CRYPTOGRAPHY

Project Report submitted to

ST. MARY'S COLLEGE (Autonomous), Tuticorin

affiliated to

Manonmaniam Sundaranar University, Tirunelveli

In participation of the requirement for the award of degree of

Bachelor of Science in Mathematics

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2022 - 2023

CERTIFICATE

We hereby declare that the project report entitled "**CRYPTOGRAPHY**" being submitted to **St. Mary's College (Autonomous), Tuticorin** affiliated to **MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI** in partial fulfilment for the award of degree of **Bachelor of Science in Mathematics** and it is a record of work done during the year 2022 – 2023 by the following students.

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DECLARATION

We hereby declare that the project reported entitled "CRYPTOGRAPHY", is our original work. It has not been submitted to any university for any degree or diploma.

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ACKNOWLEDGEMENT

First of all, we thank the Almighty for showering his blessing to undergo this project.

With immense pleasure, we register our deep sense of gratitude to our guidenand Head of the Department Dr.V.L. Stella Arputha Mary M.Sc., M.Phil., B.Ed., Ph.D for having suggested this project topic and imparted necessary guidelines throughout the period of our investigation.

We thank our beloved Principal Dr.Sr.A.S.J. Lucia Rose M.Sc., M.Phil., Ph.D., PGDCA., providing us help to carry out our research work successfully. Finally, we thank all those who extended their helping hands regarding this project.

CONTENTS

INTRODUCTION

Cryptography, or cryptology is the practice and study of techniques for secure communication in the presence of adversarial behavior. More generally, cryptography is about constructing and analyzing protocols that prevent third parties or the public from reading private messages; various aspects in information security such as data confidentiality, data integrity, authentication, and non-repudiation are central to modern cryptography. Modern cryptography exists at the intersection of the disciplines of mathematics, computer science, electrical engineering, communication science, and physics.

- ## What is cryptography?

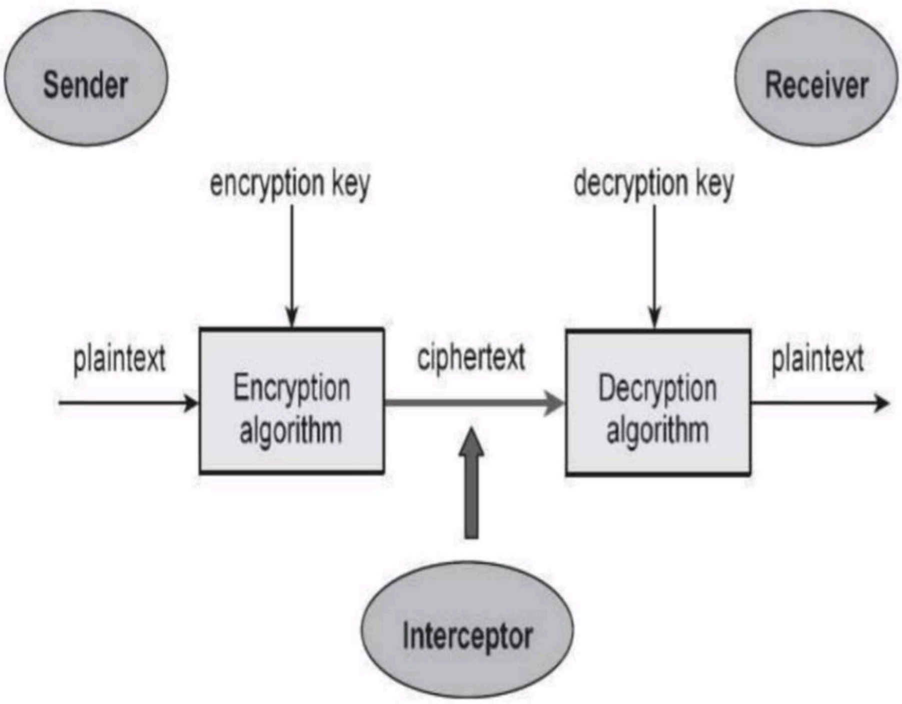
Cryptography is the science of using mathematics to encrypt and decrypt data. Cryptography enables you to store sensitive information or transmit it across insecure networks (like the Internet) so that it cannot be read by anyone except the intended recipient. It is a method of storing and transmitting data in a particular form so that only those for whom it is intended can read and process it. Cryptography not only protects data from theft or alteration, but can also be used for user authentication.

In Cryptography the techniques which are used to protect information are obtained from mathematical concepts and a set of rule based calculations known as algorithms to convert messages in ways that make it hard to decode it. These algorithms are used for cryptographic key generation, digital signing, and verification to protect data privacy, web browsing on internet and to protect confidential transactions such as credit card, debit card and for transactions.

ENCRYPTION AND DECRYPTION

Data that can be read and understood without any special measures is called plaintext or clear text. The method of disguising plaintext in such a way as to hide its substance is called encryption. Encrypting plaintext results in unreadable gibberish called cipher text. You use encryption to make sure that information is hidden from anyone for whom it is not intended, even those who can see the encrypted data. The process of reverting cipher text to its original plaintext is called decryption.

The following figure shows this process:



How does cryptography work?

A Cryptographic algorithm or cipher,is a mathematical function used in the encryption and decryption process.A cryptographic algorithm works in combination with a key,a word,number,or phrase to encrypt the plain text. The same plaintext encrypts to different cipher text with different keys.

The security of encrypted data is entirely dependent on two things:

1. The strength of the cryptographic algorithm.
2. The secrecy of the key.

- ## Why is Cryptography important?

As our business processes become increasingly more digitalized and web-based practices like online shopping became more mainstream, much bigger amounts of sensitive information circulate. That is why keeping personal data private has gained significant importance and nowadays, cyber security professionals are putting great emphasis on encryption and cryptography.

Few decades ago, hackers would only target big organizational but as the circulation of private information has become more common and information itself has turned into one of the biggest assets one can have, hackers have been targeting organizations of every size, even individuals. In fact, recent research shows that smaller organizations have been attracting hackers even more since most of them do not allocate much resource and human power to their cyber security operations. In other words, they are easier to take down. If you want to keep your business safe, you definitely need proper cryptography and encryption practices in order to keep your personnel information, customer data, business communications and such safe from the malicious attackers.

CHAPTER - I
PRELIMINARIES

Definition:

The greatest common divisor of two non-zero integers a and b is the largest integer c such that c divides both a and b. This is denoted by gcd(a, b) = c or sometimes by (a, b) = c, however we will use the former notation in this text. If the greatest common divisor of a and b is 1 then we say that a and b are **relatively prime**.

Example:

Find gcd(522, 213).

First divide 522 by 213.

$$522 = 213(2) + 96$$

Next, divide 213 by the remainder 96 and continue this process.

$$213 = 96(2) + 21$$
$$96 = 21(4) + 12$$
$$21 = 12(1) + 9$$
$$12 = 9(1) + 3$$
$$9 = 3(3) + 0.$$

So gcd(522, 213) = 3.

Definition:

We say that a number p is **prime** if it is an integer greater than 1, whose only positive divisors are 1 and itself. An integer greater than 1 which is not prime is said to be **composite.**

Definition:

For a positive integer m, which we will call our modulus, we say that two integers a and b are **congruent modulo** m if m|(a - b) or equivalently if a and b have the same remainder when divided by m. Symbolically this is written as **$a \equiv b \pmod{m}$** which is read as "a is congruent to b mod m."

Example:

23 is congruent to 3 modulo 10 since $10 \mid (23 - 3) = 20$. Also we have $59 \equiv -6 \pmod{13}$ because $13 \mid (59 - (-6)) = 65$. We find though, $7 \not\equiv 3 \pmod{5}$ since $5 \nmid (7 - 3) = 4$.

Definition:

Given an integer a and a positive integer n, satisfying gcd(a; n) = 1, we define the **multiplicative inverse of a modulo n** to be an integer d such that $ad \equiv 1 \pmod{n}$. This d is sometimes represented symbolically by $d = a^{-1}$.

The Euclidean Algorithm which we described earlier can provide a convenient way of finding multiplicative inverse modulo n. The way we can do this is by first using the algorithm to show that gcd(a,n) = 1. We then work backwards through the equations that were found in order to represent $1 = ad + nc$ for some integers d and c. We then will have that the multiplicative inverse of a modulo n is d.

Example:

Find the multiplicative inverse of 9 modulo 32. First let us perform which can be seen in the right hand column below. These remainder equations are the Euclidean Algorithm to show that gcd(32; 9) = 1; this is seen in the left hand column below. At each step we will also solve for the remainder in the equation, then labeled in a reverse ordering for later reference.

