}$ which is also to tensor, called the **inner product of the given tensors**.

Hence the inner product of two tensors is obtained by first taking outer product and then contracting it.

Example : 3.2.7

If A^i and B_i are the components of a contravariant and covariant tensors of rank are respectively then prove that A^iB_i is scalar or invariant.

Solution :

As A^i and B_i are the components of a contravariant and covariant tensor of rank one respectively, then according to the law of the transformation.

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^p}A^p \text{ and } \bar{B}_i = \frac{\partial x^q}{\partial \bar{x}^i}B_q$$

Multiplying these, we get

$$\bar{A}^i\bar{B}^i = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^i}A^pB_q$$

$$= \frac{\partial x^q}{\partial x^p}A^pB_q, \text{ Since} \frac{\partial x^q}{\partial x^p} = \delta_p^q$$

$$= \delta_p^q A^p B_q$$

$\overline{A}^i \overline{B}_i = A^p B_p$ This shows that $A^i B_i$ is scalar or Invariant.

Example : 3.2.8

If A_j^i is mixed tensor of rank 2 and B_m^{kl} is mixed tensor of rank 3. Prove that $A_j^i B_m^{jl}$ is mixed tensor of rank 3.

Solution :

As A_j^i is mixed tensor of rank 2 and B_m^{kl} is mixed tensor of rank 3. Then by law of transformation

$$\bar{A}_j^i = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j} A_q^p \text{ and } \bar{B}_m^{kl} = \frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial \bar{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \bar{x}^m} B_t^{rs} \qquad \text{... (1)}$$

Put $k = j$ then

$$\bar{B}_m^{jl} = \frac{\partial \bar{x}^j}{\partial x^r}\frac{\partial \bar{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \bar{x}^m} B_t^{rs} \qquad \text{... (2)}$$

Multiplying (1) & (2) we get

$$\bar{A}_j^i \bar{B}_m^{jl} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^r}\frac{\partial \bar{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \bar{x}^m} A_q^p B_t^{rs}$$

$$= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \bar{x}^m} \delta_r^q A_q^p B_t^{rs} \qquad \text{Since } \frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^r} = \frac{\partial x^q}{\partial x^r} = \delta_r^q$$

$$\bar{A}_j^j \bar{B}_m^{jl} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \bar{x}^m} A_q^p B_t^{qs} \qquad \text{Since } \delta_r^q B_t^{rs} = B_t^{qs}$$

This is the law of transformation of a mixed tensor of rank three. Hence $A_j^i B_m^{jl}$ is a mixed tensor of rank three.

CHAPTER - 4

TYPES OF TENSORS AND THE EINSTEIN SUMMATION CONVENTION

In this chapter, I have discussing about symmetric, skew-symmetric, conjugate tensor, relative tensor, tensor density, tensor absolute and the Einstein summation convention.

Definition : 4.1

A tensor is said to be **symmetric** with respect to two contravariant (or two covariant) indices if its components remain unchanged on an interchange of the two indices.

Example : 4.2

(1) The tensor A^{ij} is symmetric if $A^{ij} = A^{ji}$

(2) The tensor A^{ijk}_{lm} is symmetric if $A^{ijk}_{lm} = A^{jik}_{lm}$

Theorem : 4.3

A symmetric tensor of rank two has only $\frac{1}{2}n(n+1)$ different components in n-dimensional space.

Proof :

Let A^{ij} be a symmetric tensor of rank two. So that $A^{ij} = A^{ji}$.

The component of A^{ij} are $\begin{bmatrix} A^{11} & A^{12} & A^{13} & \dots & A^{1n} \\ A^{21} & A^{22} & A^{23} & \dots & A^{2n} \\ A^{31} & A^{32} & A^{33} & \dots & A^{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \dots & A^{nn} \end{bmatrix}$

i.e., A^{ij} will have n^2 components. Out of these n^2 components, n components $A^{11}, A^{22}, A^{33}, \dots, A^{nn}$ are different. Thus remaining components are $n^2 - n$. In which $A^{12} = A^{21}$, $A^{23} = A^{32}$ etc. due to symmetry.

So, the remaining different components are $\frac{1}{2}(n^2 - n)$. Hence the total number of different components,

$$= n + \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n+1)$$

Definition : 4.4

A tensor is said to be **skew- symmetric** with respect to two contravariant (or two covariant) indices if its components change sign on interchange of the two indices.

Example : 4.5

(1) The tensor A^{ij} is Skew-symmetric if $A^{ij} = -A^{ji}$

(2) The tensor A^{ijk}_{lm} is Skew-symmetric if $A^{ijk}_{lm} = -A^{jik}_{lm}$

Theorem: 4.6

A Skew symmetric tensor of second order has only $\frac{1}{2}n(n-1)$ different non-zero components.

Proof :

Let A^{ij} be a skew-symmetric tensor of order two. Then $A^{ij} = -A^{ji}$.

The component of A^{ij} are $\begin{bmatrix} 0 & A^{12} & A^{13} & \ldots & A^{1n} \\ A^{21} & 0 & A^{23} & \ldots & A^{2n} \\ A^{31} & A^{32} & 0 & \ldots & A^{3n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \ldots & 0 \end{bmatrix}$

[Since $A^{ii} = -A^{ii} \Rightarrow 2A^{ii} = 0 \Rightarrow A^{ii} = 0 \Rightarrow A^{11} = A^{22} = \ldots. = A^{nn} = 0$]

i.e., A^{ij} will have n^2 components. Out of these n^2 components, n components A^{11}, A^{22}, A^{33}, …., A^{nn} are zero. Omitting there, then the remaining components are $(n^2 - n)$. In which $A^{12} = -A^{21}$, $A^{13} = -A^{31}$ etc. Ignoring the sign. Their remaining the different components are $\frac{1}{2}(n^2 - n)$.

Hence the total number of different non-zero components = $\frac{1}{2}(n^2 - n)$.

Note : 4.7

Skew-symmetric tensor is also called **anti-symmetric tensor**.

Theorem : 4.8

A covariant or contravariant tensor of rank two say A_{ij} can always be written as the sum of a symmetric and skew- symmetric tensor.

Proof :

Consider a covariant tensor A_{ij}. We can write A_{ij} as

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$$

$$A_{ij} = S_{ij} + T_{ij}$$

Where $S_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$ and $T_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$

Now,

$$S_{ji} = \frac{1}{2}(A_{ji} + A_{ij})$$

$$S_{ji} = S_{ij}$$

So, S_{ij} is symmetric tensor,

And

$$T_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$$

$$T_{ij} = \frac{1}{2}(A_{ji} - A_{ij})$$

$$= -\frac{1}{2}(A_{ij} - A_{ji})$$

$$T_{ji} = -T_{ij} \quad (or)$$

$$T_{ij} = -T_{ji}$$

So, T_{ij} is Skew-symmetric Tensor,

Example : 4.9

If $\phi = a_{jk}A^jA^k$. Show that we can always write $\phi = b_{jk}A^jA^k$ where b_{jk} is symmetric.

Solution :

As given

$$\phi = a_{jk}A^jA^k \qquad \ldots\ldots (1)$$

Interchange the indices *i* and *j*

$$\phi = a_{jk}A^k A^j \qquad \ldots\ldots (2)$$

Adding (1) and (2)

$$2\phi = (a_{jk}+ a_{kj})\, A^jA^k$$

$$\phi = \frac{1}{2}(a_{jk} + a_{kj})\, A^jA^k$$

$$\phi = b_{jk}A^jA^k$$

Where $b_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$

To show that b_{jk} is symmetric

Since

$$b_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$$

$$b_{kj} = \frac{1}{2}(a_{kj} + a_{jk})$$

$$= \frac{1}{2}(a_{jk} + a_{kj})$$

$b_{jk} = b_{kj}$ So, b_{jk} is Symmetric.

Example : 4.10

If T_i be the component of a covariant vector show that $\left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}\right)$are component of a Skew-symmetric covariant tensor of rank two.

Solution :

As T_i is covariant vector. Then by the law of transformation

$$\overline{T}_i = \frac{\partial x^k}{\partial \overline{x}^i} T_k$$

Differentiating it w.r.t. to $\overline{x}^j$ partially

$$\frac{\partial \overline{T}_i}{\partial \overline{x}^j} = \frac{\partial}{\partial \overline{x}^j}\left(\frac{\partial x^k}{\partial \overline{x}^i} T_k\right)$$

$$= \frac{\partial^2 x^k}{\partial \overline{x}^j \partial \overline{x}^i} T_k + \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial T_k}{\partial \overline{x}^j}$$

$$\frac{\partial \overline{T}_j}{\partial \overline{x}^j} = \frac{\partial^2 x^k}{\partial \overline{x}^j \partial \overline{x}^i} T_k + \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial T_k}{\partial \overline{x}^l} \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial \overline{T}_j}{\partial \overline{x}^i} = \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j} T_k + \frac{\partial x^k}{\partial \overline{x}^j}\frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial T_k}{\partial x^l}$$

Interchanging the dummy indices k &l,

$$\frac{\partial \overline{T}_j}{\partial \overline{x}^i} = \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j} T_k + \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial T_l}{\partial x^k} \qquad \text{... (2)}$$

Substituting (1) and (2), we get

$$\frac{\partial \overline{T}_i}{\partial \overline{x}^j} - \frac{\partial \overline{T}_j}{\partial \overline{x}^i} = \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial T^l}{\partial \overline{x}^j} \left(\frac{\partial T_k}{\partial x^l} - \frac{\partial T_l}{\partial x^k} \right)$$

This is law of transformation of covariant tensor of rank two. So, $\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ are component of a covariant tensor of rank two.

To show that $\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ is Skew-symmetric tensor.

Let

$$A_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$$

$$A_{ji} = \frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j}$$

$$= -\left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i} \right)$$

$$A_{ji} = -A_{ij}$$

or $$A_{ij} = -A_{ji}$$

So, $A_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ is Skew-symmetric

So, $\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ are component of a Skew-symmetric covariant tensor of rank two.

Definition : 4.11

Consider a covariant symmetric tensor A_{ij} of rank two. Let d denote the determinant $|A_{ij}|$ with the elements A_{ij}

i.e., $d = |A_{ij}|$ and $d \neq 0$.

Now, define A^{ij} by

$$A^{ij} = \frac{Cofactor\ of\ Aij\ \ is\ the\ \ det\,er\,min\,ant|A_{ij}|}{d}$$

A^{ij} is a contravariant symmetric tensor of rank two which is called **conjugate (or Reciprocal)tensor** of A_{ij}.

Theorem : 4.12

If B_{ij} is the cofactor of A_{ij} in the determinant $d = |A_{ij}| \neq 0$ and A^{ij} defined as

Then prove that $A_{ij}A^{kj} = \delta_i^k$

Proof :

From the properties of the determinants, we have two results.

(i) $A_{ij}B_{ij} = d$

$\Rightarrow$ $$A_{ij}\frac{B_{ij}}{d} = 1$$

$$A_{ij}A^{ij} = 1, \text{ given } A^{ij} = \frac{B_{ij}}{d}$$

(ii) $A_{ij}B_{kj} = 0$

$$A_{ij}\frac{B_{kj}}{d} = 0, \qquad d \neq 0$$

$$A_{ij}A^{kj} = 0 \qquad \text{if } i \neq k$$

From (i) & (ii)

$$A_{ij}A^{kj} = \begin{cases} 1 & \text{if } \; i=k \\ 1 & \text{if } \; i\neq k \end{cases}$$

$$A_{ij}A^{kj} = \delta_i^k$$

Definition : 4.13

If the components of a tensor $A^{i_1i_2\cdots i_r}_{j_1j_2\cdots j_s}$ transform according to the equation

$$A^{k_1k_2\ldots k_r}_{l_1l_2\ldots l_s} = \left|\frac{\partial x}{\partial \bar{x}}\right|^{\omega} A^{i_1i_2\ldots i_r}_{j_1j_2\cdots j_s}\frac{\partial \bar{x}^{k_1}}{\partial x^{i_1}}\frac{\partial \bar{x}^{k_2}}{\partial x^{i_2}}\cdots\frac{\partial \bar{x}^{k_r}}{\partial x^{i_r}}\cdot\frac{\partial x^{j_1}}{\partial \bar{x}^{l_1}}\frac{\partial x^{j_2}}{\partial \bar{x}^{l_2}}\cdots\frac{\partial x^{j_s}}{\partial \bar{x}^{l_s}}$$

Hence $A^{i_1i_2\cdots i_r}_{j_1j_2\cdots j_s}$ is called a **relative tensor** of weight ω, where $\left|\frac{\partial x}{\partial \bar{x}}\right|$ is the Jacobian of transformation. If ω = 1, the relative tensor is called a **tensor density**. If $w = 0$ tensor is said to be **absolute**.

Example : 4.14

If $A\,(i, j, k)$is a scalar for arbitrary vectors A^i, B^j, C_k . Show that $A\,(i, j, k)$is a tensor of type (1, 2)

Solution :

Let X and Y be two coordinate systems. As given $A\ (i, j, k) A^i B^j C_k$ is scalar. Then

$$\overline{A}\ (i, j, k)\, \overline{A}^i \overline{B}^j \overline{C}_k = A(p,q,r) A^p B^q C_r \qquad \text{...... (1)}$$

Since A^i, B^j and C_k are vectors. Then

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^p} A^P \qquad or \qquad A^P = \frac{\partial x^p}{\partial \overline{x}^i} \overline{A}^i$$

$$\overline{B}^j = \frac{\partial \overline{x}^j}{\partial x^q} B^q \qquad or \qquad B^q = \frac{\partial x^q}{\partial \overline{x}^j} \overline{B}^j$$

$$\overline{C}^k = \frac{\partial \overline{x}^k}{\partial x^r} C^r \qquad or \qquad C^r = \frac{\partial x^r}{\partial \overline{x}^k} \overline{C}^k$$

So, from (1)

$$\overline{A}\ (i, j, k)\, \overline{A}^i \overline{B}^j \overline{C}_k = A\ (p, q, r)\ \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^k}{\partial x^r} \overline{A}^i\, \overline{B}^j\, \overline{C}_k$$

As $\overline{A}^i\ \overline{B}^j\ \overline{C}_k$ are arbitrary

Then

$$\overline{A}\ (i, j, k) = \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^k}{\partial x^r}\ A\ (p, q, r)$$

So, $A\ (i, j, k)$ is tensor of type (1, 2).

The Einstein summation convention

Definition : 4.15

The rule that Einstein invented is to allow the repeated index to become itself the designation for the summation over that particular index. This rule is what is referred to as the **Einstein summation convention** in the literature. For example, the expression

$$a_1 x_1 + a_2 x_2 + \ldots + a_N x_N , \qquad (1)$$

which traditionally is denoted by

$$\sum_{i=1}^{N} a_i x_i \qquad (2)$$

is now simply written as

$$a_i x_i \, . \qquad (3)$$

Repeated index i indicates the sum over i making it unnecessary to write the symbol Σ_i explicitly. The only ambiguity in the the expression above is that the range of the index i is not clear. More rigorously, one should write:

$$a_i x_i \, , \; (1 \le i \le N) \, , \qquad (4)$$

but this is almost never necessary, since all the indices run from one till, a priori, specified number. In these notes, I reserve letter N to denote this maximum value that indices can take on and I refer to the set {1, 2, …,N} as the index range. I also refer to expression (1) as the full form of the expression (4).

The letter i used in expression (3) to indicate the sum over i, which is repeated exactly twice, can be equally interchanged by any other letter such as j, k, … to address the same sum. That is, expressions a_jx_j, a_kx_k, … all refer to precisely the same expression (1). This is why index $i$in (3) is called a dummy index. Changing a dummy index to a new letter is called relabeling.

As the rule of the game, repetition of an index more than twice is forbidden provided that you are counting the repetitions in a single-term expression. For example, an expression like $a_{ii}\, x_i$ is not allowed and, in fact, it does not occur in any consistent calculations, but the expression $a_{ij}x_j+a_{ij}y_i$ is allowed, since although, say i, appears in three places, but no index is repeated more than two times in each individual term.

Example : 4.16

Express the expression $a_{ij}x_j$ in its full form for N = 3.

Solution :

Here we have two letters i and j as indices, i is not repeated but j is repeated (precisely) twice. Hence, we have a meaningful expression and a sum over index j is to be understood. Therefore,

$$a_{ij}x_j = a_{i1}\, x_1 + a_{i2}\, x_2 + a_{i3}\, x_3\, .$$

Here i can be 1, 2, or 3.

An index such as i in the expression which can take on freely any number in the index range, is called a free index. The characteristic of a free index is that it is not repeated in any single term expression. In contrast to the case of a dummy index, one cannot change the letter used for a free index in a given expression . For instance, two expressions $a_{ij}x_j$ and $a_{kj}x_j$ are not in general equal, unless $i = k$.

Example : 4.17

Write down the full form of $a_{ii}b_{jk}c_j$, when $N = 2$.

Solution :

Here i and j are dummy indices and k is the free index. Thus, a sum over i and another sum over j is to be understood and we are confronted with a double sum for any value of the free index k. Hence,

$$a_{ii}b_{jk}c_j = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ii}b_{jk}c_j$$

$$= \sum_{j=1}^{2} a_{11}b_{jk}c_j + \sum_{j=1}^{2} a_{22}b_{jk}c_j$$

$$= a_{11}b_{1k}c_1 + a_{11}b_{2k}c_2 + a_{22}b_{1k}c_1 + a_{22}b_{2k}c_2.$$

I did the sum over i first followed by the sum over j. Of course, it could have been done the other way around with the same result.

One point that is good to note and it is clearly seen in the last two examples is that after writing an expression in its full form, no dummy index would be present in the outcome anymore and we get an expression that depends only on free index or indices. This point could be very helpful to detect blunders in our calculations, somewhat similar to dimensional analysis of physical equations. Based on this similarity, let us call this rule the free index analysis rule. For instance, employing this rule, it is readily seen that the equation $x_n y_n = a_{ijk}b_i c_j$ is not a consistent one and, if this equation is a product of our calculations, we realize that there should be some mistake somewhere. The reason is as follows. On the left hand side n is the dummy index and after being summed over we get an expression that is not dependent on any index, while on the right hand side after the expression is summed over two dummy indices i and j, we are left with an expression that depends on the free index k. Thus, from free indices point of view this equation is not consistent.

Example : 4.18

Let A with entries a_{ij}, B with entries b_{ij}, and C with entries c_{ij} be $m \times n$, $n \times p$, and $m \times p$ matrices, respectively, such that $C = AB$. From elementary linear algebra we know that for any $1 \le i \le m$ and any $1 \le j \le p$, we have

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Using summation convention, the formula above is written simply as

$$c_{ij} = a_{ik} b_{kj}$$

CHAPTER - 5

THE *e*-SYSTEMS AND THE GENERALIZED KRÖNECKER DELTAS

In this chapter, I have discussing about completely symmetric, completely skew-symmetric and the generalized krönecker deltas.

Definition : 5.1

The system of quantities $A^{i_1 \ldots i_k}$ (or $A_{i_1 \ldots i_k}$) depending on k indices, is said to be **completely symmetric** if the value of the symbol A is unchanged by any permutation of the indices.

Definition : 5.2

The systems $A^{i_1 \ldots i_k}$ or ($A_{i_1 \ldots i_k}$) depending on k indices, is said to be **completely skew-symmetric** if the value of the symbol A is unchanged by any even permutation of the indices and A merely changes the sign after an odd permutation of the indices.

Any permutation of n distinct objects say a permutation of n distinct integers, can be accomplished by a finite number of interchanges of pairs of these objects and that the number of interchanges required to bring about a given permutation form a perscribed order is always even or always odd.

In any skew-symmetric system, the term containing two like indices is necessarily zero. Thus if one has a skew-symmetric system of quantities $A_{ijk}$where i, j, k assume value 1, 2, 3. Then

$$A_{122} = A_{112} = 0$$

$$A_{123} = -A_{213},\ A_{312} = A_{123} \text{ etc.}$$

In general, the components A_{ijk} of a skew-symmetric system satisfy the relations.

$$A_{ijk} = -A_{ikj} = -A_{jik}$$

$$A_{ijk} = A_{jki} = A_{kij}$$

Definition : 5.3

Consider a skew-symmetric system of quantities $e_{i_1 \dots i_n}$ (or $e^{i_1 \cdots i_n}$) in which the indices $e^{i_1 \cdots i_n}$ assume values $1,2,\dots,n$. The system $e_{i_1 \dots i_n}$ (or $e^{i_1 \cdots i_n}$) is said to be the ***e*-system** if

$$e_{i_1 \dots i_n} \ (\text{or } e^{i_1 \cdots i_n}) \begin{cases} = +1; & \text{when } i_1, i_2, \dots, i_n \\ & \text{an even permutation of number } 1, 2, \dots, n \\ = -1; & \text{when } i_1, i_2, \dots, i_n \\ & \text{an odd permutation of number } 1, 2, \dots, n \\ = 0; & \text{in all other cases} \end{cases}$$

Example : 5.4

Find the components of system e_{ij} when i, j takes the value 1,2.

Solution :

The components of system e_{ij} are

$$e_{11},\ e_{12},\ e_{21},\ e_{22}.$$

By definition of e-system, we have

$e_{11} = 0,$ indices are same

$e_{12} = 1,$ since $i\,j$ has even permutation of 12

$e_{21} = -e_{21} = -1$ since $i\,j$ has odd permutation of 12

$e_{22} = 0,$ indices are same

Example : 5.5

Find the components of the system e_{ijk}.

Solution :

By the definition of e-system,

$$e_{123} = e_{231} = e_{321} = 1$$

$$e_{213} = e_{132} = e_{321} = -1$$

$$e_{ijk} = 0 \text{ if any two indices are same.}$$

Definition : 5.6

A symbol $\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ depending on k superscripts and k subscripts each of which take values from 1 to n, is called a **generalized Krönecker delta** provided that

(a) it is completely skew-symmetric in superscripts and subscripts

(b) if the superscripts are distinct from each other and the subscripts are the same set of numbers as the superscripts.

The value of symbol, $\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ is

$$\begin{cases} = 1; \text{ an even number of transposition is required to arrange superscripts in the superscripts in the same order as subscripts.} \\ = -1; \text{ where odd number of transpositions arrange the superscripts in the same order as subscripts} \\ = 0; \text{ in all other cases the value of the symbolis zero} \end{cases}$$

Example : 5.7

Find the values of δ^{ij}_{kl} .

Solution :

By definition of generalised Kronecker Delta, $\delta^{ij}_{kl} = 0$ if $i = j$ or $k = l$ or if the set *ij*is not the set *kl.*

i.e., $\delta^{11}_{pq} = \delta^{22}_{pq} = \delta^{23}_{13} = \cdots = 0$

δ^{ij}_{kl} = 1 if *kl* is an even permutation of *ij*

i.e., $\delta^{12}_{12} = \delta^{21}_{21} = \delta^{13}_{13} = \delta^{31}_{31} = \delta^{23}_{23} = \cdots = 1$

and $\delta^{ij}_{kl} = -1$ if kl is an odd permutation of ij.

i.e., $$\delta^{12}_{21} = \delta^{31}_{13} = \delta^{13}_{31} = \delta^{21}_{12} = \cdots = -1$$

Theorem : 5.8

To prove that the direct product $e^{i_1 i_2 \cdots i_n} e_{j_1 j_2 \cdots j_n}$ of two systems $e^{i_1 \cdots i_n}$ and $e_{j_1 j_2 \cdots j_n}$ is the generalized Krönecker delta.

Proof :

By definition of generalized Krönecker delta, the product $e^{i_1 i_2 \cdots i_n} e_{j_1 j_2 \cdots j_n}$ has the following values.

(i) Zero if two or more subscripts or superscripts are same.

(ii) +1, if the difference in the number of transpositions of $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ from $1,2,\ldots,n$ is an even number.

(iii) –1, if the difference in the number of transpositions of $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ from $1, 2,\ldots,n$ an odd number.

Thus

$$e^{i_1 i_2 \cdots i_n} e_{j_1 j_2 \cdots j_n} = \delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}$$

Theorem : 5.9

To prove that

(i) $e^{i_1 i_2 \ldots i_n} = \delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}$

(ii) $e_{i_1 i_2 \ldots i_n} = \delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}$

Proof :

By Definition of e-system, $e^{i_1 i_2 \ldots i_n}$(or $e_{i_1 i_2 \ldots i_n}$) has the following values.

(i) +1; if $i_1, i_2, \ldots, i_n$ is an even permutation of numbers $1,2,\ldots,n$.

(ii) −1; if $i_1, i_2, \ldots, i_n$ is an odd permutation of numbers $1,2,\ldots,n$

(iii) 0; in all other cases

Hence by Definition of generalized krönecker delta, we can write

(1) $e^{i_1 i_2 \ldots i_n} = \delta^{i_1 i_2 \ldots i_n}_{1,2,\ldots,n}$

and

(2) $e_{i_1 i_2 \ldots i_n} = \delta^{1,2,\ldots,n}_{i_1 i_2 \ldots i_n}$

Contraction of $\delta^{ijk}_{\alpha\beta\gamma}$: 5.10

Let us contract $\delta^{ijk}_{\alpha\beta\gamma}$ on k and γ. For $n = 3$, the result is

$$\delta^{ijk}_{\alpha\beta\gamma} = \delta^{ij1}_{\alpha\beta 1} + \delta^{ij2}_{\alpha\beta 2} + \delta^{ij3}_{\alpha\beta 3} = \delta^{ij}_{\alpha\beta}$$

This expression vanishes if i and j are equal or if α and β are equal.

If $i = 1$, and $j = 2$, we get $\delta^{123}_{\alpha\beta 3}$

Hence

$$\delta^{12}_{\alpha\beta} = \begin{cases} +1; & \text{if } \alpha\beta \text{ is an even permutation of } 12 \\ -1; & \text{if } \alpha\beta \text{ is an odd permutation of } 12 \\ 0; & \text{if } \alpha\beta \text{ is not permutation of } 12 \end{cases}$$

Similarly results hold for all values of α and β selected from the set of numbers 1, 2, 3.

Hence

$$\delta^{ij}_{\alpha\beta} = \begin{cases} +1; & \text{if } ij \text{ is an even permutation of } \alpha\beta \\ -1; & \text{if } ij \text{ is an odd permutation of } \alpha\beta \\ 0; & \text{if two of the subscripts or superscripts are equal or when the} \\ & \text{subscripts and superscripts are not formed from the same numbers.} \end{cases}$$

If we contract $\delta^{ij}_{\alpha\beta}$. To contract $\delta^{ij}_{\alpha\beta}$ first contract it and the multiply the result by $\frac{1}{2}$. We obtain a system depending on two indices

$$\delta^{i}_{\alpha} = \frac{1}{2}\,\delta^{ij}_{\alpha j} = \frac{1}{2}\,(\delta^{i1}_{\alpha 1} + \delta^{i2}_{\alpha 2} + \delta^{i3}_{\alpha 3})$$

It $i = 1$ in δ^{i}_{α} then we get $\delta^{1}_{\alpha} = \frac{1}{2}\,(\delta^{12}_{\alpha 2} + \delta^{13}_{\alpha 3})$

This vanishes unless α = 1 and if α = 1 then δ^{1}_{1} = 1.