$32 = 9(3) + 5 \quad \rightarrow \quad 5 = 32 - 9(3) \quad$ (iii)

$9 = 5(1) + 4 \quad \rightarrow \quad 4 = 9 - 5(1) \quad$ (ii)

$5 = 4(1) + 1 \quad \rightarrow \quad 1 = 5 - 4(1) \quad$ (i)

Now we work backwards through these equations. First we use the last equation (i) which states 1 = [5-4(1)]: Next we use the second to last equation (ii) to substitute 4 = [9 - 5(1)] into our previous expression. Lastly we will replace 5 = [32-9(3)] (iii) and again group our terms to obtain the desired equationin terms of 32 and 9.

1 = [5 - 4(1)] (i)

= 5 - [9 - 5(1)] (ii)

= 5 - 9 + 5

= 9(-1) + 5(2) (group terms)

= 9(-1) + 2[32 - 9(3)] (iii)

= 9(-1) + 32(2) + 9(-6) (distribute)

= 32(2) + 9(-7) (group terms)

Thus we see that $9^{-1} \equiv -7 \equiv 25 \pmod{32}$.

$9(-7) \equiv -63 \equiv 1 \pmod{32}$

Definition:

A number α is said to be a primitive root modulo n if every number co prime to n is congruent to a power of α modulo n

In other words a positive integer g is said to be a primitive root of prime number 'p', if $\alpha^1 \pmod{p}$, $\alpha^2 \pmod{p}$, $\alpha^3 \pmod{p}$… $\alpha^{p-1} \pmod{p}$ are distinct

Example:

Is 2 a primitive root of prime number 5?

Here α = 2 and p = 5

$2^1 \bmod 5 \equiv 2$

$2^2 \bmod 5 \equiv 4$

$2^3 \bmod 5 \equiv 3$

$2^4 \bmod 5 \equiv 1$

Since all values are distinct, 2 is a primitive root of prime number 5.

CHAPTER - II

SHIFT CIPHER

In cryptography, a Caesar cipher, also known as Caesar's cipher, the shift cipher, Caesar's code or Caesar shift, is one of the simplest and most widely known encryption techniques. It is a type of substitution cipher in which each letter in the plaintext is replaced by a letter some fixed number of positions down the alphabet. For example, with a left shift of 3, D would be replaced by A, E would become B, and so on. The method is named after Julius Caesar, who used it in his private correspondence.

If you have a message you want to transmit securely, you can encrypt it (translate it into a secret code). One of the simplest ways to do this is with a shift cipher.

A shift cipher involves replacing each letter in the message by a letter that is some fixed number of positions further along in the alphabet. We will call this number the encryption key. It is just the length of the shift we are using. For example, upon encrypting the message "COOKIE" using a shift cipher with encryption key 3, we obtain the encoded message (or cipher text): FRRNLH.

To make all of this more mathematical, consider the following conversion table for the English alphabet:

0	1	2	3	4	5	6	7	8	9	10	11	12
A	B	C	D	E	F	G	H	I	J	K	L	M
13	14	15	16	17	18	19	20	21	22	23	24	25
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

● Using the table, we can represent the letters in our message “COOKIE” with their corresponding numbers: 2 14 14 10 8 4.

● Now add 3 (the encryption key) to each number to get: 5 17 17 13 11 7

● Now use the table to replace these numbers with their corresponding letters: FRRNLH.

ENCRYPTION

Encryption using the Shift Cipher is very easy. First we must create the ciphertext alphabet, which as discussed above is simply found by 'shifting' the alphabet to the left by the number of places given by the key. Thus a shift of 1 moves "A" to the end of the ciphertext alphabet, and "B" to the left one place into the first position. As the key gets bigger, the letters shift further along, until we get to a shift of 26, when "A" has found it's way back to the front. We have already seen a shift of 3 in the table above, and below we have a shift of 15.

Plain text Letter	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Cipher text Letter	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Once we have created the table, the encryption process is easy, as we just replace each occurrence within the plaintext of a letter with the corresponding cipher text letter as given by the cipher text alphabet.

Hence, if we wanted to encrypt the plaintext "JULIUS CAESAR" with the key he himself used, namely 3, we look along the plaintext alphabet row in the first table to find "J", and note that this encrypts to "M".

We then look for "U", and take the cipher text letter "X". Continuing in this way, we finally encrypt to "MXOLXV FDHVDU"

DECRYPTION

Decryption by the intended recipient of a cipher text received that has been encrypted using the Shift Cipher is also very simple. One can either use the table already created above, and find each letter of the cipher text in the bottom row, and replace with the corresponding plaintext letter directly above it, or the recipient could create the inverse table, with the cipher text alphabet on top, and using a shift of -3 on it, which gives the table below.

Cipher text Letter	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Plain text Letter	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W

Clearly, the encryption table and its inverse are the same as each other, only reordered. If we have received the cipher text "PDUFXV EUXWXV", and we know that it has been enciphered using the key 3, then we can use the table to decipher the message. We see that "P" represents the plaintext letter "M", "D" represents "A" and so on. Continuing in this way we retrieve the plaintext "MARCUS BRUTUS", the name of the famous conspirator in the assassination of Julius Caesar**.**

Example:

There is a small complication when we want to encrypt a message that contains a letter near the end of the alphabet. For example, if we consider the new message "PIZZA," then what letter should we use to replace the "Z" when we encrypt?

After performing a shift cipher encryption with encryption key 3, the message "PIZZA" becomes SLCCD. The letter "Z" was replaced with the letter "C," which we can view as being 3 places further along than "Z" if, after we reach "Z," we cycle the alphabet around to the beginning again.

In terms of the numerical representations of our letters, the encryption of the message "PIZZA" looks this way:

$$15\ 8\ 25\ 25\ 0 \rightarrow 18\ 11\ 2\ 2\ 3.$$

There is a handy mathematical concept that describes this very nicely. Define the following notation for integers a and b and integer $m > 1$:

$$a \equiv b \pmod{m} \text{ means m is a divisor of } a - b.$$

In our situation, we take the number m (the modulus), to be equal to the size of our character set, so $m = 26$.

Now take each number x from the representation of the message and perform the following arithmetic:

Add 3 to x, and if the result is between 0 and 25, stop; otherwise, replace $x + 3$ with the integer y between 0 and 25 that satisfies $y \equiv x + 3 \pmod{26}$.

In summary, our encryption of the message “pizza” using a shift cipher with encryption key 3 looks like this:

$$
\begin{array}{lllllll}
\mathbf{P} & \rightarrow & 15 & \rightarrow & 15+3 \equiv 18 \pmod{26} & \rightarrow & \mathbf{S} \\
\mathbf{I} & \rightarrow & 8 & \rightarrow & 8+3 \equiv 11 \pmod{26} & \rightarrow & \mathbf{L} \\
\mathbf{Z} & \rightarrow & 25 & \rightarrow & 25+3 \equiv 2 \pmod{26} & \rightarrow & \mathbf{C} \\
\mathbf{Z} & \rightarrow & 25 & \rightarrow & 25+3 \equiv 2 \pmod{26} & \rightarrow & \mathbf{C} \\
\mathbf{A} & \rightarrow & 0 & \rightarrow & 0+3 \equiv 3 \pmod{26} & \rightarrow & \mathbf{D}
\end{array}
$$

How is the original (plaintext) message recovered from the ciphertext if the encryption key is known?

The following cipher text was produced using a shift cipher with encryption key 9:

LQXLXUJCN.

To decrypt it (i.e., to recover the plaintext message), we need to add 17 (. . . or subtract 9 . . . why is that the same?) to each of the numbers representing the ciphertext letters.
Here 17 is the decryption key for the shift cipher with encryption key 9.

Again, we must sometimes replace the result of this addition with the appropriate number between 0 and 25:

$$
\begin{array}{lllllll}
\mathbf{L} & \rightarrow & 11 & \rightarrow & 11+17 \equiv 2 \pmod{26} & \rightarrow & \mathbf{C} \\
\mathbf{Q} & \rightarrow & 16 & \rightarrow & 16+17 \equiv 7 \pmod{26} & \rightarrow & \mathbf{H} \\
\mathbf{X} & \rightarrow & 23 & \rightarrow & 23+17 \equiv 14 \pmod{26} & \rightarrow & \mathbf{O} \\
\mathbf{L} & \rightarrow & 11 & \rightarrow & 11+17 \equiv 2 \pmod{26} & \rightarrow & \mathbf{C} \\
\mathbf{X} & \rightarrow & 23 & \rightarrow & 23+17 \equiv 14 \pmod{26} & \rightarrow & \mathbf{O} \\
\mathbf{U} & \rightarrow & 20 & \rightarrow & 20+17 \equiv 11 \pmod{26} & \rightarrow & \mathbf{L} \\
\mathbf{J} & \rightarrow & 9 & \rightarrow & 9+17 \equiv 0 \pmod{26} & \rightarrow & \mathbf{A} \\
\mathbf{C} & \rightarrow & 2 & \rightarrow & 2+17 \equiv 19 \pmod{26} & \rightarrow & \mathbf{T} \\
\mathbf{N} & \rightarrow & 13 & \rightarrow & 13+17 \equiv 4 \pmod{26} & \rightarrow & \mathbf{E}
\end{array}
$$

CHAPTER - III

AFFINE CIPHER

The affine cipher is a type of mono alphabetic substitution cipher , where each letter in an alphabet is mapped to its numerical equivalent, encrypted using a simple mathematical function, and converted back to a letter.