Similar result can be obtained by setting i = 2 or i = 3. Thus δ^{i}_{α} has the values.

(i) 0 if $i \neq \alpha$, ($\alpha, i = 1,2,3$)

(ii) 1 if $i = \alpha$.

By counting the number of terms appearing in the sums. In general we have

$$\delta^{i}_{\alpha} = \frac{1}{n-1}\,\delta^{ij}_{\alpha j} \text{ and } \delta^{ij}_{ij} = n(n-1) \qquad \text{….. (1)}$$

We can also deduce that

$$\delta^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r} = \frac{(n-k)!}{(n-r)!}\,\delta^{i_1 i_2 \ldots i_r i_{r-1} \ldots i_k}_{j_1 j_2 \ldots j_r j_{r-1} \ldots j_k} \qquad \text{….. (2)}$$

and

$$\delta^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r} = n(n-1)\,(n-2)\,\ldots\,(n-r+1) = \frac{n!}{n-r!} \qquad \text{….(3)}$$

or

$$e^{i_1 i_2 \ldots i_n} e_{i_1\, i_2 \ldots\, i_n} = \text{n!} \qquad \text{….. (4)}$$

and from (2) we deduce the relation

$$e^{\,i_1 i_2 \ldots i_r i_{r+1} \ldots i_n} e_{j_1 j_2 \ldots j_r j_{r+1} \ldots j_n} = \text{n!}$$

A STUDY ON MEAN CORDIAL LABELING AND GEOMETRIC MEAN CORDIAL LABELING OF GRAPHS

A project submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

K. S. YAAMINI

Reg.No: 19SPMT28

Under the guidance of

Ms. M. PARVATHI BANU M.Sc., M.Phil.,

DEPARTMENT OF MATHEMATICS

St. Mary's College (Autonomous), Thoothukudi

April- 2021

CERTIFICATE

This is to certify that this project work entitled **"A STUDY ON MEAN CORDIAL LABELING AND GEOMETRIC MEAN CORDIAL LABELING OF GRAPHS"** is submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of the degree of **Master of Science in Mathematics** and is the work done during the year **2020-2021** by **K. S. YAAMINI (REG. NO: 19SPMT28)**

M. Parvathi

Signature of the Guide

Signature of the Coordinator

Joyce

Signature of the Director

Director
Self Supporting Courses
St. Mary's College (Autonomous,
Thoothukudi - 628 001.

Lucia Rose

Signature of the Principal

Principal
St. Mary's College (Autonomous)
Thoothukudi - 628 001.

Signature of the Examiner

DECLARATION

I hereby declare that, the project entitled **"A STUDY ON MEAN CORDIAL LABELING AND GEOMETRIC MEAN CORDIAL LABELING OF GRAPHS"** submitted for the degree of Master of Science is my work carried out under the guidance of **Ms. M. Parvathi Banu M.Sc., M.Phil.,** Assistant Professor, Department of Mathematics(SSC), **St.Mary's College (Autonomous), Thoothukudi.**

Yaami

Station: Thoothukudi **Signature of the Student**

Date: 10.04.2021

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Place: Thoothukudi

Date: 10.04.2021

CONTENT

CHAPTER 1

PRELIMINARIES

Definition 1.1:

A ***graph*** G is an ordered triple $(V(G), E(G), \chi_G)$, where

(i) $V(G)$ is a non- empty set of vertices

(ii) $E(G)$ is the set of edges disjoint from $V(G)$

(iii) χ_G is a function from $E(G)$ to the set of all unordered pairs of elements of V.

Note 1.2:

An edge starting and ending with the same vertex is called a ***loop***. An edge with distinct ends is called a ***link***.

Definition 1.3:

A graph G is called a ***simple graph*** if

(i) it has no loops

(ii) no two of links join the same pair of vertices.

Definition 1.4:

Let $G = (V, E, \chi_G)$ be a graph. A graph $H = (V', E', \chi_H)$ is a ***subgraph*** of G if

i) $V' \subseteq V$

ii) $E' \subseteq E$

iii) χ_H is a restriction of χ_G to E'.

Definition 1.5:

A subgraph $H = (V', E', \chi_H)$ is called a ***spanning subgraph*** of $G = (V, E, \chi_G)$ if $V' = V$.

Definition 1.6:

The ***degree*** or ***valency*** of a vertex v in a graph G is the number of edges of G incident with v, counting each loop twice.

Remark 1.7:

(i) A vertex of degree 0 is called an ***isolated vertex***.

(ii) A vertex of degree 1 is called a ***end vertex*** (or) an ***pendant vertex*** (or) ***leaf***.

Note 1.8:

A graph G is ***regular*** if degree of each vertex is the same.

Definition 1.9:

A simple graph G is said to be a ***complete graph*** if every vertex is adjacent to all other vertices. A complete graph with n vertices is denoted by K_n.

Definition 1.10:

A graph G is said to be ***bipartite graph*** if $V(G)$ is partitioned into two sets X and Y such that every edge of G has one end in X and another end in Y. The pair $(X, Y)$is called a bipartition of V.

Definition 1.11:

If (X, Y) is a bipartition of a graph G such that every vertex in X is adjacent to every vertex in Y. Then the graph G is called a ***complete bipartite graph.***

If $|X| = m$ and $|Y| = n$, then the complete bipartite graph is denoted by $K_{m,n}$.

Definition 1.12:

Two vertices u and v in a graph G are said to be connected if there is a $(u, v) -$ path in G. A graph G is ***connected*** if any two vertices are connected. A graph which is not connected is said to be ***disconnected***.

Definition 1.13:

A **subdivision of an edge** e of a graph G is the subdivision of edge by introducing new vertices.

A **subdivision of a graph** G denoted by $S(G)$ is a graph resulting from the subdivision of edges in G. The subdivision of some edge with end points u and v yields a graph containing one new vertex w and with the edge set replacing e by two edges uw and wv.

Definition 1.14:

Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. We define

(i) The union of $G_1 \cup G_2$ to be (V, X) where $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$

(ii) The sum $G_1 + G_2$ as $G_1 \cup G_2$ togather with all lines joining points of V_1 to points of V_2.

(iii) The product $G_1 \times G_2$ as a graph having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$

Definition 1.15:

A finite sequence in which vertices and edges occur alternatively and which begins and ends with vertices is called a ***walk***.

The starting point is called ***origin*** and end point is called ***terminus***. The vertices in between the origin and the terminus are called ***internal vertices.***

Note 1.16:

(i) If the origin and terminus coincide in a walk, then it is called ***closed walk***.

(ii) A walk in which edges are not repeated is called a ***trail***.

(iii) A walk in which vertices are not repeated is called a ***path***.

Definition 1.17:

A closed trail in which the origin and internal vertices are distinct is called a ***cycle***. A cycle of length *n* is called a *n-cycle* and is denoted by C_n.

Definition 1.18:

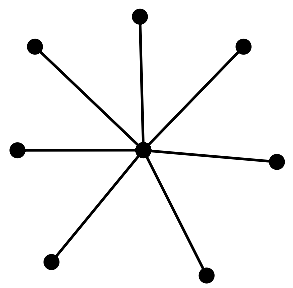
A ***directed graph*** (or) ***digraph*** D is a pair (V, A) where V is a finite non-empty set and A is a subset of $V \times V - \{(x, x)/x \in V\}$. The elements of V and A are called vertices and arcs respectively. If $(u, v) \in A$, the the arc (u, v) is said to have u as its initial vertex and v as its terminal vertex. The arc is represented by means of an arrow from *u* to *v*.

An ***undirected graph*** is a graph whose edges are not directed.

Definition 1.19:

A complete bipartite graph $K_{1,n-1}$ is called a ***star graph*** with n vertices. It is denoted by S_n.

Example 1.20:



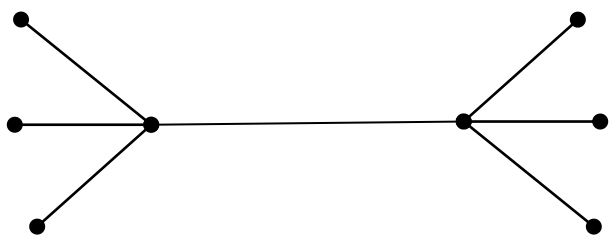
S_7

Figure 1.1

Definition 1.21:

The **bistar** $B_{n,n}$ is a graph obtained by joining the centre vertices of two copies of $K_{1,n}$ by an edge.

Example 1.22:



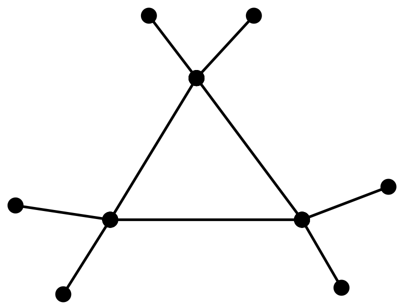
$B_{3,3}$

Figure 1.2

Definition 1.23:

Corona of two graphs $G_1 \odot G_2$, where G_1 with (m_1 edges, n_1 vertices) and G_2 with (m_2 edges, n_2 vertices) is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Example 1.24:



$C_3 \odot \overline{K_2}$

Figure 1.3

CHAPTER 2

MEAN CORDIAL LABELING OF GRAPHS

2.1 INTRODUCTION

The graphs considered here are finite, undirected and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The concept of cordial labeling was introduced by Cahit in the year 1987. Let f be a function from $V(G)$ to $\{0,1,2\}$. For each edge uv of G, assign the label $|f(u) - f(v)|$. f is called a cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i,j \in \{0, 1\}$ where, $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x respectively. A graph which admits a cordial labeling is called a cordial graph.

In this chapter, we study the concept of mean cordial labeling and the mean cordial labeling behaviour of some graphs. The symbol $\lceil x \rceil$ stands for smallest integer greater than or equal to x.

2.2 MEAN CORDIAL LABELING OF SOME STANDARD GRAPHS

Definition 2.2.1:

Let f be a function from $V(G)$ to $\{0,1,2\}$. For each edge uv of G, assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. Then, f is called a ***mean cordial labeling*** of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i,j \in \{0,1,2\}$ where, $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x respectively.

A graph with a mean cordial labeling is called a mean cordial graph.

Example 2.2.2:

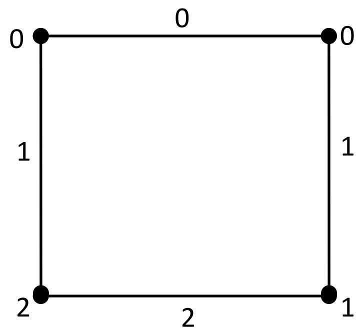


Figure 2.1

$v_f(0) = 2, v_f(1) = 1, v_f(2) = 1$

$e_f(0) = 1, e_f(1) = 2, e_f(2) = 1$

Here , $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1,2\}$

Hence f is a mean cordial labeling and G is a mean cordial graph.

Remark 2.2.3:

If we try to extend the range set of f to $\{0,1,2, \dots, k\}$ $(k > 2)$, the definition shall not workout since $v_f(0)$ becomes very small.

Theorem 2.2.4:

Any Path P_n is mean cordial.

Proof:

Let P_n be the path $u_1u_2 \dots u_n$.

Case (i): $n \equiv 0(mod\ 3)$

Let $n = 3t$

Define $f\left(u_i\right) = 2$, $1 \leq i \leq t$

$f\left(u_{t+i}\right) = 1$, $1 \leq \mathrm{i} \leq \mathrm{t}$

$f\left(u_{2t+i}\right) = 0$, $1 \leq \mathrm{i} \leq \mathrm{t}$

Then, $v_f(0) = v_f(1) = v_f(2) = t$ and $e_f(0) = t - 1, e_f(1) = e_f(2) = t$

Hence f is mean cordial labeling.

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$. Assign labels to the vertices u_i $(1 \leq i \leq n - 1)$ as in case (i). Then assign the label 0 to the vertex u_n.

Here, $v_f(0) = t + 1, v_f(1) = v_f(2) = t$ and $e_f(0) = e_f(1) = e_f(2) = t$

Hence f is mean cordial labeling.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$. Assign labels to the vertices u_i $(1 \leq i \leq n - 1)$ as in case (ii). Then assign the label 1 to the vertex u_n.

Here, $v_f(0) = v_f(1) = t + 1, v_f(2) = t$ and $e_f(0) = e_f(2) = t, e_f(1) = t + 1$

Hence f is mean cordial labeling.

Theorem 2.2.5:

The Star $K_{1,n}$ is a mean cordial iff $n \leq 2$.

Proof:

Let $V(K_{1,n}) = \{u, u_i / 1 \leq \mathrm{i} \leq n\}$ and

$E(K_{1,n}) = \{uu_{\mathrm{i}} / 1 \leq \mathrm{i} \leq n\}$

For $n \leq 2,$ the result follows from theorem 2.2.4.

Assume $n > 2$. If possible let there be a mean cordial labeling f.

Case (i): $f(u) = 0$

Then $f(u) + f(v) \leq 2$ for all edge uv

Hence $e_f(2) = 0$ which is a contradiction.

Case (ii): $f(u) = 2$

In this case, $e_f(0) = 0$ which is a contradiction.