The formula used means that each letter encrypts to one other letter, and back again, meaning the cipher is essentially a standard substitution cipher with a rule governing which letter goes to which.

As such, it has the weaknesses of all substitution ciphers. Each letter is enciphered with the function (ax + b) mod 26, where b is the magnitude of the shift.

ALGORITHM

The 'key' for the Affine cipher consists of 2 numbers, we will call them a and b. The following discussion assumes the use of a 26 character alphabet (m = 26). a should be chosen to be relatively prime to m (i.e. a should have no factors in common with m). For example 15 and 26 have no factors in common, so 15 is an acceptable value for a, however 12 and 26 have factors in common (e.g. 2) so 12 cannot be used for a value of a.

When encrypting, we first convert all the letters to numbers ('a'=0, 'b'=1, ... , 'z'=25). The ciphertext letter c, for any given letter p is (remember p is the number representing a letter):

The encryption function is

$$\mathbf{c = ap + b \ (mod\ m)\ ,\ 1 \le a \le m\ ,\ 1 \le b \le m}$$

The decryption function is:

$$p = a^{-1}(c-b) \pmod{m}$$

where a^{-1} is the multiplicative inverse of a in the group of integers modulo m. To find a multiplicative inverse, we need to find a number x such that:

$$ax \equiv 1 \pmod{m}$$

Example:

In these two examples, one encrypting and one decrypting, the alphabet is going to be the letters A through Z, and will have the corresponding values found in the following table:

A	0	**N**	13
B	1	**O**	14
C	2	**P**	15
D	3	**Q**	16
E	4	**R**	17
F	5	**S**	18
G	6	**T**	19
H	7	**U**	20
I	8	**V**	21
J	9	**W**	22
K	10	**X**	23
L	11	**Y**	24
M	12	**Z**	25

ENCRYPTING

In this encrypting example, the plaintext to be encrypted is AFFINE CIPHER using the table mentioned above for the numeric values of each letter, taking a to be 5, b to be 8, and m to be 26 since there are 26 characters in the alphabet being used. Only the value of a has a restriction since it has to be coprime with 26. The possible values that a could be are 1,3,5,7,9,11,15,17,19,21,23 and 25. The value for b can be orbitary as long as a does not equal to 1 since this is the shift of the cipher. Thus the encryption function for this example will be y= E(x) =(5x+8) mod 26. The first step in encrypting the message is to write the numeric values of each letter.

Plaintext	**A**	**F**	**F**	**I**	**N**	**E**	**C**	**I**	**P**	**H**	**E**	**R**
X	0	5	5	8	13	4	2	8	15	7	4	17

Now, take each value of x and solve the first part of the equation (5x+8). After finding the value of (5x+8) for each character, take the remainder when dividing the result of (5x+8) by 26. The following table shows the first 4 steps of the encrypting process.

Plaintext	A	F	F	I	N	E	C	I	P	H	E	R
X	0	5	5	8	13	4	2	8	15	7	4	17
(5x+8)	8	33	33	48	73	28	18	48	83	43	28	93
(5x+8)mod26	8	7	7	22	21	2	18	22	5	17	2	15

The final step in encrypting the message is to look up each numeric value in the table for the corresponding letters. In this example, the encrypted text would be IHHWVCSWFRCP. The table below

Plaintext	A	F	F	I	N	E	C	I	P	H	E	R
X	0	5	5	8	13	4	2	8	15	7	4	17
(5x+8)	8	33	33	48	73	28	18	48	83	43	28	93
(5x+8)mod26	8	7	7	22	21	2	18	22	5	17	2	15
Ciphertext	**I**	**H**	**H**	**W**	**V**	**C**	**S**	**W**	**F**	**R**	**C**	**P**

DECRYPTING

In this decryption example, the cipher text that will be decrypted is the cipher text from the encryption example. The corresponding decryption function is D(y) = 21(y-8) mod 26, where a^{-1} is calculated to be 21, and b is 8. To begin write the numeric equivalents to each letter in the cipher text are shown in the table below.

Ciphertext	**I**	**H**	**H**	**W**	**V**	**C**	**S**	**W**	**F**	**R**	**C**	**P**
Y	8	7	7	22	21	2	18	22	5	17	2	15

Now the next step is to compute 21(y-8), and then take the remainder when that result is divided by 26. The following table shows the results of both computations.

Ciphertext	**I**	**H**	**H**	**W**	**V**	**C**	**S**	**W**	**F**	**R**	**C**	**P**
Y	8	7	7	22	21	2	18	22	5	17	2	15
21(y-8)	0	-21	-21	294	273	-126	210	294	-63	189	-126	147
21(y-8)mod26	0	5	5	8	13	4	2	8	15	7	4	17

The final step in decrypting the cipher text is to use the table to convert numeric values back into letters. The plaintext in this decryption is AFFINE CIPHER. Below is the table with the final step completed.

Ciphertext	**I**	**H**	**H**	**W**	**V**	**C**	**S**	**W**	**F**	**R**	**C**	**P**
Y	8	7	7	22	21	2	18	22	5	17	2	15
21(y-8)	0	-21	-21	294	273	-126	210	294	-63	189	-126	147
21(y-8)mod26	0	5	5	8	13	4	2	8	15	7	4	17
Plaintext	**A**	**F**	**F**	**I**	**N**	**E**	**C**	**I**	**P**	**H**	**E**	**R**

CHAPTER - IV

APPLICATION OF LINEAR ALGEBRA IN CRYPTOGRAPHY

In crytography, encryption is the process of concealing information – which we call plaintext – in a way that makes it unrecognisable at first glance. In order to encrypt information,we use a cipher ,or a set of steps to encode the data. In order to make the information legible, we use decryption, which is recreating the original message from the encrypted data , known as *ciphertext*. In order to decrypt information, we take the cipher that was used in the encryption process, reverse it,and apply it to the coded message. The reversed cipher is known as the key.

The concept of encryption and decryption applies to Linear Algebra through the use of matrices as plaintext/ciphertext, and matrix algebra as the cipertext/key. The matrix we use for the plaintext/ciphertext must be invertible in order for the ciphertext to be decrypted.

In order for a matrix to be invertible it has to be a square matrix, and it can't be a zero matrix so our plaintext has to be a nonzero, square matrix. In this example , we will have our plaintext can be the same message as before**, "Leave the door unlocked"**. To convert this plaintext to a matrix, we assign each character of the plaintext to an integer. For this example we'll use number 0-52:

- 0: Spaces
- 1-26: Uppercase 'A'-'Z' in the alphabet
- 27-52: Lowercase 'a'-'z' in the alphabet

‘’=0	A=1	B=2	C=3	D=4	E=5	F=6	G=7	H=8
I=9	J=10	K=11	L=12	M=13	N=14	O=15	P=16	Q=17
R=18	S=19	T=20	U=21	V=22	W=23	X=24	Y=25	Z=26
a=27	b=28	c=29	d=30	e=31	f=32	g=33	h=34	i=35
j=36	k=37	l=38	m=39	n=40	o=41	p=42	q=43	r=44
s=45	t=46	u=47	v=48	w=49	x=50	y=51	z=52	

In setting up our matrix, we have to be mindful that the matrix needs to be square, and since the amount of character in our plaintext is 23, we add 2 spaces at the end of the plaintext to get a square amount of character, 25. Now we have our plaintext matrix(which we will denote as matrix P):

$$P = \begin{bmatrix} 12 & 31 & 27 & 48 & 31 \\ 0 & 46 & 34 & 31 & 0 \\ 30 & 41 & 41 & 44 & 0 \\ 47 & 40 & 38 & 41 & 29 \\ 37 & 31 & 30 & 0 & 0 \end{bmatrix}$$

Next we choose our cipher. We will perform elementary row operations on the identity matrix,

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now performing row operations on the identity matrix we get our cipher matrix E

- $R_1 \leftrightarrow R_3$

$$I \sim \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- $R_2 \leftrightarrow R_4$

$$I \sim \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- $R_5 \rightarrow R_5 + R_1$

$$I \sim \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

- $R_4 \rightarrow R_4 + R_3$

$$I \sim \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

- $R_1 \rightarrow R_1 + R_3$

$$I \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Let this matrix be E (i.e Cipher matrix)

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Now, when we multiply our matrices P and E we get our cipher text matrix (which we will call C)

$$P * E = C = \begin{bmatrix} 87 & 48 & 43 & 31 & 31 \\ 65 & 31 & 0 & 46 & 0 \\ 115 & 44 & 30 & 41 & 0 \\ 126 & 41 & 76 & 40 & 29 \\ 67 & 0 & 37 & 31 & 0 \end{bmatrix}$$