Case (iii): $f(u) = 1$

In this case, $e_f(0) = 0$ which is again a contradiction.

Hence f is not a mean cordial labeling.

$K_{1,n}$ is not a mean cordial for all $n > 2$.

Theorem 2.2.6:

The cycle C_n is mean cordial iff $n \equiv 1,2(mod\ 3)$

Proof:

Let C_n be the cycle $u_1u_2 \ldots u_nu_1$

Case (i): $n \equiv 0(mod\ 3)$

Let $n = 3t$. Then, $v_f(0) = v_f(1) = v_f(2) = t$

Here, $e_f(0) \leq t - 1$. This is a contradiction.

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$

Define $f\left(u_i\right) = 0, \quad 1 \le i \le t + 1$

$f\left(u_{t+1+i}\right) = 1, \quad 1 \le i \le t$

$f\left(u_{2t+1+i}\right) = 2, \quad 1 \le i \le t$

Then, $v_f(0) = t + 1, v_f(1) = v_f(2) = t$ and $e_f(0) = e_f(2) = t, e_f(1) = t + 1$

Hence f is a mean cordial labeling.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$

Define $f\left(u_i\right) = 0, \ 1 \le i \le t + 1$

$f\left(u_{t+1+i}\right) = 1, \ 1 \le i \le t$

$f\left(u_{2t+1+i}\right) = 2, \ 1 \le i \le t + 1$

Then, $v_f(1) = t, v_f(0) = v_f(2) = t + 1$ and $e_f(1) = e_f(2) = t + 1, e_f(0) = t$

Hence f is a mean cordial labeling.

Theorem 2.2.7:

The complete graph K_n is mean cordial iff $n \le 2$.

Proof:

Clearly, K_1 and K_2 are mean cordial by theorem 2.2.4

Assume $n > 2$. If possible let there be a mean cordial labeling f.

Case (i): $n \equiv 0(mod\ 3)$

Let $n = 3t , t \geq 1$

Then, $v_f(0) = v_f(1) = v_f(2) = t$

$$e_f(0) = \binom{t}{2},\ e_f(1) = \binom{t}{2} + t^2 + t^2,\ e_f(2) = \binom{t}{2} + t^2$$

Then, $e_f(1) - e_f(0) = 2t^2 > 1$

which is a contradiction.

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$

Subcase 1: $v_f(0) = t + 1, v_f(1) = v_f(2) = t$

$$e_f(0) = \binom{t+1}{2},\ e_f(1) = \binom{t}{2} + t(t+1) + t(t+1),\ e_f(2) = \binom{t}{2} + t^2$$

Here, $e_f(1) - e_f(2) = t^2 + 2t > 1$

which is a contradiction.

Subcase 2: $v_f(1) = t + 1, v_f(0) = v_f(2) = t$

$$e_f(0) = \binom{t}{2},\ e_f(1) = \binom{t+1}{2} + t(t+1) + t^2,\ e_f(2) = \binom{t}{2} + t(t+1)$$

Here, $e_f(2) - e_f(0) = t^2 + t > 1$

which is a contradiction.

Subcase 3: $v_f(2) = t + 1, v_f(1) = v_f(0) = t$

$$e_f(0) = \binom{t}{2}, \quad e_f(1) = \binom{t}{2} + t(t+1) + t^2, \quad e_f(2) = \binom{t+1}{2} + t(t+1)$$

Here, $e_f(1) - e_f(0) = 2t^2 + t > 1$

which is a contradiction.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$

Subcase 1: $v_f(0) = t, v_f(1) = v_f(2) = t + 1$

$$e_f(0) = \binom{t}{2}, \quad e_f(1) = \binom{t+1}{2} + t(t+1) + t(t+1), \quad e_f(2) = \binom{t+1}{2} + (t+1)^2$$

Here, $e_f(1) - e_f(0) = 2t^2 + 3t > 1$

which is a contradiction.

Subcase 2: $v_f(1) = t, v_f(0) = v_f(2) = t + 1$

$$e_f(0) = \binom{t+1}{2}, \quad e_f(1) = \binom{t}{2} + t(t+1) + (t+1)^2, \quad e_f(2) = \binom{t+1}{2} + t(t+1)$$

Here, $e_f(2) - e_f(0) = t^2 + t > 1$

which is a contradiction.

Subcase 3: $v_f(2) = t, v_f(1) = v_f(0) = t + 1$

$$e_f(0) = \binom{t+1}{2}, \quad e_f(1) = \binom{t+1}{2} + t(t+1) + (t+1)^2, \quad e_f(2) = \binom{t}{2} + t(t+1)$$

Here, $e_f(1) - e_f(0) = (t+1)^2 + t^2 + t > 1$

which is a contradiction.

Theorem 2.2.8:

$S(K_{1,n})$ is mean cordial, where $S(G)$ denotes subdivision of G.

Proof:

Let $V(S(K_{1,n})) = \{u, u_i, v_i / 1 \leq i \leq n\}$ and

$E(S(K_{1,n})) = \{uu_i, u_iv_i / 1 \leq i \leq n\}$

Case(i): $n \equiv 0(mod\ 3)$

Let $n = 3t$

Define $f(u) = 0, f\left(u_i\right) = 0,\ 1 \leq i \leq t,\ f(u_{t+i}) = 1,\ 1 \leq i \leq 2t$

$f\left(v_i\right) = 0,\ 1 \leq i \leq t, f(v_{t+i}) = 2,\ 1 \leq i \leq 2t$

Then, $v_f(0) = 2\mathrm{t} + 1, v_f(1) = v_f(2) = 2t$

and $e_f(0) = e_f(1) = e_f(2) = 2t.$

Hence f is mean cordial labeling.

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$. Assign lables to vertices u, u_i and v_i $(1 \leq i \leq n - 1)$ as in case (i).

Then assign the label 1 and 2 to the vertices u_n and v_n respectively.

Here , $v_f(0) = v_f(1) = v_f(2) = 2t + 1$ and $e_f(0) = 2t, e_f(1) = e_f(2) = 2t + 1$

Hence f is mean cordial labeling.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$. Assign labels to vertices u, u_i and $v_i\ (1 \leq i \leq n-1)$ as in case (ii).

Then assign the label 0 and 2 to the vertices u_n and v_n respectively.

Here , $v_f(0) = v_f(2) = 2t + 2, v_f(1) = 2t + 1$

and $e_f(0) = e_f(2) = 2t + 1, e_f(1) = 2t + 2$

Hence f is mean cordial labeling.

2.3 MEAN CORDIAL LABELING OF CERTAIN GRAPHS

Theorem 2.3.1:

The graph $S\ (P_n \Theta K_1)$ is a mean cordial graph.

Proof:

Let $|V(P_n \odot K_1)| = 2n.$

Subdividing the edges of $(P_n \Theta K_1)$, we get $|V(S(P_n \odot K_1))| = 4n - 1 = m.$

Let $v_1, v_2, \ldots, v_m$ be the vertices of $V(S(P_n \Theta K_1))$.

Label the vertices of $S\ (P_n \Theta K_1)$ as it is shown the figure 2.2.

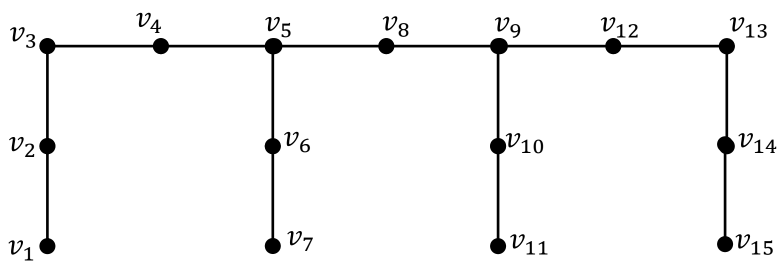


Figure 2.2

Case (i): $m \equiv 0 (mod\ 3)$

Let $m = 3t$

Define $f: V(S(P_n \odot K_1)) \rightarrow \{0,1,2\}$ by

$f(v_i) = 0,\ 1 \leq i \leq t,$

$f(v_i) = 1,\ t+1 \leq i \leq 2t,$

$f(v_i) = 2,\ 2t+1 \leq i \leq 3t.$

Then, $v_f(0) = v_f(1) = v_f(2) = t$

and $e_f(0) = t - 1,\ e_f(1) = e_f(2) = t$

Hence f is a mean cordial labeling.

Case (ii): $m \equiv 1 (mod\ 3)$

Let $m = 3t - 2$

Define $f: V(S(P_n \odot K_1)) \rightarrow \{0,1,2\}$ by

$f(v_i) = 0,\ \ 1 \leq i \leq t$

$f(v_i) = 1,\ \ t+1 \leq i \leq 2t-1$

$f(v_i) = 2,\ 2t \leq i \leq 3t-2$

Then, $v_f(0) = t,\ v_f(1) = v_f(2) = t-1$

and $e_f(0) = e_f(1) = e_f(2) = t$

Hence f is a mean cordial labeling.

Case (iii): $m \equiv 2(mod\ 3)$

Let $m = 3t - 1$

Define $f: V(S(P_n \odot K_1)) \to \{0,1,2\}$ by

$f(v_i) = 0,\ 1 \le i \le t$

$f(v_i) = 1,\ t+1 \le i \le 2t$

$f(v_i) = 2,\ 2t+1 \le i \le 3t-1$

Then, $v_f(0) = v_f(1) = t, v_f(2) = t-1$

and $e_f(0) = t-1,\ e_f(1) = t, e_f(2) = t-1$

Hence f is a mean cordial labeling.

Theorem 2.3.2:

The graph $S(B_{n,n})$ is a mean cordial graph.

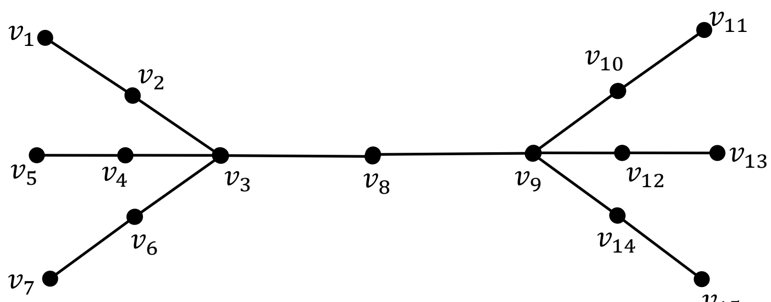
Proof:

Let $|V(B_{n,n})| = 2n + 2.$

Subdividing the edges of $B_{n,n}$, we get $|V(S(B_{n,n}))| = 4n + 3 = m.$

Let $v_1, v_2, \dots, v_m$ be the vertices of $S(B_{n,n})$.

Label the vertices of $S(B_{n,n})$ as shown in the figure 2.3.



$S(B_{3,3})$

Figure 2.3

Case (i) : $m \equiv 0(mod\ 3)$

Let $m = 3t$

Define $f: V(S(B_{n,n})) \rightarrow \{0,1,2\}$ by

$f(v_i) = 0,\ 1 \leq i \leq t,$

$f(v_i) = 1,\ t+1 \leq i \leq 2t,$

$f(v_i) = 2,\ 2t+1 \leq i \leq 3t.$

Then, $v_f(0) = v_f(1) = v_f(2) = t$

and $e_f(0) = t-1,\ e_f(1) = e_f(2) = t$

Hence f is a mean cordial labeling.

Case (ii) : $m \equiv 1(mod\ 3)$

Let $m = 3t - 2$

Define $f: V(S(B_{n,n})) \rightarrow \{0,1,2\}$ by

$f(v_i) = 0,\ 1 \leq i \leq t$

$f(v_i) = 1,\ t+1 \leq i \leq 2t-1$

$f(v_i) = 2,\ 2t \leq i \leq 3t-2$

Then, $v_f(0) = t, v_f(1) = v_f(2) = t-1$

and $e_f(0) = e_f(1) = e_f(2) = t$

Hence f is a mean cordial labeling.

Case (iii) : $m \equiv 2(mod\ 3)$

Let $m = 3t - 1$

Define $f: V(S(B_{n,n})) \to \{0,1,2\}$ by

$f(v_i) = 0, \quad 1 \le i \le t$

$f(v_i) = 1, \quad t+1 \le i \le 2t$

$f(v_i) = 2, \quad 2t+1 \le i \le 3t-1$

Then, $v_f(0) = v_f(1) = t, v_f(2) = t - 1$

and $e_f(0) = t - 1, \ e_f(1) = t, e_f(2) = t - 1$

Hence f is a mean cordial labeling.

Theorem 2.3.3:

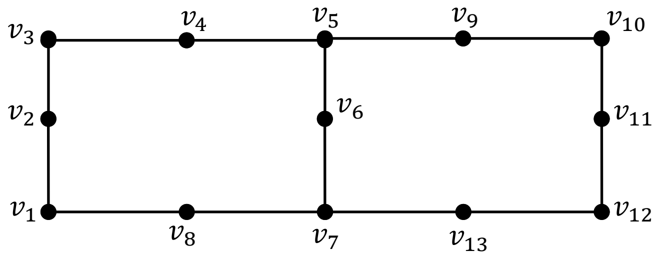
The graph $S(P_2 \times P_n), n \ge 3$ is mean cordial if $m \equiv 1(mod\ 3)$ and $m \equiv 2(mod\ 3)$ where $|V(S(P_2 \times P_n))| = 5n - 2 = m$.

Proof:

Let $\left|V(P_2 \times P_n)\right| = 2n$.