Now, to convert our cipher matrix to cipher text, we simply type out the characters of the corresponding integers. For the integers that exceed 52, we subtract 52 from the integer and use the result (think of the interval as a circular list, where the front and end are connected). Here is our cipher text:

"ivqeeMe t KrXncO ke"

In order to decrypt this cipher text, we first find the ***key matrix***, which is the cipher matrix's inverse E^{-1} :

$$\text{Let } E^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 & x_9 & x_{10} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{16} & x_{17} & x_{18} & x_{19} & x_{20} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \end{bmatrix}$$

We know that $\mathbf{EE^{-1}=I}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 & x_9 & x_{10} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{16} & x_{17} & x_{18} & x_{19} & x_{20} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Next, we multiply our key matrix and our cipher matrix to get the original plaintext matrix:

So we want to prove that $C * E^{-1} = P$

$$\mathbf{C * E^{-1}} = \begin{bmatrix} 35 & 48 & 43 & 31 & 31 \\ 133 & 31 & 0 & 46 & 0 \\ 11 & 44 & 30 & 41 & 0 \\ 22 & 41 & 24 & 40 & 29 \\ 15 & 0 & 37 & 31 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 31 & -25 & 48 & 31 \\ 0 & 46 & -18 & 31 & 0 \\ 30 & 41 & -63 & 44 & 0 \\ -5 & 40 & -14 & 41 & 29 \\ 37 & 31 & -22 & 0 & 0 \end{bmatrix}$$

(Applying mod 52 on all negative values in the matrix, we get the following matrix)

$$= \begin{bmatrix} 12 & 31 & 27 & 48 & 31 \\ 0 & 46 & 34 & 31 & 0 \\ 30 & 41 & 41 & 44 & 0 \\ 47 & 40 & 38 & 41 & 29 \\ 37 & 31 & 30 & 0 & 0 \end{bmatrix}$$

$$= \mathbf{P}$$

We know that:

1. P = Plaintext matrix
2. E = Cipher matrix
3. C = Cipher text matrix = P * E
4. E^{-1} = Key matrix

So we can rewrite the equation C * E^{-1} = P as

$$\mathbf{P*E*E^{-1} = P}$$

The product of a matrix and its inverse is the identity matrix (according to the definition of an inverse), so we can simplify this to :

$$\mathbf{P*I = P}$$

And multiplying plaintext matrix P by the identity matrix I returns P (i.e P = P)

So multiplying the key matrix and the ciphertext matrix returns the plaintext matrix:

$$P = \begin{bmatrix} 12 & 31 & 27 & 48 & 31 \\ 0 & 46 & 34 & 31 & 0 \\ 30 & 41 & 41 & 44 & 0 \\ 47 & 40 & 38 & 41 & 29 \\ 37 & 31 & 30 & 0 & 0 \end{bmatrix}$$

And when we convert the plaintext matrix to plaintext we get the original message:

"Leave the door unlocked"

Thus completing the decryption.

CHAPTER - V

THE RSA CIPHER

One of the most well-known and widely used public-key cipher systems is the RSA Cipher. It is named for its authors Ron Risvest, Adi Shamir, and Leonard Adle-man who first publicly described the system in 1977. Clifford Cocks, a cryptographer working for the British government, independently discovered an equivalent encryption cipher to RSA in 1973, but his work was not declassified until 1997, so Risvest,Shimir and Adleman are commonly credited with the discovery of the cipher. The security of this system is based upon the difficulty of factoring large numbers into their prime factorizations. This cipher is a wonderful example of an application of elementary number theory topics to the realm of cryptography. The cipher utilizes topics such as congruence , modular exponentiation and modular multiplicative inverses. We will first explain how the cipher works and work through an example of its. We then will delve into some of the mathematics behind how it works.

Bob wishes to establish a public encryption key so that people may send him encrypted messages which only he can decrypt. To do so, first Bob will choose two secret large prime numbers, p and q. Bob then forms his modulus n by computing n = pq. Next, Bob will choose an integer e such that gcd (e, (p -1)(q -1)) = 1. This e will serve as his encryption key. Bob then computes his decryption key d such that de ≡ 1 (mod (p-1) (q-1)). That is, d is the multiplicative inverse of e modulo (p-1)(q-1). Bob then makes n and e public, keeping p, q, and d secret.

If Alice wants to send a secret message to Bob, she will first convert her plaintext message into an integer m. (Note that if m ≤ n then Alice should break up m into several blocks which are each smaller than n and send the blocks individually. For now we will assume m < n). Alice then encrypts her plaintext message m into her ciphertext c by computing,

$$\mathbf{c \equiv m^e (mod\ n)}$$

and choosing the value for c such that 0 < c < n. Alice will send the ciphertext c to Bob.

In order to read Alice's message, Bob will then decrypt her ciphertext by computing

$$\mathbf{m \equiv c^d \ (mod\ n)}$$

and choosing the value m which satisfies $0 < m < n$.We will do an example now to see how this cipher will work before we continue on to explore the mathematics behind why it works.

Example:

For this example we will use small numbers in order to simplify the work. In practice, however, numbers such as the choice of n will need to be on the order of 10^{160}. For a few of the computations we will still likely need the use of a computer.

Bob chooses his prime numbers p = 47 and q = 67 and then computes n = pq =3149.
Next he needs to choose an encryption key e so that gcd (e,Φ(n)) = 1.
We see that since p = 47 and q = 67 then Φ(n)=(p - 1)(q - 1) = 3036. Bob chooses e = 5.We can use the Euclidean Algorithm to verify that gcd(5; 3036) = 1.

We write:

$$3036 = (607)5 + 1$$

$$5 = (5)1 + 0.$$

So indeed we have gcd (5, 3036) = 1. Now Bob must compute his decryption exponent d by computing d as the multiplicative inverse of e modulo (p - 1)(q - 1). That is, we need:

$$d \equiv 5^{-1} \ (\text{mod } 3036)$$

We can compute d working backwards through Euclidean algorithm. We can also use the following formula:

$$\mathbf{d=(1+k\Phi(n))/e}$$

Try for each integer k until we receive an integer for d

k = 1 d = (1+1(3036))/5

d = 607.4

$k = 2$ $d = (1+2(3036))/5$

$d = 1214.6$

$k = 3$ $d = (1+3(3036))/5$

$d = 1821.8$

$k = 4$ $d = (1+4(3036))/5$

$\mathbf{d = 2429}$

Thus we have that d = 2429. Bob now has his secret primes p and q, his modulus n, his encryption key e and his decryption key d. Bob keeps p, q, and d secret, and he makes n and e public so that Alice can send him an encrypted message.

Suppose Alice wants to send the plaintext message “HI" to Bob. One way Alice could convert her message into an integer m is to use a basic mapping of $A \to 01$, $B \to 02$, etc. So she gets “ HI ” becomes m = 0809 = 809 which is strictly less than n = 3149. Then Alice takes Bob's encryption key and computes

$$\mathbf{c \equiv m^e \ (mod\ n)}$$

obtaining

$$c \equiv 809^5 \ (\text{mod } 3149) \equiv 2522 \ (\text{mod } 3149).$$

An efficient technique for computing powers modulo n is through the technique of successive squaring. To do this we rewrite the exponent as a sum of powers of 2. So for the above example we would write $5 = 4 + 1 = 2^2 + 2^0$. Then we would compute

$$809^5 \equiv 809^{4+1} \equiv (809^2)^2\,(809)^1 \equiv (654481)^2\,(809) \ \ (\text{mod } 3149).$$

To simplify the process we continually reduce modulo 3149 as we compute multiplication. We see that $654481 \equiv 2638 \ (\text{mod } 3149)$. So by reducing, next we will obtain:

$$(654481)^2\,(809) \equiv (2638)^2\,(809) \equiv (6959044)(809) \equiv (2903)(809)$$
$$\equiv 2348527 \equiv 2522 \ \ (\text{mod } 3149).$$

So Alice has computed that

$$c \equiv 2522 \quad (\text{mod } 3149) \qquad \text{and since } 0 < 2522 < 3149$$

she will choose c = 2522. Alice then sends her cipher text c to Bob.

Bob can now decrypt Alice's message as

$$\mathbf{m \equiv c^{d} \pmod{n}}$$

$$m \equiv 2522^{2429} \pmod{3149} \equiv 809$$

He notes that $0 < 809 < 3149$ so Bob knows that Alice's message m must be $m = 809$.

So we see Bob was able to recover and read Alice's message "HI"

CHAPTER - VI

DIFFIE – HELLMAN KEY EXCHANGE

One benefit of private-key ciphers is that they are often much faster computationally than public-key ciphers are. So for this benefit they are still widely used for communication. A very real issue can arise, however, when trying to use a private-key cipher.

Suppose Bob and Alice want to communicate privately using a symmetric cipher. To do this they both need to know a shared key which will allow them to encrypt and decrypt the information that they send to one another. But, they currently do not have any secure way of communicating (it is possible they have never even met each other before!), so they cannot just publicly discuss what key to use as it might be overheard and intercepted by Eve the eavesdropper. They need a way to securely establish a secret shared key which they can use for their private-key cipher without Eve (who presumably can read/hear all of their current communication) being able to find out what the key is.