Subdividing the edges of $(P_2 \times P_n)$, we get $\left|V\left(S(P_2 \times P_n)\right)\right| = 5n - 2 = m$ and $\left|E\left(S(P_2 \times P_n)\right)\right| = 6n - 4$.

Label the vertices of $S(P_2 \times P_n)$ as it is in the figure2.4.



$S(P_2 \times P_3)$

Figure 2.4

Let $v_1, v_2, \ldots, v_m$ be the vertices of $S(P_2 \times P_n)$

Case 1: $m \equiv 1(mod\ 3)$

Let $m = 3t - 2$

Define f: $V(S(P_2 \times P_n)) \rightarrow \{0,1,2\}$ by

$f(v_i) = 0,\ 1 \leq i \leq t$

$f(v_i) = 1,\ t + 1 \leq i \leq 2t - 1$

$f(v_i) = 2,\ 2t \leq i \leq 3t - 2$

Then, $v_f(0) = t, v_f(1) = v_f(2) = t - 1$

and $e_f(0) = t - 1,\ e_f(1) = e_f(2) = t$

Hence f is a mean cordial labeling.

Case (2) : $m \equiv 2(mod\ 3)$

Let $m = 3t - 1$

Define f: $V(S(P_2 \times P_n)) \rightarrow \{0,1,2\}$ by

$f(v_i) = 0,\ 1 \leq i \leq t$

$f(v_i) = 1,\ t + 1 \leq i \leq 2t$

$f(v_i) = 2,\ 2t + 1 \leq i \leq 3t - 1$

Then, $v_f(0) = v_f(1) = t, v_f(2) = t - 1$

and $e_f(0) = t,\ e_f(1) = e_f(2) = t + 1$

Hence f is a mean cordial labeling.

Theorem 2.3.4:

The graph $S(P_2 \times P_n)$, $n \geq 3$ is not mean cordial if $m \equiv 0(mod\ 3)$ where $\left|V\left(S(P_2 \times P_n)\right)\right| = 5n - 2 = m.$

Proof:

Let $\left|V\left(S(P_2 \times P_n)\right)\right| = 5n - 2 = m$ and $\left|E\left(S(P_2 \times P_n)\right)\right| = 6n - 4.$

Let $m = 3t.$

Labeling $t = \frac{m}{3}$ vertices of $S(P_2 \times P_n)$ with 0, we get $e_f(0) = \mathrm{t} - 1.$

Since $\left|E\left(S(P_2 \times P_n)\right)\right| = 6n - 4 \geq 3t$, it is a contradiction.

Thus, the graph is not a mean cordial graph.

CHAPTER 3

GEOMETRIC MEAN CORDIAL LABELING OF GRAPHS

3.1 INTRODUCTION

In this chapter, we study the concept of geometric mean cordial labeling and the geometric mean cordial labeling behaviour of some standard graphs. The graphs considered here are finite, undirected and simple.

3.2 GEOMETRIC MEAN CORDIAL LABELING OF GRAPHS

Definition 3.2.1:

Let $G = (V, E)$ be a *(p, q)* graph. Let f be a function from $V(G)$ to $\{0,1,2\}$. For each edge uv of G, assign the label $\left\lceil\sqrt{f(u)f(v)}\,\right\rceil$, f is called a ***geometric mean cordial labeling*** of G if $\left|v_f(i) - v_f(j)\right| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$, $i, j \in \{0,1,2\}$ where, $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with $x, x \in \{0, 1, 2\}$ respectively. A graph which admits a geometric mean cordial labeling is called a geometric mean cordial graph.

Example 3.2.2:

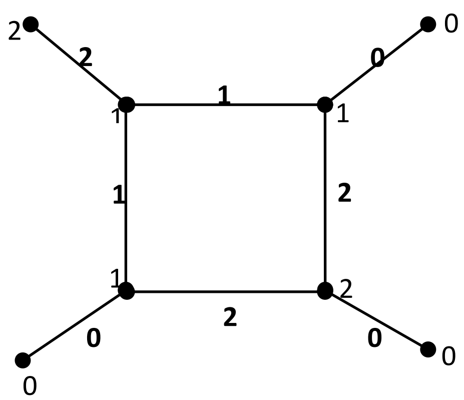


Figure 3.1

Here $v_f(0) = v_f(1) = 3$, $v_f(2)$ 2 and $e_f(0) = e_f(2) = 3$, $e_f(1) = 2$

$\therefore \left|v_f(i) - v_f(j)\right| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$, $i, j \in \{0, 1, 2\}$

Hence f is a geometric mean cordial labeling and the above graph is a geometric mean cordial graph.

Theorem 3.2.3:

Any Path P_n is geometric mean cordial.

Proof:

Let P_n be the path $u_1 u_2 ... u_n$.

Define $f: V(P_n) \longrightarrow \{0, 1, 2\}$ as follows

Case (i): $n \equiv 0 (mod\ 3)$

Let $n = 3t$

Define $f\left(u_i\right) = 0$, $1 \leq \text{i} \leq \text{t}$,

$f\left(u_{t+i}\right) = 1$, $1 \leq \text{i} \leq \text{t}$,

$f\left(u_{2t+i}\right) = 2$, $1 \leq \text{i} \leq \text{t}$.

Then $v_f(0) = v_f(1) = v_f(2) = t$ and $e_f(0) = e_f(2) = t, e_f(1) = t - 1$

Case (ii): $n \equiv 1 (mod\ 3)$

Let $n = 3t + 1$.

Define $f\left(u_i\right) = 2$, $1 \leq \text{i} \leq \text{t}$,

$f\left(u_{t+i}\right) = 1$, $1 \leq \text{i} \leq \text{t} + 1$,

$f\left(u_{2t+1+i}\right) = 0$, $1 \leq \text{i} \leq \text{t} + 1$.

Then $v_f(0) = v_f(2) = t, v_f(1) = t + 1$ and $e_f(0) = e_f(1) = e_f(2) = t$

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$

Define $f(u_i) = 2, \ 1 \leq \mathrm{i} \leq \mathrm{t},$

$f(u_{t+i}) = 1, \ 1 \leq \mathrm{i} \leq \mathrm{t} + 1,$

$f(u_{2t+1+i}) = 0, \ 1 \leq \mathrm{i} \leq \mathrm{t} + 1.$

Then $v_f(0) = v_f(1) = t + 1, v_f(2) = t$ and $e_f(0) = t + 1, e_f(1) = e_f(2) = t$

From all the above cases, we see that $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$,

for all $i, j \in \{0, 1,2\}$ and hence f is a geometric mean cordial labeling.

Example 3.2.4:

Geometric mean cordial labeling of P_6 is given below

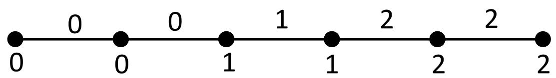


Figure 3.2

Here $v_f(0) = v_f(1) = v_f(2) = 2$ and $e_f(0) = e_f(2) = 2, e_f(1) = 1$

Theorem 3.2.5:

The Star $K_{1,n}$ is geometric mean cordial.

Proof:

Let $V(K_{1,n}) = \{u, u_i / 1 \leq \mathrm{i} \leq n\}$ and

$E(K_{1,n}) = \{uu_\mathrm{i} / 1 \leq \mathrm{i} \leq n\}$

$K_{1,n}$ has $n + 1$ vertices and n edges.

Let u be the centre of $K_{1,n}$.

Define $f: V(K_{1,n}) \to \{0, 1, 2\}$ as follows: Let $f(u) = 1$.

Case (i): $n \equiv 0(mod\ 3)$

Let $n = 3t$. Assign the labels 0, 1, 2 to each of the t vertices respectively.

Then, $v_f(0) = v_f(2) = t, v_f(1) = t + 1$ and $e_f(0) = e_f(1) = e_f(2) = t.$

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$. Assign the labels 0 to $t + 1$ vertices and the labels 1 and 2 to the remaining each of t vertices respectively.

Then, $v_f(0) = v_f(1) = t + 1, v_f(2) = t$ and $e_f(0) = t + 1, e_f(1) = e_f(2) = t.$

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$. Assign the labels 1 to t vertices and the labels 0 and 2 to the remaining each of the $t + 1$ vertices respectively.

Then, $v_f(0) = v_f(1) = v_f(2) = t + 1$ and $e_f(0) = e_f(2) = t + 1, e_f(1) = t.$

From all the above three cases, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, for all $i, j \in \{0, 1,2\}$ and hence f is a geometric mean cordial labeling.

Example 3.2.6:

Geometric mean cordial labeling of the star $K_{1,8}$ is given below

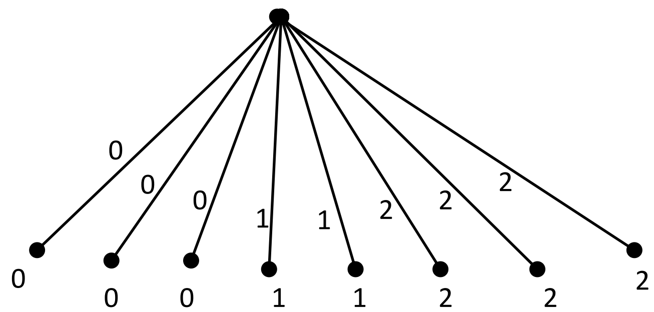


Figure 3.3

Here $v_f(0) = v_f(1) = v_f(2) = 3$ and $e_f(0) = e_f(2) = 3, e_f(1) = 2$.

Theorem 3.2.7:

The cycle C_n is geometric mean cordial if $n \equiv 1,2(mod\ 3)$.

Proof:

Let C_n be the cycle $u_1u_2 \ldots u_nu_1$. It has *n* vertices and *n* edges.

Case (i): $n \equiv 0(mod\ 3)$

Let $n = 3t$.

If C_n admits geometric mean cordial labeling *f*, then the only possibility is

$v_f(0) = v_f(1) = v_f(2) = t$ and $e_f(0) = e_f(1) = e_f(2) = t$.

If we assign 0's to *t* number of vertices in C_n, then we get $e_f(0) > t$.

Hence f is not a geometric mean cordial labeling.

For remaining two cases, define $f: V(K_{1,n}) \to \{0, 1, 2\}$ as follows:

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$. Assign the label 1 to $t + 1$ vertices and the labels 0 and 2 to the remaining each of *t* vertices.

Then $v_f(0) = v_f(2) = t, v_f(1) = t + 1$ and $e_f(0) = t + 1$, $e_f(1) = e_f(2) = t$.

Hence f is a geometric mean cordial labeling.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$. Assign the label 0 to t vertices and the labels 1 and 2 to the remaining each of $t + 1$ vertices.

Then $v_f(0) = t, v_f(1) = v_f(2) = t + 1$ and $e_f(0) = e_f(2) = t + 1,\ e_f(1) = t$.

Hence f is a geometric mean cordial labeling.

Theorem 3.2.8:

The complete graph K_n is geometric mean cordial if $n \leq 2$.

Proof:

By theorem 3.2.3, it is clear that K_1 and K_2 are geometric mean cordial.

Assume $n > 2$.

If possible let there be a geometric mean cordial labeling $f: V(K_{1,n}) \rightarrow \{0, 1, 2\}$.

Case (i): $n \equiv 0 (mod\ 3)$

Let $n = 3t, t \geq 1$.

Then, $v_f(0) = v_f(1) = v_f(2) = t$.

Consider the edges having end vertices with label 0 only. These edges contribute 1 to $e_f(0)$ and clearly there are $\binom{t}{2}$ edges having label 0.

Now consider the vertices having label 1, which are adjacent to *t* vertices having label 0. Each of these edges contributes 1 to $e_f(0)$ and clearly t^2 edges are having the label 0. The same is true for the vertices having label 2 only. Then, $e_f(0) = \binom{t}{2} + t^2 + t^2$

Consider the edges having end vertices with label 1 only. These edges contribute 1 to $e_f(1)$ and clearly there are $\binom{t}{2}$ edges having label 1. The edges incident with the vertices having the label 2 have no contribution to $e_f(1)$ and clearly $e_f(1) = 0$.The same is true for edges incident with the vertices having the labels 0 and $e_f(1) = 0$. Then, $e_f(1) = \binom{t}{2}$

Consider the edges having end vertices with label 2 only. These edges contribute 1 to $e_f(2)$ and there are $\binom{t}{2}$ edges having label 2. Consider the edges having end vertices with labels 1 and 2. These edges contribute t^2 1's to $e_f(2)$. The edges having ends with label 0 contribute 0 to $e_f(2)$. Then $e_f(2) = \binom{t}{2} + t^2$

Hence $e_f(0) - e_f(1) = 2t^2 > 1$.

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$.

Subcase (i): $v_f(0) = v_f(2) = t, v_f(1) = t + 1$

Then by the argument as in case (i), we have

$e_f(0) = \binom{t}{2} + t^2 + t^2$, $e_f(1) = \binom{t+1}{2}$, $e_f(2) = t^2 + t\binom{t}{2}$.

Hence $e_f(0) - e_f(2) = t^2 > 1$.

Subcase (ii): $v_f(0) = t + 1, v_f(1) = v_f(2) = t$

Then, we have $e_f(0) = \binom{t+1}{2} + (t^2 + t) + (t^2 + t)$, $e_f(1) = \binom{t}{2}$, $e_f(2) = t^2 + \binom{t}{2}$

Hence $e_f(1) - e_f(2) = t^2 > 1$.

Subcase (iii): $v_f(0) = v_f(1) = t, v_f(2) = t + 1$

Then, we have $e_f(0) = \binom{t}{2} + t^2 + (t^2 + t)$, $e_f(1) = \binom{t}{2}$, $e_f(2) = (t^2 + t) + \binom{t+1}{2}$

Hence $e_f(0) - e_f(1) = 2t^2 + t > 1$.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$.