One way that this problem can be solved is with the Diffie-Hellman Key Exchange. This key exchange was first published by Whitefield Diffie and Martin Hellman in 1976 . The idea is that we can use a type of dual public-key cipher in order to create a shared key for a private-key cipher. We will explain how this cipher works and then discuss some of the mathematical applications that we can see are used. Diffie-Hellman Key Exchange: Suppose Bob and Alice wish to establish a shared secret key for use in a private-key cipher. They can do so using the following method.

A large prime number p is chosen and a primitive root g modulo p is chosen. Both numbers p and g can be made public, and so Alice and Bob can share these with each other through insecure channels.

Once p and g are established then Alice will choose a secret large integer x and Bob will choose a secret large integer y. They can choose these such that $1 \leq x < p-1$ and $1 \leq y < p - 1$.

ENCRYPTING

Alice computes $\mathbf{X = g^x \ (mod\ p)}$, chooses the value of X satisfying $0 < X < p$, and sends X to Bob.

Similarly, Bob computes $\mathbf{Y = g^y (mod\ p)}$, chooses the value of Y satisfying $0 < Y < p$, and sends Y to Alice.

DECRYPTING

Once they have received these messages each of Alice and Bob can compute a shared private-key K.

Alice does this by computing $\mathbf{K \equiv Y^x \ (mod\ p)}$ and Bob does this by computing $\mathbf{K \equiv X^y \ (mod\ p)}$ and each chooses K such that $0 < K < p$. We can see that they have computed the same key K by observing that

$$\mathbf{Y^x \ (mod\ p) \ \equiv \ (g^y)^x \ (mod\ p) \ \equiv \ (g^x)^y \ (mod\ p) \ \equiv X^y \ (mod\ p).}$$

Let us do an example so that we can see how this system works. For our example we will be using relatively small numbers so that the computations we need to make do not get out of hand. A true implementation of the Diffie-Hellman Key Exchange would need to use a very large value of p and would use a computer to carry out all of the computations.

Example:

Bob and Alice agree to use p = 17. They need to find a primitive root g modulo p = 17. They try g = 3.

$3^1 \bmod 17 \equiv 3$	$3^9 \bmod 17 \equiv 14$
$3^2 \bmod 17 \equiv 9$	$3^{10} \bmod 17 \equiv 8$
$3^3 \bmod 17 \equiv 10$	$3^{11} \bmod 17 \equiv 7$
$3^4 \bmod 17 \equiv 13$	$3^{12} \bmod 17 \equiv 4$
$3^5 \bmod 17 \equiv 5$	$3^{13} \bmod 17 \equiv 12$
$3^6 \bmod 17 \equiv 15$	$3^{14} \bmod 17 \equiv 2$
$3^7 \bmod 17 \equiv 11$	$3^{15} \bmod 17 \equiv 6$
$3^8 \bmod 17 \equiv 16$	$3^{16} \bmod 17 \equiv 1$

Since all the values are distinct 3 is a primitive root of 17

g = 3 is a primitive root modulo p = 17, Alice then chooses her secret integer. She picks x = 12 such that. Similarly Bob chooses his secret integer y = 11 which is in the interval such that x < p and y < p. x and y are **private** keys.

Calculating public keys X and Y:

Alice computes X

$$\mathbf{X \equiv g^x \pmod p}$$

$$X \equiv 3^{12} \pmod{17}$$

$$\mathbf{X = 4}$$

Bob computes Y

$$\mathbf{Y \equiv g^y \pmod p}$$

$$Y \equiv 3^{11} \pmod{17}$$

$$\mathbf{Y = 7}$$

Then Alice and Bob exchange their public keys with each other

Once Alice receives Bob's message containing Y = 7, she computes

$$\mathbf{K_1 \equiv Y^x \pmod{p}}$$

$$K_1 \equiv 7^{12} \mod 17$$

$$\mathbf{K_1 = 13}$$

Similarly once Bob receives Alice's X = 4, he computes

$$\mathbf{K_2 \equiv X^y \pmod{p}}$$

$$K_2 \equiv 4^{11} \mod 17$$

$$\mathbf{K_2 = 13}$$

We see that $K_1 = K_2$. Now Bob and Alice have a secret shared key of K = 13. Hence their key exchange is successful.

Why is it that the Diffie-Hellman key exchange is secure? That is, if Eve is an eavesdropper listening in on Bob and Alice's communications, why is Eve not able to find K for herself? Eve will be able to know p, g, X and Y since all of these are sent via insecure communication channels. If Eve wanted to compute K she would need to compute $g^x \pmod{p}$ or $g^y \pmod{p}$.

Eve does not know x or y, however, unless she can solve the discrete logarithm problem to obtain either y from $Y \equiv g^y \pmod{p}$ or x from $X \equiv g^x \pmod{p}$. Thus the security of the Diffie-Hellman key exchange is based on the difficulty of computing discrete logarithms over finite groups. The procedure to find the exponent x and y is very difficult, in case of large prime numbers p it will take thousands of years to check all the possibilities. Since Diffie Hellman does not deal with encryption and decryption it is usually implemented along with some means of authentication, such as RSA. In real life application p is taken 2048 bit long prime number for strong security purposes.

CONCLUSION

Cryptography is the use of a series of complex puzzles to conceal and uncover messages. Equations and computer coding convert plain, readable data into a format that only an authorized system or person can read. This allows the information to remain secure and enables parties to send and receive complex messages.

As we toward a society where automated information resources are increased and cryptography will continue to increase in importance as a security mechanism.

Electronic networks for banking, shopping, inventory control, benefit and service delivery, information storage and retrieval, distributed processing, and government applications will need improved methods for access control and data security.

Cryptography is the practice of secure communication in the presence of third parties. Its objective is to make it difficult for an eavesdropper to understand the communication. Cryptography is used in a variety of applications, including email, file sharing, and secure communications. The conclusion of cryptography is that it is a powerful tool for secure communication, but it is not perfect. There are a number of ways to attack a cryptographic system, and new attacks are constantly being discovered. Cryptography is an important part of security, but it is not the only factor to consider.

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A STUDY ON BIOSTATISTICS

Project report submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI.

Affiliated to

MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI.

In partial fulfilment of the requirement for the award of degree of

Bachelor of Science in Mathematics

Submitted by

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(2022– 2023)

CERTIFICATE

We hereby declare that the project report entitled **"A STUDY ON BIOSTATISTICS"** being submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli,** in partial fulfilment for the award of degree of **Bachelor of Science in Mathematics** and it is record of work done during the year **2022 -2023** by the following students:

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DECLARATION

We hereby declare that the project report entitled "A **STUDY ON BIOSTATISTICS**" is our original work. It has not been submitted to any University for any degree or diploma.

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ACKNOWLEDGEMENT

First of all, we thank the Almighty for showering his blessing to undergo this project.

With immense pleasure, we register our deep sense of gratitude to our guide **Dr. Sr. S. Kulandai Therese M.Sc., B.Ed., M.Phil., Ph.D.,** Assistant Professor in Mathematics for having suggested the project topic and imported necessary guidelines throughout the period of our investigation.

Our sincere thanks goes to **Dr. V.L. Stella Arputha Mary M.Sc., M.Phil., B.Ed., Ph.D.,** Head of the Department of Mathematics and other staff.

We thank our beloved Principal **Dr. Sr. A.S.J. Lucia Rose M.Sc., M.Phil., Ph.D., PGDCA.,** for providing the help to carry out our project work successfully.

Finally, we thank all those who extended their helping hands regarding this project.

CONTENT

CHAPTER 1

Introduction

Statistics simply means numerical data, and is field of math that generally deals with collection of data, tabulation, and interpretation of numerical data. It is actually a form of mathematical analysis that uses different quantitative models to produce a set of experimental data or studies of real life. It is an area of applied mathematics concern with data collection analysis, interpretation, and presentation. Statistics deals with how data can be used to solve complex problems. Some people consider statistics to be a distinct mathematical science rather than a branch of mathematics. Statistics makes work easy and simple and provides a clear and clean picture of work you do on a regular basis.

Biostatistics is a branch of biological science which deals with the study and methods of collection, presentation, analysis and interpretation of data of biological research. Biostatistics is also called as biometrics since it involves man measurements and calculations. In biostatistics, the statistical methods are applied to solve biological problems.

Biostatistics, a portmanteau word constructed from biology and statistics, is defined as per the etymology; application of statistics in biology. Historically the field of statistics was emerged and systematically developed to answer various problems in biology. Later it was found that the field started having applications in various other disciplines, notably in quantitative fields of humanities (psychology and economics) such that the original meaning of statistics got steadily expanded necessitating the coinage of biostatistics to refer biological statistics. The term statistics now acquired a new meaning, "branch of mathematics that deals with the experimental design, the collection of numerical data, summarization of the data, analysis and interpretation of the data for drawing inferences on the basis of the probability."