Subcase (i): $v_f(0) = t, v_f(1) = v_f(2) = t + 1$.

$e_f(0) = \binom{t}{2} + (t^2 + t) + (t^2 + t),\ e_f(1) = \binom{t+1}{2},\ e_f(2) = (t + 1)^2 + \binom{t+1}{2}$.

Hence $e_f(1) - e_f(2) = (t + 1)^2 > 1$.

Subcase (ii): $v_f(0) = v_f(2) = t + 1, v_f(1) = t$.

$e_f(0) = \binom{t+1}{2} + (t^2 + t) + (t + 1)^2,\ e_f(1) = \binom{t}{2},\ e_f(2) = (t^2 + t) + \binom{t+1}{2}$

Hence $e_f(0) - e_f(2) = (t + 1)^2 > 1$.

Subcase (iii): $v_f(0) = v_f(1) = t + 1, v_f(2) = t$.

$e_f(0) = \binom{t+1}{2} + (t + 1)^2 + (t^2 + t),\ e_f(1) = \binom{t+1}{2},\ e_f(2) = (t^2 + t) + \binom{t}{2}$

Hence $e_f(0) - e_f(1) = (t + 1)^2 + t^2 + t > 1$.

In all the above cases, we see that K_n is not geometric mean cordial

Theorem 3.2.9:

The complete bipartite graph $K_{2,n}$ is not geometric mean cordial for $n > 2$.

Proof:

Let $V(K_{2,n}) = A \cup B$, where $A = \{u, v\}$ and $B = \{u_1, u_2, u_3, \dots, u_n\}$.

Then $E(K_{2,n}) = \{uu_i, vu_i: 1 \le i \le n\}$.

From theorem 3.2.3 and theorem 3.2.7, it follows that $K_{2,1}$ and $K_{2,2}$ are geometric mean cordial.

Assume $n > 2$.

Then $K_{2,n}$ has $2 + n = m$ (say) vertices and $2n = 2(m - 2)$ edges.

Case (i): $m \equiv 0 (mod\ 3)$

Let $m = 3t$.

Then $K_{2,n}$ has $6t - 4$ edges.

The only possibility of assigning labels to the vertices without violating the vertex label difference is $v_f(0) = v_f(1) = v_f(2) = t$.

Subcase (i): $f(u) = f(v) = 0$

Then, $e_f(0) = (t - 2) + t + t + (t - 2) + t + t = 6t - 4, e_f(1) = e_f(2) = 0$

Subcase (ii): $f(u) = 0, f(v) = 1$

Then, $e_f(0) = (t - 1) + (2t - 2) + t = 4t - 3, e_f(1) = t - 1, e_f(2) = t$

Subcase (iii): $f(u) = 1, f(v) = 0$. This is similar to subcase (ii).

Subcase (iv): $f(u) = 0, f(v) = 2$

Then, $e_f(0) = (t - 1) + (2t - 2) + t = 4t - 3, e_f(1) = 0, e_f(2) = 2t - 1$.

Subcase (v): $f(u) = 2, f(v) = 0$. This is similar to subcase (iv).

Subcase (vi): $f(u) = 1, f(v) = 1$

Then, $e_f(0) = t + t = 2t, e_f(1) = (t - 2) + (t - 2) = 2t - 4, e_f(2) = t + t = 2t$.

Subcase (vii): $f(u) = 1, f(v) = 2$

Then,$e_f(0) = t + t = 2t, e_f(1) = t - 1, e_f(2) = (t - 1) + (t - 1) + (t - 1) = 3t - 3$.

Subcase (viii): $f(u) = 2, f(v) = 1$. This is similar to subcase (vii).

Subcase (ix): $f(u) = 2, f(v) = 2$

Then, $e_f(0) = t + t = 2t, e_f(1) = 0, e_f(2) = t + (t - 2) + t + (t - 2) = 4t - 4$.

In all the above subcases, we observe that if atleast any one of the labeling of the vertices u and v is zero, then $e_f(0) \geq 3t - 2$. Also if either or both of the vertices u and v are having the label 2, then $e_f(2) \geq 3t - 3$.

Hence, in all the subcases, we see that $K_{2,n}$ is not geometric mean cordial. Thus in the following cases, we consider only the subcases in which both the vertices u and v are having label 1.

Case (ii): $m \equiv 1(mod\ 3)$

Let $m = 3t + 1$. Then $K_{2,n}$ has $6t - 2$ edges.

Suppose $f(u) = f(v) = 1$.

Subcase (a): $v_f(0) = t + 1,\ v_f(1) = v_f(2) = t.$

Then, $e_f(0) = (t + 1) + (t + 1) = 2t + 2, e_f(1) = (t - 2) + (t - 2) = 2t - 4,$

$e_f(2) = 2t$

Subcase (b): $v_f(0) = v_f(2) = t,\ v_f(1) = \mathrm{t} + 1.$

Then, $e_f(0) = t + t = 2t, e_f(1) = (t - 1) + (t - 1) = 2t - 2, e_f(2) = t + t = 2t$

Subcase (c): $v_f(0) = v_f(1) = t,\ v_f(2) = \mathrm{t} + 1.$

Then, $e_f(0) = t + t = 2t, e_f(1) = (t - 2) + (t - 2) = 2t - 4,$

$e_f(2) = (t + 1) + (t + 1) = 2t + 2.$

Case (iii): $m \equiv 2(mod\ 3)$

Let $m = 3t + 2$. Then $K_{2,n}$ has $6t$ edges.

Suppose $f(u) = f(v) = 1$.

Subcase (a): $v_f(0) = v_f(1) = t + 1,\ v_f(2) = t.$

Then, $e_f(0) = (t + 1) + (t + 1) = 2t + 2, e_f(1) = (t - 1) + (t - 1) = 2t - 2,$

$e_f(2) = 2t.$

Subcase (b): $v_f(0) = t, v_f(1) = v_f(2) = \mathrm{t} + 1$.

Then, $e_f(0) = t + t = 2t, e_f(1) = (t-1) + (t-1) = 2t - 2$,

$e_f(2) = (t+1) + (t+1) = 2t + 2$.

From all the above cases, $K_{2,n}$ is not a geometric mean cordial graph for $n > 2$.

Theorem 3.2.10:

The graph $K_{n,n}$ is not geometric mean cordial for $n \geq 3$.

Proof:

Let $V(K_{n,n}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$. Then , $K_{n,n}$ has $2n$ vertices and n^2 edges.

Case (i): $n \equiv 0 (mod\ 3)$.

Let $n = 3t$, where $t \geq 1$. Then $K_{n,n}$ has $6t$ vertices and $9t^2$ edges. If $K_{n,n}$ admits a geometric mean cordial labeling, then we must have

$$v_f(0) = v_f(1) = v_f(2) = 2t \qquad \dots(1)$$

$$e_f(0) = e_f(1) = e_f(2) = 3t^2 \qquad \dots(2)$$

Suppose (1) holds.

Since $v_f(0) = 2t$, $2t$ vertices of $V(K_{n,n})$ are labeled with 0. If these $2t$ vertices are in V_1, then they are adjacent with $3t$ vertices in V_2 and hence $2t \times 3t = 6t^2$ edges have the label 0. Similar case arises when these $2t$ vertices are in V_2.

If t vertices are in V_1 and the remaining t vertices are in V_2, then $(t \times 3t) + (t \times 2t) = 5t^2$ edges have the label 0.

In general, if $2t - i$ vertices are in V_1 and i vertices are in V_2, where $0 \leq i \leq 2t$, then $e_f(0) = (2t - i)3t + i\{3t - (2t - i)\} = 6t^2 - 2it + i^2$. The maximum value for $e_f(0) = 6t^2$ is obtained by putting $i = 0$ or $2t$ and the minimum value for $e_f(0) = 5t^2$ is obtained by putting $i = t$. Thus $5t^2 \leq e_f(0) \leq 6t^2$. This is a contradiction to $e_f(0) = 3t^2$ from (2). Thus $K_{n,n}$ is not a geometric mean cordial graph.

Case (ii): $n \equiv 1(mod\ 3)$.

Let $n = 3t + 1$, where $t \geq 1$. Then $K_{n,n}$ has $6t + 2$ vertices and $9t^2 + 6t + 1$ edges.

If $K_{n,n}$ admits a geometric mean cordial labeling, then we must have

(i) $v_f(0) = 2t,\ v_f(1) = v_f(2) = 2t + 1$ and

$e_f(0) = e_f(1) = 3t^2 + 2t, e_f(2) = 3t^2 + 2t + 1$

(ii) $v_f(0) = v_f(2) = 2t + 1,\ v_f(1) = 2\text{t}$ and

$e_f(0) = 3t^2 + 2t + 1, e_f(1) = e_f(2) = 3t^2 + 2t$

(iii) $v_f(0) = v_f(1) = 2t + 1,\ v_f(2) = 2\text{t}$ and

$e_f(0) = e_f(2) = 3t^2 + 2t, e_f(1) = 3t^2 + 2t + 1$

Suppose (i) holds.

Since $v_f(0) = 2t$, $2t$ vertices of $V(K_{n,n})$ are labeled with 0. If these $2t$ vertices are in V_1, then they are adjacent with $3t + 1$ vertices in V_2 and hence $2t \times (3t + 1) = 6t^2 + 2t$ edges have the label 0. Similar case arises when these $2t$ vertices are in V_2.

If t vertices are in V_1 and the remaining t vertices are in V_2, then

$t \times (3t + 1) + t \times (2t + 1) = 5t^2 + 2t$ edges have the label 0.

In general, if $2t - i$ vertices are in V_1 and i vertices are in V_2, where $0 \leq i \leq 2t$, then $e_f(0) = (2t - i)(3t + 1) + i\{3t + 1 - (2t - i)\} = 6t^2 + 2t - 2it + i^2$. The maximum value for $e_f(0) = 6t^2 + 2t$ is obtained by putting $i = 0$ or $2t$ and the minimum value for $e_f(0) = 5t^2 + 2t$ is obtained by putting $i = t$. Thus $5t^2 + 2t \leq e_f(0) \leq 6t^2 + 2t$.

This is a contradiction to $e_f(0) = 3t^2 + 2t$ from (i).

Suppose (ii) holds.

Since $v_f(0) = 2t + 1$, $2t + 1$ vertices of $V(K_{n,n})$ are labeled with 0. If these $2t + 1$ vertices are in V_1, then they are adjacent with $3t + 1$ vertices in V_2 and hence $(2t + 1) \times (3t + 1) = 6t^2 + 5t + 1$ edges have the label 0. Similar case arises when these $2t + 1$ vertices are in V_2.

If $2t$ vertices are in V_1 and the remaining 1 vertex is in V_2, then $2t \times (3t + 1) + 1 \times (t + 1) = 6t^2 + 3t + 1$ edges have the label 0.

In general, if $2t + 1 - i$ vertices are in V_1 and i vertices are in V_2, where $0 \leq i \leq 2t + 1$,then $e_f(0) = (2t + 1 - i)(3t + 1) + i\{(3t + 1) - ((2t + 1) - i)\}$ $= 6t^2 + 5t + 1 - 2it - i + i^2$ .The maximum value for $e_f(0) = 6t^2 + 5t + 1$ is obtained by putting $i = 0$ or $2t + 1$ and the minimum value for $e_f(0) = 6t^2 + 3t + 1$ is obtained by putting $i = 1$. Thus $6t^2 + 3t + 1 \leq e_f(0) \leq 6t^2 + 5t + 1$. This is a contradiction to $e_f(0) = 3t^2 + 2t + 1$ from (ii).

Suppose (iii) holds.

Since $v_f(1) = 2t + 1$, $2t + 1$ vertices of $V(K_{n,n})$ are labeled with 1. If t vertices are in V_1 and the remaining $t + 1$ vertices are in V_2, then $t \times (t + 1) = t^2 + t$

edges have the label 1. If $t-1$ vertices are in V_1 and the remaining $t+2$ vertices are in V_2, then $(t-1)\times(t+2)=t^2+t-2$ edges having the label 1.

In general, if $2t+1-i$ vertices are in V_1 and i vertices are in V_2, where $0 \le i \le t+2$, then $e_f(0)=(2t+1-i)\times i=2it+i-i^2$. The maximum value for $e_f(0)=t^2+t$ is obtained by putting $i=0$ or $t+2$ and the minimum value for $e_f(0)=t^2+t-2$ is obtained by putting $i=t-1$.

Thus $t^2+t-2 \le e_f(0) \le t^2+t$. This is a contradiction to $e_f(0)=3t^2+2t+1$ from (i).

Then $K_{n,n}$ is not a geometric mean cordial labeling.

Case (iii): $n \equiv 2(mod\ 3)$.

Let $n=3t+2$, where $t \ge 1$.

Then $K_{n,n}$ has $6t+4$ vertices and $9t^2+12t+4$ edges. If $K_{n,n}$ admits a geometric mean cordial labeling, then we must have

(i) $v_f(0)=v_f(1)=2t+1, v_f(2)=2t+2$ and

$e_f(0)=e_f(1)=3t^2+4t+1, e_f(2)=3t^2+2t+1$

(ii) $v_f(0)=v_f(2)=2t+1,\ v_f(1)=2\mathrm{t}+2$ and

$e_f(0)=3t^2+4t+2, e_f(1)=e_f(2)=3t^2+4t+1$

(iii) $v_f(0)=2t+2, v_f(1)=v_f(2)=2t+1$, and

$e_f(0)=e_f(2)=3t^2+4t+1, e_f(1)=3t^2+4t+2$

Suppose (i) holds.