Biostatistics (also known as biometry) are the development and application of statistical methods to a wide range of topics in biology. It encompasses the design of biological experiments, the collection and analysis of data from those experiments and the interpretation of the results. It is a specialized discipline of statistics that deals with statistical applications in the biological and health sciences. The design of health surveys, clinical trials, vital statistics, cancer survivorship studies and biological field studies are some specific biostatistical applications.

Francis Galton is called as the 'Father of Biostatistics'. He created the statistical concept 'correlation'. Sir Galton for the first time used statistical tools to study differences among human population. He also invented the use of questionnaires and surveys for collecting data on human communities.

On the other hand, the term 'mathematical biology' is defined as an interdisciplinary field encompassing all applications of mathematics to the biology. Development of this field is concurrent with that of biostatistics.

Vital statistics is that branch of statistics that deals mainly with births, deaths, human populations and the incidence of disease. **Medical statistics** is a further specially of biostatistics when the mathematical facts and data are related to health, preventive medicine and disease.

CHAPTER 2

Preliminaries

Definition 2.1

Arithmetic mean of n observations $x_1, x_2, \ldots\ldots, x_n$ is defined by

$$\text{Then} \qquad \overline{x} = \frac{x_1 + x_2 + \ldots\ldots\ldots + x_n}{n} = \frac{\sum x_i}{n}$$

This definition is useful when n is so small that grouping of the values into a frequency distribution is not necessary. Suppose $x_1, x_2, \ldots\ldots\ldots, x_n$ be the distinct values of a variate with corresponding frequencies $f_1, f_2, \ldots\ldots\ldots, f_n$.

$$\overline{x} = \frac{\sum f_i x_i}{\sum f_i} \qquad i = 1,2,\ldots\ldots.,n$$

Example 2.1.1

Consider the 10 numbers 18 ,15 , 18 , 16 , 17 , 18 , 15 , 19 ,17 , 17 .

$$\text{Then} \quad \overline{x} = \frac{18+15+18+16+17+18+15+19+17+17.}{10}$$

$$= \frac{170}{10} = 17$$

The frequency distribution for the above data is

x_i	15	16	17	18	19
f_i	2	1	3	3	1

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} \quad i = 1,2,..,5\,.$$

$$= \frac{(2 \times 15) + (1 \times 16) + (3 \times 17) + (3 \times 18) + (1 \times 19)}{2+1+3+3+1}$$

$$\bar{x} = \frac{170}{10} = 17$$

Theorem 2.1.2

If $x_1, x_2, \ldots \ldots \ldots .., x_k$ are the arithmetic means of $n_1, n_2, \ldots \ldots \ldots \ldots, n_k$

Observations then the arithmetic mean of the combined set of observation is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots\ldots + n_k\bar{x}_k}{n_1 + n_2 + \ldots\ldots + n_k}$$

Proof:

$\boldsymbol{n_1\bar{x}_1}$ is the sum of all n_1 observation in the first set.

$n_2\bar{x}_2$ is the sum of all $n_2$observation in the second set .

……… …… ……… ……… ……… ……… ……

$\boldsymbol{n_k\bar{x}_k}$ is the sum of all $n_k$observation in the k^{th} set

$\sum_{i=1}^{k} n_i\bar{x}_i$ Is the sum of all $n_1 + n_2 + \ldots \ldots + n_k$ observation in the combination set.

$$\therefore \bar{x} = \frac{1}{N}\left(\sum_{i=1}^{k} n_i\bar{x}_i\right) \text{ Where } N = \sum_{i=1}^{k} n_i\,.$$

Hence the theorem

Definition 2.2

Median of a frequency distribution is the value of the variate which divides the total frequency into two equal parts . In other words median is the value of the variate for which the cumulative frequency is $\frac{1}{2}N$ where N is the total **Median** $= \boldsymbol{l} + \frac{(\boldsymbol{N/2 - m})\boldsymbol{h}}{\boldsymbol{f_k}}$

Where l is the lower boundary of the median class , m is the $c.f.$ above the median class , f_k is the frequency corresponding to the median class and h is the width of the class .

Definition 2.3

In the case of grouped frequency distribution the mode is computed by the formula

Mode $= \boldsymbol{l} + \frac{(f - f_1)h}{2f - f_1 - f_2}$

Where l is the lower boundary of the modal class; f is the maximum frequency; f_1 and f_2 are the frequencies of the classes preceeding and following the modal class; h is the width of the class.

An alternate formula for finding the mode is also given by

Mode $= \boldsymbol{l} + \frac{hf_2}{f_1 + f_2}$ with the above notations.

Definition 2.4

The **Standard deviation** $\boldsymbol{\sigma}$ of a frequency distribution is defined by

$\boldsymbol{\sigma} = \left[\frac{\sum f_i(x_i - \bar{x})^2}{N}\right]^{1/2}$ Where $N = \sum f_i$ and $\bar{x}$ is the arithmetic mean the frequency distribution.

Definition 2.5

The square of the standard deviation of a frequency distribution called the **Variance** of the frequency distribution.

Hence **Variance** $= \boldsymbol{\sigma}^2$

Definition 2.6

Let S be a sample space associated with a random experiment. A function $X : S \to \boldsymbol{R}$ which assigns to each element $\omega \in S$ one and only one real number is called a **Random variable** $(\boldsymbol{r} . \boldsymbol{v})$ **.** Thus $X(\omega)$ represents a real number.

Definition 2.7

If a random variable X takes at most a countable number of values $x_1, x_2, \ldots, x_n, \ldots \ldots \ldots$ it is called a **discrete random variable.**

Let $P(X = x_i) = p_i$. Then by definition of probability it follows that

$\sum p_i = 1$ and for any subset A of ξ, $P(A) = \sum_{x_i \in A} p(x_i)$.

Definition 2.8

A **Frequency distribution** is an organized tabulation showing exactly how many individuals are located in each category on the scale of measurement. A frequency distribution presents an organized picture of the entire set of scores, and it shows where each individual is located related to other in the distribution

Definition 2.9

Range is the most simple and obvious measure of dispersion. It is the difference between the maximum and the minimum value of the variate.

Definition 2.10

The **Mean deviation** of a frequency distribution from any average A is defined by

M.D = $\frac{\sum f_i |x_{i} - A|}{N}$ where $N = \sum f_i$.

Definition 2.11

The **root mean square deviation** of a frequency distribution is defined to be

$s = \left[\frac{\sum f_i(x_i - A)^2}{N}\right]^{1/2}$ where A is any arbitrary origin and s^2 is called the **mean square deviation.**

Definition 2.12

Coefficient of variation of a frequency distribution is defined to be

$$\boldsymbol{C.V = \frac{\sigma}{x} \times 100}$$

Definition 2.13

Karl person's **Coefficient of correlation** between the variables x and y is defined by $\gamma_{xy} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{n\sigma_x \sigma_y}$ where $\bar{x}, \bar{y}$ are the arithmetic means and σ_x, σ_y the standard deviation of the variables x and y respectively.

Definition 2.14

A collection of random variables is **independent and identically distributed** if each random variable has the same probability distribution as the others and all are mutually independent. This property is usually abbreviated as ***i.i.d.***, ***iid***, or ***IID***. IID was first defined in statistics and finds application in different fields such as data mining and signal processing.

Definition 2.15 (Bayes' rule)

For any events A and B in a probability space (Ω, $\mathcal{F}$,P)

$$P(A|B) = \frac{P(A)\,P(B|A)}{P(B)}$$

as long as P (B) > 0.

CHAPTER 3

Kappa Statistics

3.1 INTRODUCTION

The kappa statistic is frequently used to test interrater reliability. The importance of rater reliability lies in the fact that it represents the extent to which the data collected in the study are correct representations of the variables measured. Measurement of the extent to which data collectors (raters) assign the same score to the same variable is called interrater reliability. While there have been a variety of methods to measure interrater reliability, traditionally it was measured as percent agreement, calculated as the number of agreement scores divided by the total number of scores. In 1960, Jacob Cohen critiqued use of percent agreement due to its inability to account for chance agreement. He introduced the Cohen's kappa, developed to account for the possibility that raters actually guess on at least some variables due to uncertainty. Like most correlation statistics, the kappa can range from -1 to +1. While the kappa is one of the most commonly used statistics to test interrater reliability, it has limitations. Judgments About what level of kappa should be acceptable for health research are questioned. Cohen's suggested interpretation may be too lenient for health related studies because it implies that a score as low as 0.41 might be acceptable. Kappa and percent agreement are compared, and levels for both kappa and percent agreement that should be demanded in healthcare studies are suggested.