Since $v_f(0) = 2t + 1$, $2t + 1$ vertices of $V(K_{n,n})$ are labeled with 0. If these $2t + 1$ vertices are in V_1, then they are adjacent with $3t + 2$ vertices in V_2 and hence $(2t + 1) \times (3t + 2) = 6t^2 + 7t + 2$ edges have the label 0. Similar case arises when these $2t + 1$ vertices are in V_2.

If $2t$ vertices are in V_1 and the remaining one vertex is in V_2, then $2\mathrm{t} \times (3\mathrm{t} + 2) + 1 \times (\mathrm{t} + 2) = 6\mathrm{t}^2 + 5\mathrm{t} + 2$ edges have the label 0.

In general, if $2t + 1 - i$ vertices are in V_1 and i vertices are in V_2, where $0 \leq i \leq 2t + 1$, then $e_f(0) = ((2t + 1) - i)(3t + 2) + i\{(3t + 2) - ((2t + 1) - i)\} = 6t^2 + 7t + 2 - 2it - i + i^2$. The maximum value for $e_f(0) = 6t^2 + 7t + 2$ is obtained by putting $i = 0$ or $2t + 1$ and the minimum value for $e_f(0) = 6t^2 + 5t + 2$ is obtained by putting $i = 1$. Thus $6t^2 + 5t + 2 \leq e_f(0) \leq 6t^2 + 7t + 2$.

This is a contradiction to $e_f(0) = 3t^2 + 4t + 1$ from (i).

Suppose (ii) holds.

Since $v_f(1) = 2t + 2$, $2t + 2$ vertices of $V(K_{n,n})$ are labeled with 1.If these $2t + 2$ vertices are in V_1, then there is no vertex labeled 1 in V_2 and hence $(2t + 1) \times 0 = 0$ edges have the label 1. Similar case arises when these $2t + 2$ vertices are in V_2. If $t + 2$ vertices are in V_1 and the remaining t vertices are in V_2, then $(t + 2) \times t = t^2 + 2t$ edges have the label 1.

In general, if $2t + 2 - i$ vertices are in V_1 and i vertices are in V_2, where $0 \leq i \leq 2t + 2$, then $e_f(1) = (2t + 2 - i) \times i = 2it + 2i - i^2$. The maximum value for

$e_f(1) = t^2 + 2t$ is obtained by putting $i = 0$ or $2t + 2$ and the minimum value for $e_f(1) = 0$ is obtained by putting $i = t$ .Thus $0 \leq e_f(1) \leq t^2 + 2t$. This is a contradiction to $e_f(1) = 3t^2 + 4t + 1$ from (ii).

Suppose (iii) holds.

Since $v_f(0) = 2t + 2$, $2t + 2$ vertices of $V(K_{n,n})$ are labeled with 0. If these $2t + 2$ vertices are in V_1, then they are adjacent with $3t + 2$ vertices in V_2 and hence $(2t + 2) \times (3t + 2) = 6t^2 + 10t + 4$ edges have the label 0. Similar case arises when these $2t + 2$ vertices are in V_2. If t vertices are in V_1 and the remaining $t + 2$ vertices are in V_2, then $\mathrm{t} \times (3\mathrm{t} + 2) + (\mathrm{t} + 2)(2\mathrm{t} + 2) = 5\mathrm{t}^2 + 8\mathrm{t} + 4$ edges have the label 0.

In general, if $2t - i$ vertices are in V_1 and i vertices are in V_2 , where $0 \leq i \leq 2t + 1$, then $e_f(0) = (2t + 2 - i)(3t + 2) + i\{(3t + 2) - ((2t + 2) - i)\} = 6t^2 + 10t + 4 - 2it - 2i + i^2$. The maximum value for $e_f(0) = 6t^2 + 10t + 4$ is obtained by putting $i = 0$ or $2t + 2$ and the minimum value for $e_f(0) = 5t^2 + 8t + 4$ is obtained by putting $i = t$.Thus $5t^2 + 8t + 4 \leq e_f(0) \leq 6t^2 + 10t + 4$. This is a contradiction to $e_f(0) = 3t^2 + 4t + 1$ from (iii). Thus $K_{n,n}$ is not a geometric mean cordial graph.

CHAPTER 4

GEOMETRIC MEAN CORDIAL LABELING OF m−SUBDIVISION OF GRAPHS

4.1 INTRODUCTION

In this chapter, we study the concept of Geometric Mean Cordial Labeling of m−Subdivision of Graphs. We construct m−subdivision of graphs for standard graphs such as path and cycle and check whether the m−subdivision of graphs are geometric mean cordial or not.

4.2 m−SUBDIVISION OF GRAPHS

Definition 4.2.1:

The operation $S_m(G)$ of a graph G is a graph G resulting from the subdivision of edges by m vertices in G. This is called ***m−subdivision*** of a graph G.

For $m = 1$, $S_1(G) = S(G)$ where $S(G)$ denotes subdivision of G.

For $m \geq 2$, $S_m(G) = S(S_{m-1}(G))$

Result 4.2.2:

The subdivision of the graph P_n is $S(P_n) \cong P_{2n-1}$ where P_{2n-1} is a path of $2n - 1$ vertices and $2n - 2$edges.

Remark 4.2.3:

From the result 4.2.2, it follows that

$S_1(P_{m+1}) = S(P_{m+1}) = P_{2(m+1)-1} = P_{2m+1}$

$S_2(P_{m+1}) = S\big(S_1(P_{m+1})\big) = S(P_{2m+1}) = P_{2(2m+1)-1} = P_{4m+1}$

$S_3(P_{m+1}) = S(S_2(P_{m+1})) = S(P_{4m+1}) = P_{2(4m+1)-1} = P_{8m+1}$

In general, we have

$S_m(P_{m+1}) = S(S_{m-1}(P_{m+1})) = S(P_{(2^{m-1}m)+1}) = P_{2(2^{m-1}m+1)-1} = P_{2^m m+1}$

Then $S_m(P_{m+1})$ is a path of $(2^m m) + 1$ vertices and $2^m m$ edges.

Result 4.2.4:

The subdivision of the graph C_n is $S(C_n) \cong C_{2n}$ where C_{2n} is a cycle of $2n$ vertices and $2n$ edges.

Remark 4.2.5:

From the result 4.2.4, it follows that

$S_1(C_n) = S(C_n) = C_{2n}$

$S_2(C_n) = S(S_1(C_n)) = S(C_{2n}) = C_{4n}$

$S_3(C_n) = S(S_2(C_n)) = S(C_{4n}) = C_{8n}$

$S_{m-1}(C_n) = S(S_{m-2}(C_n)) = S(C_{2^{m-2}n}) = C_{2^{m-1}n}$

In general, we have, $S_m(C_n) = C_{2^m n}$

Then $S_m(C_n)$ is a cycle of $2^m n$ vertices and $2^m n$ edges.

4.3 GEOMETRIC MEAN CORDIAL LABELING OF m−SUBDIVISION OF GRAPHS

Theorem 4.3.1:

$S(P_n)$ is geometric mean cordial.

Proof:

Let $P_n: u_1, u_2, \dots, u_n$ be the path of n vertices and $n - 1$ edges.

We subdivide the $n-1$ edges of P_n. Now, we get $n-1$ subdivisional vertices.

Let $s_1, s_2, \dots, s_{n-1}$ be the subdivisional vertices of P_n.

From the result 4.2.2, it follows that $S(P_n) \cong P_{2n-1}$ where P_{2n-1} is a path of $2n-1$ vertices and $2n-2$ edges. From the theorem 3.2.3, it follows that $S(P_1) \cong P_1$ and $S(P_2) \cong P_3$ which are geometric mean cordial.

Case (i): $n \equiv 0 (mod\ 3)$.

Let $n = 3t, t \geq 1$

Now the path P_{2n-1} has $6t-1$ vertices and $6t-2$ edges.

Let $V(P_{2n-1}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{s_1, s_2, \dots, s_{n-1}\}$

Define the function $f: V_1 \longrightarrow \{0,1,2\}$ for $3t$ vertices of P_n by

$$f(u_i) = 2, \quad 1 \leq i \leq t,$$

$$f(u_{i+t}) = 1, \quad 1 \leq i \leq t,$$

$$f(u_{i+2t}) = 0, \quad 1 \leq i \leq t,$$

Consider vertices of V_2. If $t = 1$, then there exists 2 subdivisional vertices. The possible labeling of these two subdivisional vertices namely s_1 and s_2 are 1 and 0, or 1 and 2 or 2 and 1. In these three combinations, we get geometric mean cordial.

If $t > 1$, $3t-1$ subdivisional vertices are labeled according to the following function

$$f(s_i) = \begin{cases} 2, & 1 \leq i \leq t-1, \\ 1, & t \leq i \leq 2t-1, \\ 0, & 2t \leq i \leq 3t-1. \end{cases}$$

Then $v_f(0) = 2t, v_f(1) = 2t, v_f(2) = 2t-1.$

$$e_f(0) = 2t, e_f(1) = 2t-1, e_f(2) = 2t-1.$$

Case(ii): $n \equiv 1(mod\ 3)$.

Let $n = 3t + 1, t \geq 1$.

Now the path P_{2n-1} has $6t + 1$ vertices and $6t$ edges.

Let $V(P_{2n-1}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{s_1, s_2, \dots, s_{n-1}\}$

Define the function $f: V_1 \longrightarrow \{0,1,2\}$ for $3t + 1$ vertices of P_n by

$$f(u_i) = 2, \quad 1 \leq i \leq t,$$

$$f(u_{i+t}) = 1, \quad 1 \leq i \leq t + 1,$$

$$f(u_{1+i+2t}) = 0, \quad 1 \leq i \leq t,$$

Consider the vertices of V_2

$$f(s_i) = \begin{cases} 2, & 1 \leq i \leq t, \\ 1, & t + 1 \leq i \leq 2t, \\ 0, & 2t + 1 \leq i \leq 3t. \end{cases}$$

Then $v_f(0) = 2t, v_f(1) = 2t + 1, v_f(2) = 2t$.

$$e_f(0) = 2t, e_f(1) = 2t, e_f(2) = 2t.$$

Case(iii): $n \equiv 2(mod\ 3)$.

Let $n = 3t + 2$.

Now the path P_{2n-1} has $6t + 3$ vertices and $6t + 2$ edges.

Let $V(P_{2n-1}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{s_1, s_2, \dots, s_{n-1}\}$

Define the function $f: V_1 \longrightarrow \{0,1,2\}$ for $3t + 2$ vertices of P_n by

$$f(u_i) = 2, \quad 1 \leq i \leq t,$$

$$f(u_{i+t}) = 1, \quad 1 \leq i \leq t + 1,$$

$$f(u_{1+i+2t}) = 0, \quad 1 \leq i \leq t + 1,$$

Consider the vertices of V_2

$$f(s_i) = \begin{cases} 2, & 1 \leq i \leq t, \\ 1, & t+1 \leq i \leq 2t, \\ 0, & 2t+1 \leq i \leq 3t+1. \end{cases}$$

Then $v_f(0) = 2t+2, v_f(1) = 2t+1, v_f(2) = 2t.$

$$e_f(0) = 2t+2, e_f(1) = 2t, e_f(2) = 2t.$$

The labeling defined does not satisfy the vertex and edge condition. To get a geometric mean cordial labeling, we change the vertex labeled 0 which is adjacent to 1 by the labeling 2.

Then, we get $v_f(0) = 2t+1, v_f(1) = 2t+1, v_f(2) = 2t+1.$

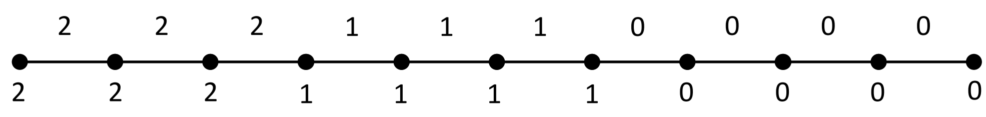
$$e_f(0) = 2t+1, e_f(1) = 2t, e_f(2) = 2t+1.$$

Now, it satisfies both the vertex and edge condition.

In all the three cases, we see that $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j \in \{0,1,2\}$, f is a geometric mean cordial labeling and hence the subdivision of a graph $S(P_n)$ is geometric mean cordial.

Example 4.3.2:

Geometric mean cordial labeling of $S(P_6)$ is given below



$S(P_6) \cong P_{11}$

Figure 4.1

In $S(P_6)$, $v_f(0) = 4, v_f(1) = 4, v_f(2) = 3$ and $e_f(0) = 4, e_f(1) = 3, e_f(2) = 3$

Theorem 4.3.3:

$S_m(P_{m+1})$ is geometric mean cordial.

Proof:

We know that $S_m(P_{m+1})$ is a graph of $2^m(m+1)$ vertices and $2^m m$ edges.

The theorem is easily verified for $m = 0,1,2$.

If $m = 0$, we get a graph P_1 of 1 vertex and no edge. Now the graph has no subdivision.

If $m = 1$, we get subdivision of a graph $S_1(P_2) = S(P_2) \cong P_3$.

From the theorem 3.2.3, it follows P_3 is geometric mean cordial, $S_1(P_2)$ is geometric mean cordial.

If $m = 2$, we get a subdivision of a graph $S_2(P_3) = S\big(S_1(P_3)\big) = S(P_3) \cong P_5$. From the theorem 3.2.3, it follows P_5 is geometric mean cordial, $S_2(P_3)$ is geometric mean cordial and hence $S(P_n)$ is geometric mean cordial.