3.2 IMPORTANCE OF MEASURING INTERRATER RELIABILITY

Many situations in the healthcare industry rely on multiple people to collect research or clinical laboratory data. The question of consistency, or *agreement* among the individuals collecting data immediately arises due to the variability among human observers. The extent of agreement among data collectors is called, "*interrater reliability*". Interrater reliability is a concern to one degree or another in most large studies due to the fact that multiple people collecting data may experience and interpret the phenomena of interest differently. Variables subject to interrater errors are readily found in clinical research and diagnostics literature. As a potential source of error, researchers are expected to implement training for data collectors to reduce the amount of variability in how they view and interpret data, and record it on the data collection instruments. Finally, researchers are expected to measure the effectiveness of their training and to report the degree of agreement (interrater reliability) among their data collectors.

3.3 MEASUREMENT OF INTERRATER RELIABILITY

There are a number of statistics that have been used to measure interrater and interrater reliability. A partial list includes percent agreement, Cohen's kappa (for two raters), the Fleiss kappa (adaptation of Cohen's kappa for 3 or more raters) the contingency coefficient, the Pearson r and the Spearman Rho, the intra-class correlation coefficient, the concordance correlation coefficient, and Krippendorff's Alpha (useful when there are multiple raters and multiple possible ratings). Use of correlation coefficients such as Pearson's r may be a poor reflection of the amount of agreement between raters resulting in extreme over or underestimates of the true level of rater agreement. In this paper, we will consider only two of the most common measures, percent agreement and Cohen's kappa.

3.3.1 Cohen's Kappa

Cohen's kappa, symbolized by the lower case Greek letter, κ is a robust statistic useful for either interrater or interrater reliability testing. Similar to correlation Coefficients, it can range from -1 to +1, where 0 represents the amount of agreement that can be expected from random chance, and 1 represents perfect agreement between the raters. While kappa values below 0 are possible, Cohen notes they are unlikely in practice .As with all correlation statistics, the kappa is a standardized value and thus is interpreted the same across multiple studies. Cohen suggested the Kappa result be interpreted as follows: values ≤ 0 as indicating no agreement and 0.01–0.20 as none to slight, 0.21–0.40 as fair, 0.41- 0.60 as moderate, 0.61–0.80 as substantial, and 0.81– 1.00 as almost perfect agreement. However, this interpretation allows for very little agreement among Raters to be described as "substantial". For percent agreement, 61% agreement can immediately be seen as problematic. Almost 40% of the data in the dataset represent faulty data. In healthcare research, this could lead to recommendations for changing practice based on faulty evidence. For a clinical laboratory, having 40% of the sample evaluations being wrong would be an extremely serious quality problem. This is the reason that many texts recommend 80% agreement as the minimum acceptable interrater agreement. Given the reduction from percent agreement that is typical in kappa results, some lowering of standards from percent agreement appears logical. However, accepting 0.40 to 0.60 as "moderate" may imply the lowest value (0.40) is adequate agreement. A more logical interpretation is suggested in given table .

Value of Kappa	Level of Agreement	% Data that are Reliable
0-.20	None	0-4%
.21-.39	Minimal	4-15%
.40-.59	Weak	15-35%
.60-.79	Moderate	35-63%
.80-.90	Strong	64-81%
Above .90	Almost Perfect	82-100%

Calculation of Cohen's kappa may be performed according to the following formula:

$$\kappa = \frac{\Pr(a) - \Pr(e)}{1 - \Pr(e)}$$

Where Pr(a) represents the actual observed agreement, and Pr(e) represents chance agreement.

Note that the sample size consists of the number of observations made across which raters are compared. Cohen specifically discussed two raters in his papers. The kappa is based on the chi-square table, and the Pr(e) is obtained through the following formula

$$\textbf{Expected (Chance) Agreement} = \frac{\left(\frac{cm^1 \text{ X } rm^1}{n}\right) + \left(\frac{cm^2 \text{X } rm^2}{n}\right)}{n}$$

where

cm^1 represents column 1 marginal

cm^2represents column 2 marginal

rm^1represents row 1 marginal,

rm^2represents row 2 marginal, and

n represents the number of observations

(not the number of raters).

Example 3.3.1.1

The kappa statistic calculated can be found in the given table. Notice that the percent agreement is 0.94 while the Kappa is 0.85 — a considerable reduction in the level of congruence. The greater the expected chance agreement, the lower the resulting value of the kappa.

DATA IN TABLE FORMAT

		Rater 1 normal	Rater 1 abnormal	Row Marginals	
Rater 2	normal	147	3	150	rm^1
	abnormal	10	62	72	rm^2
Column Marginals		157	65	222	n
		cm^1	cm^2		

Solution:

$$K = \frac{Pr(a) - Pr(e)}{1 - Pr(e)}$$

Raw % Agreement

$$\frac{147+62}{222} = .94$$

Pr(e) Calculation

$$\text{Expected Agreement} = \frac{\left(\frac{cm^1 \ X \ rm^1}{n}\right) + \left(\frac{cm^2 X \ rm^2}{n}\right)}{n}$$

$$\text{Expected Agreement} = \frac{\left(\frac{157 \ X \ 150}{222}\right) + \left(\frac{65 \ X \ 72}{222}\right)}{222}$$

$$\text{Expected Agreement} = \frac{108.08+21.08}{222} = .85$$

$$\text{Kappa} = \frac{.94 - .57}{1 - .57} = .85$$

This indicates **strong Agreement** based on table

Example 3.3.1.2

Part of the hope for people using the kappa statistic is to argue that diagnostics criteria can be used consistently, and that they therefore measure something "real". The easiest case is if there are two doctors .Suppose two doctors are diagnosing patients as either HDL Cholesterol or Serum creatinine. Then we can represent the proportions of diagnoses in a table:

Doctor B	**Doctor A**		Row Total
	HDL Cholesterol	Serum Creatinine	
HDL Cholesterol	57.9	3.4	61 .3
Serum creatinine	24.9	9.36	34 .26
Column Total	82.8	12.76	95 .56

Solution :

$$K = \frac{Pr(a) - Pr(e)}{1 - Pr(e)}$$

$$Pr(a) = \frac{57.9 + 9.36}{95.56}$$

$$\boxed{Pr(a) = 0.7039}$$

$$\textbf{Expected (Chance) Agreement} = \frac{\left(\frac{cm^1 \ X \ rm^1}{n}\right) + \left(\frac{cm^2 X \ rm^2}{n}\right)}{n}$$

$$Pr(e) = \frac{\frac{82.8(61.3)}{95.56} + \frac{12.76(34.26)}{95.56}}{95.56}$$

$$Pr(e) = \frac{53.1147 + 4.5747}{95.56}$$

$$Pr(e) = \frac{57.6894}{95.56}$$

$$\boxed{Pr(e) = 0.6037}$$

$$\text{k} = \frac{0.7039 - 0.6037}{1 - 0.6037}$$

$$k = \frac{0.1002}{0.3963}$$

$$\boxed{k = 0.2528}$$

This indicates **Minimal Agreement** based on table.

3.4 SCOTT'S π STATISTIC

Two statistics are often used in practice for evaluating the extent of agreement between raters. These are the Kappa statistic suggested by Cohen (1960) and the π -statistic (should be read "pi-statistic") suggested by Scott (1955) suggested computing the extent of agreement between raters A and B using the $\pi -$ statistic *PI*, which is defined as follows:

$$PI = \frac{P - e(\pi)}{1 - e\ (\pi)} \quad \cdots (1)$$

Where $p = (A + D)/N$ is the overall agreement propensity and $e\ (\pi)$ (should be read e of PI) is given by

$$e\ (\pi) = \left(\frac{(A1 + B1)/2}{N}\right)^2 + \left(\frac{(A2 + B2)/2}{N}\right)^2 \quad \cdots (2)$$

It should be noted that $e(\pi)$ designates the propensity for both raters to agree by chance without having the same assessment of a subject.

The *p* component of equation (1) only involves subjects both raters have classified in the same category. It could be used as a naïve measure of the extent of agreement. However, there are reasons to believe that raters A and B would classify some subjects into the same category not for the same reasons. Subjects classified in the same category for different reasons correspond to an agreement by chance. Because chance agreement does not measure consistency in the rating, it is not of interest and the *p* component should be adjusted accordingly. Gwet (2001) discusses extensively about the motivation of the form of equation (1) and explains why statistics of this type provides the desired adjustment.