Case(i): $m \equiv 0(mod\ 3)$.

Let $m = 3t, t \geq 1$.

Now the path has $(2^{3t}.3t) + 1$ vertices and $2^{3t}.3t$ edges. The labeling is as follows.

If we assign $0'^s$ to $2^{3t}.t$ vertices, $1'^s$ to $2^{3t}.t + 1$ vertices and $2'^s$ to $2^{3t}.t$ vertices, then

$$v_f(0) = 2^{3t}.t, v_f(1) = 2^{3t}.t + 1, v_f(2) = 2^{3t}.t \text{ and}$$

$$e_f(0) = 2^{3t}.t, e_f(1) = 2^{3t}.t, e_f(2) = 2^{3t}.t$$

In this case, $\left|v_f(i) - v_f(j)\right| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$ for all $i,j \in \{0,1,2\}$, f is a geometric mean cordial labeling and hence the $m -$ subdivision of a graph $S_m(P_{m+1})$ is a geometric mean cordial graph.

Case(ii): $m \equiv 1(mod\ 3)$

Let $m = 3t + 1, t \geq 1$.

Now the path has $(2^{3t+1}(3t + 1)) + 1$ vertices and $2^{3t+1}(3t + 1)$ edges. The labeling is as follows. There are two subcases.

Subcase(i): $v_f(0) = \frac{(2^{3t+1}(3t+1))-1}{3} + 1$, $v_f(1) = \frac{(2^{3t+1}(3t+1))-1}{3} + 1$,

$$v_f(2) = \frac{\left(2^{3t+1}(3t+1)\right)-1}{3}, t = 1, 3, 5, \dots$$

In this subcase, we get

$e_f(0) = \frac{(2^{3t+1}(3t+1))-1}{3} + 1$, $e_f(1) = \frac{(2^{3t+1}(3t+1))-1}{3}$, $e_f(2) = \frac{(2^{3t+1}(3t+1))-1}{3}$

Subcase(ii): $v_f(0) = \frac{(2^{3t+1}(3t+1))-2}{3} + 1$, $v_f(1) = \frac{(2^{3t+1}(3t+1))-2}{3} + 1$,

$$v_f(2) = \frac{\left(2^{3t+1}(3t+1)\right)-2}{3} + 1, t = 2, 4, 6, \dots$$

In this subcase, we get

$e_f(0) = \frac{(2^{3t+1}(3t+1))-2}{3} + 1$, $e_f(1) = \frac{(2^{3t+1}(3t+1))-2}{3}$, $e_f(2) = \frac{(2^{3t+1}(3t+1))-2}{3} + 1$.

In all the subcases, $\left|v_f(i) - v_f(j)\right| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$ for all $i, j \in \{0,1,2\}$, f is a geometric mean cordial labeling and hence the $m -$ subdivision of a graph $S_m(P_{m+1})$ is a geometric mean cordial graph.

Case(iii): $m \equiv 2(mod\ 3)$

Let $m = 3t + 2$.

Now the path has $(2^{3t+2}(3t + 2)) + 1$ vertices and $2^{3t+2}(3t + 2)$ edges. The labeling is as follows. There are two subcases.

Subcase(i): $v_f(0) = \frac{(2^{3t+2}(3t+2))-1}{3} + 1$, $v_f(1) = \frac{(2^{3t+2}(3t+2))-1}{3} + 1$,

$$v_f(2) = \frac{\left(2^{3t+2}(3t+2)\right)-1}{3}, t = 1, 3, 5, \ldots$$

In this subcase, we get

$e_f(0) = \frac{(2^{3t+2}(3t+2))-1}{3} + 1$, $e_f(1) = \frac{(2^{3t+2}(3t+2))-1}{3}$, $e_f(2) = \frac{(2^{3t+2}(3t+2))-1}{3}$

Subcase(ii): $v_f(0) = \frac{(2^{3t+2}(3t+2))-2}{3} + 1$, $v_f(1) = \frac{(2^{3t+2}(3t+2))-2}{3} + 1$,

$$v_f(2) = \frac{\left(2^{3t+2}(3t+2)\right)-2}{3} + 1, t = 2, 4, 6, \ldots$$

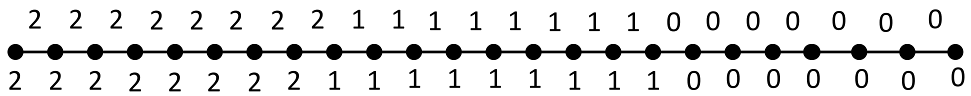
In this subcase, we get

$e_f(0) = \frac{(2^{3t+2}(3t+2))-2}{3} + 1$, $e_f(1) = \frac{(2^{3t+2}(3t+2))-2}{3}$, $e_f(2) = \frac{(2^{3t+2}(3t+2))-2}{3} + 1$.

In all the subcases, $\left|v_f(i) - v_f(j)\right| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$ for all $i, j \in \{0,1,2\}$, f is a geometric mean cordial labeling and hence the $m-$ subdivision of a graph $S_m(P_{m+1})$ is a geometric mean cordial graph.

Example 4.3.4:

Geometric mean cordial labeling of $S_3(P_4)$ is given below



$S_3(P_4) \cong P_{25}$

Figure 4.2

Here $v_f(0) = 8, v_f(1) = 9, v_f(2) = 8$ and $e_f(0) = 8, e_f(1) = 8, e_f(2) = 8$

Theorem 4.3.5:

$S(C_n)$ is geometric mean cordial iff $n \equiv 1,2(mod\ 3)$

Proof:

Let $C_n: u_1, u_2, \dots, u_n$ be the cycle of n vertices and n edges.

Let $s_1, s_2, \dots, s_n$ be the subdivisional vertices of C_n.

From the result 4.2.4, it follows that $S(C_n)$ is C_{2n} and has $2n$ vertices and $2n$ edges.

Case (i): $n \equiv 0(mod\ 3)$.

Let $n = 3t$.

Now the cycle C_{2n} has $6t$ vertices and $6t$ edges. Here C_{2n} consists of $3t$ vertices of C_n and $3t$ subdivisional vertices. If $S(C_n)$ admits a geometric mean cordial labeling f, then we should have

$$v_f(0) = v_f(1) = v_f(2) = 2t \text{ and } e_f(0) = e_f(1) = e_f(2) = 2t \quad \dots(1)$$

Consider $v_f(0) = 2t$. If we assign $0's$ to $2t$ number of vertices in $S(C_n)$, then we get $e_f(0) > 2t$ a contradiction to (1). Hence f is not a geometric mean cordial labeling.

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1$.

Now the cycle C_{2n} has $6t + 2$ vertices and $6t + 2$ edges. Here C_{2n} consists of $3t + 1$ vertices of C_n and $3t + 1$ subdivisional vertices.

Assign the label 1 to $t + 1$ vertices, and the labels 0 and 2 to remaining each of the t vertices in C_n and orderly we assign the same labeling to $3t + 1$ subdivisional vertices such that 0 to 1^{st} t subdivisional vertices $s_1, s_2, \dots, s_t$ and 1 and 2 to remaining $s_{t+1}, s_{t+2}, \dots, s_{2t+1}$ and $s_{2t+2}, s_{2t+3}, \dots, s_{3t+1}$ respectively.

Then, we get

$v_f(0) = 2t, v_f(1) = 2t + 2, v_f(2) = 2t$ and

$e_f(0) = 2t + 1, e_f(1) = 2t + 1, e_f(2) = 2t$

It does not satisfy vertex labeling. To get geometric mean cordial labeling, we change the one vertex labeled 1 adjacent to 2 by the labeling 2, then we get

$v_f(0) = 2t, v_f(1) = 2t + 1, v_f(2) = 2t + 1$ and

$e_f(0) = 2t + 1, e_f(1) = 2t, e_f(2) = 2t + 1$

If we change the one vertex labeled 1 adjacent to 0 by the labeling 2, then it would not affect the previous edge labeling, it would give the same result.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2$.

Now the cycle C_{2n} has $6t + 4$ vertices and $6t + 4$ edges. Here C_{2n} consists of $3t + 2$ vertices of C_n and $3t + 2$ subdivisional vertices.

Assign the label 0 to t vertices, and the labels 1 and 2 to remaining each of the $t + 1$ vertices in C_n and orderly we assign the same labeling to $3t + 2$ subdivisional vertices such that 0 to 1^{st} t subdivisional vertices $s_1, s_2, \dots, s_t$ and 1 and 2 to remaining $s_{t+1}, s_{t+2}, \dots, s_{2t+1}$ and $s_{2t+2}, s_{2t+3}, \dots, s_{3t+2}$ respectively.

Then, we get

$v_f(0) = 2t, v_f(1) = v_f(2) = 2t + 2$ and

$e_f(0) = 2t + 1, e_f(1) = 2t + 1, e_f(2) = 2t + 2$

It does not satisfy vertex labeling.

To get geometric mean cordial labeling, we change the one vertex labeled 1 adjacent to a vertex labeled 0 by the labeling 0 and one vertex labeled 2 adjacent to a vertex labeled 1 by the labeling 1, then we get

$v_f(0) = 2t + 1, v_f(1) = 2t + 2, v_f(2) = 2t + 1$ and

$e_f(0) = 2t + 2, e_f(1) = e_f(2) = 2t + 1$

In all cases, we see that $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j \in \{0,1,2\}$, f is a geometric mean cordial labeling and hence the subdivision of a graph $S(C_n)$ is geometric mean cordial graph.

Theorem 4.3.6:

$S_m(C_n)$ is geometric mean cordial iff $n \equiv 1,2(mod\ 3)$

Proof:

We know that $S_m(C_n)$ is the graph of $2^m n$ vertices and $2^m n$ edges.

Case (i): $n \equiv 0(mod\ 3)$

Let $n = 3t, t \geq 1$.

Now the graph consists of $2^m 3t$ vertices and $2^m 3t$ edges. If f admits a geometric mean cordial labeling, then it should be

$v_f(0) = v_f(1) = v_f(2) = 2^m t$ and $e_f(0) = e_f(1) = e_f(2) = 2^m t$

When we assign $0'^s$ to $2^m t$ vertices, we get $e_f(0) > 2^m t$.

Hence f is not geometric mean cordial labeling.

Case (ii): $n \equiv 1(mod\ 3)$

Let $n = 3t + 1, t \geq 1$.

Now the graph $S_m(C_n)$ consists of $2^m(3t + 1)$ vertices and $2^m(3t + 1)$ edges.

In this case, $S_m(C_{3t+1}) \cong C_{2^m(3t+1)}$ is a cycle that is geometric mean cordial.

Case (iii): $n \equiv 2(mod\ 3)$

Let $n = 3t + 2, t \geq 1$.

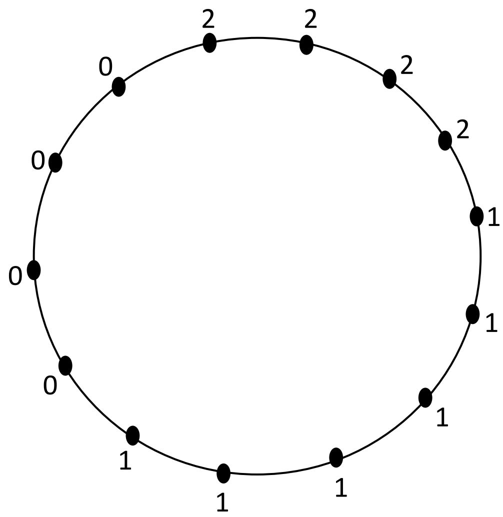
Now the graph $S_m(C_n)$ consists of $2^m(3t + 2)$ vertices and $2^m(3t + 2)$ edges.

In this case, $S_m(C_{3t+2}) \cong C_{2^m(3t+2)}$ is a cycle that is geometric mean cordial.

In all the cases, we see that $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j \in \{0,1,2\}$, f is a geometric mean cordial labeling and hence the $m -$ subdivision of a graph $S_m(C_n)$ is a geometric mean cordial graph.

Example 4.3.7:

Geometric mean cordial labeling of $S(C_7)$ is given below



$S(C_7) \cong C_{14}$

Figure 4.3

Here $v_f(0) = 4, v_f(1) = 6, v_f(2) = 4$ and $e_f(0) = 5, e_f(1) = 5, e_f(2) = 4$

The above labeling is not geometric mean cordial labeling. To make the geometric mean cordiality, we make the following changes as in figure 4.4

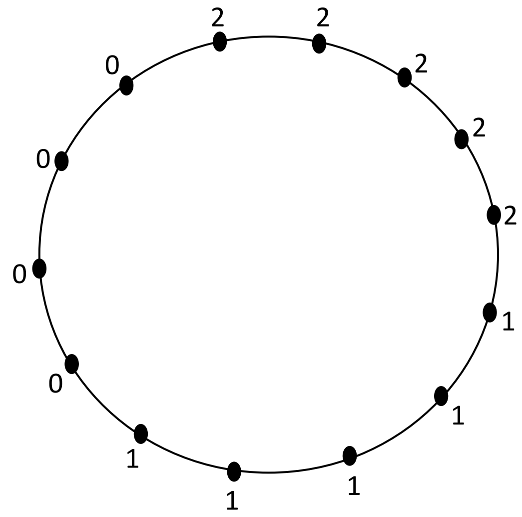


Figure 4.4

Here $v_f(0) = 4, v_f(1) = 5, v_f(2) = 5$ and $e_f(0) = 5, e_f(1) = 4, e_f(2) = 5$