Example 3.4.1

Part of the hope for people using the kappa statistic is to argue that diagnostics criteria can be used consistently, and that they therefore measure something "real". The easiest case is if there are two doctors .Suppose two doctors are diagnosing patients as either HDL Cholesterol or Serum creatinine. Then we can represent the proportions of diagnoses in a table:

Doctor B	**Doctor A**		Row Total
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HDL Cholesterol	57.9	3.4	61 .3
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Column Total	82.8	12.76	95 .56

Solution:

$$PI = \frac{p - e(\pi)}{1 - e\,(\pi)} \quad \cdots (1)$$

Where $p = (A + D\,)/N$ is the overall agreement propensity and $e\,(\pi)$ (should be read e of PI) is given by

$$e\,(\pi) = \left(\frac{(A1 + B1\,)/2}{N}\right)^2 + \left(\frac{(\,A2 + \;B2)/2}{N}\right)^2 \quad \cdots (2)$$

$$\left(\frac{(A1 + B1\,)/2}{N}\right)^2 = \left(\frac{(82.8 + 61.3\,)/2}{95.56}\right)^2 + \left(\frac{(12.76 + 34.26\,)/2}{95.56}\right)^2$$

$$e\,(\pi) = \quad (0.75397)^2 + \ (0.24602)^2$$

$$= \quad 0.5685 \; + 0.0605$$

$$e\ (\pi) = \quad 0.6290$$

To find P

$$p = \ (A + D\ )/N$$

$$p \ = \frac{57.9 + 9.36}{95.56}$$

$$p \quad = \quad 0.7039$$

From ….. (1)

$$PI \ = \frac{p - e(\pi)}{1 - e\ (\pi)}$$

$$PI \ = \frac{0.7039 - 0.6290}{1 - 0.6290}$$

$$\mathrm{P}I \ = \frac{0.0749}{0.371}$$

$$\mathrm{P}I \ = 0.2019$$

CHAPTER 4

Frequentist Statistics

4.1 INTRODUCTION

The goal of statistical analysis is to extract information from data by computing statistics, which are deterministic functions of the data.

In this chapter we model the data-acquisition process probabilistically. This allows to analyze statistical techniques and derive theoretical guarantees on their performance. The data are interpreted as realizations of random variables, vectors or processes (depending on the dimensionality). The information that we want to extract can then be expressed in terms of the joint distribution of these quantities. We consider this distribution to be unknown but fixed, taking a frequentist perspective.

4.2 INDEPENDENT IDENTICALLY-DISTRIBUTED SAMPLING

In this chapter we consider one-dimensional real-valued data, modeled as the realization of an iid sequence. Figure 4.2 depicts the corresponding graphical model. This is a very popular assumption, which holds for controlled experiments, such as randomized trials to test drugs, and can often be a good approximation in other settings. However, in practice it is crucial to evaluate to what extent the independence assumptions of a model actually hold. The following example shows that measuring a quantity by sampling a subset of individuals randomly from a large population produces data satisfying the iid assumption, as long as we sample with replacement (if the population is large, sampling without replacement will have a negligible effect).

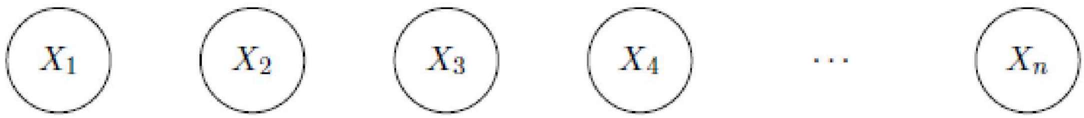


Fig 4.2

Figure 4.2 : Directed graphical model corresponding to an independent sequence. If the sequence is also identically distributed, then $X_1, X_2, \ldots, X_n$ all have the same distribution.

4.3 MEAN SQUARE ERROR

The mean square error (MSE) of an estimator Y that approximates a deterministic quantity $\gamma \in \mathbb{R}$ is

$$\text{MSE}(Y) := \mathrm{E}\left((Y-\gamma)^2\right)$$

The MSE can be decomposed into a bias term and a variance term. The bias term is the difference between the quantity of interest and the expected value of the estimator. The variance term corresponds to the variation of the estimator around its expected value.

Where $Y := h(X_1, X_2, \ldots, X_n)$. We define an estimator as a deterministic function of the available data $x_1, x_2, \ldots, x_n$

4.3.1 Lemma (Bias-variance decomposition)

The MSE of an estimator Y that approximates $\gamma \in \mathbb{R}$ satisfies

$$MSE(Y) = \underbrace{\mathrm{E}\left((Y - \mathrm{E}(Y))^2\right)}_{\text{variance}} + \underbrace{(\mathrm{E}(Y) - \gamma)^2}_{\text{bias}}.$$

Proof:

The lemma is a direct consequence of linearity of expectation. If the bias is zero, then the estimator equals the quantity of interest on average.

4.4 CONSISTENCY

If we are estimating a scalar quantity, the estimate should improve as we gather more data. Ideally the estimate should converge to the true value in the limit when the number of data n $\to\infty$. Estimators that achieve this are said to be consistent.

4.4.1 Definition (Consistency)

An estimator $\widetilde{Y}(n) := h(\widetilde{X}(1), \widetilde{X}(2), \ldots, \widetilde{X}(n))$ that approximates $\gamma \in \mathbb{R}$ is consistent if it converges to γ *as* n $\to\infty$ in mean square, with probability one or in *probability*.

4.4.2 Theorem (The sample mean is consistent).

The sample mean is a consistent estimator of the mean of an iid sequence of random variables as long as the variance of the sequence is bounded.

Proof :

We consider the sample mean of an iid sequence $\widetilde{X}$ *with* mean μ,

$$\widetilde{Y}(n) := \frac{1}{n}\sum_{i=1}^{n} \widetilde{X}(i).$$

The estimator is equal to the moving average of the data. As a result it converges to μ in mean square (and with probability one) by the law of large numbers as long as the variance σ^2 of each of the entries in the iid sequence is bounded.

4.4.3 Example (Estimating the average height)

In this example we illustrate the consistency of the sample mean. Imagine that we want to estimate the mean height in a population. To be concrete we consider a population of m := 25000 people. Figure 4.2 shows a histogram of their heights if we sample n individuals from this population with replacement, then their heights form an iid sequence $\widetilde{X}$. The mean of this sequence is

$$\begin{aligned} \mathrm{E}\left(\widetilde{X}\,(i)\right) &:= \sum_{j=1}^{m} \mathrm{P}\,(\text{Person } j \text{ is chosen}) \cdot \text{height of person } j \\ &= \frac{1}{m}\sum_{j=1}^{m} h_j \\ &= \mathrm{av}\,(h_1, \ldots, h_m) \end{aligned}$$

for $1 \leq i \leq n$, where $h_1, \ldots , h_m$ are the heights of the people. In addition, the variance is bounded because the heights are finite. By Theorem 4.4.2 the sample mean of the n data should converge to the mean of the iid sequence and hence to the average height over the whole population..

If the mean of the underlying distribution is not well defined, or its variance is unbounded, then the sample mean is not necessarily a consistent estimator. This is related to the fact that

The data are available here:

wiki.stat.ucla.edu/socr/index.php/SOCR_Data_Dinov_020108_HeightsWeights.

Sample for 200 individuals here

INDEX	HEIGHT (INCHES)	INDEX	HEIGHT(INCHES)
1	65.78	46	68.67
2	71.52	47	66.88
3	69.40	48	67.70
4	68.22	49	69.82
5	67.79	50	69.09
6	68.70	51	69.91
7	69.80	52	67.33
8	70.01	53	70.23
9	67.90	54	70.41
10	66.78	55	66.54
11	66.49	56	70.18
12	67.62	57	70.41
13	68.30	58	66.54
14	67.12	59	66.36
15	68.28	60	67.54
16	71.09	61	66.50
17	66.46	62	69.00
18	68.65	63	68.30
19	71.23	64	67.07
20	67.13	65	70.81
21	67.83	66	68.22
22	68.88	67	69.06
23	63.48	68	67.73
24	68.42	69	67.22
25	67.63	70	67.37
26	67.21	71	65.27
27	70.84	72	70.84
28	67.49	73	69.92
29	66.53	74	64.29
30	65.44	75	68.25
31	69.52	76	66.36
32	65.81	77	68.36
33	67.82	78	65.48
34	70.60	79	69.72
35	71.80	80	67.73

36	69.21	81	68.64
37	66.80	82	66.78
38	67.66	83	70.05
39	67.81	84	66.28
40	64.05	85	69.20
41	68.57	86	69.13
42	65.18	87	76.36
43	69.66	88	70.09
44	67.97	89	70.18
45	65.98	90	68.23
91	68.13	137	65.92
92	70.24	138	67.44
93	71.49	139	73.90
94	69.23	140	69.98
95	70.06	141	69.52
96	70.56	142	65.18
97	69.29	143	68.10
98	63.43	144	68.34
99	66.77	145	65.18
100	68.89	146	68.26
101	64.87	147	68.57
102	67.09	148	64.50
103	68.35	149	68.71
104	65.61	150	68.89
105	67.76	151	69.54
106	68.02	152	67.40
107	67.66	153	66.48
108	66.31	154	66.01
109	69.44	155	72.44
110	63.84	156	64.13
111	67.72	157	70.98
112	70.05	158	67.50
113	70.19	159	72.02
114	65.95	160	65.31
115	70.01	161	67.08
116	68.61	162	64.39
117	68.81	163	69.37
118	69.76	164	68.38
119	65.46	165	65.31

120	68.83	166	67.14
121	65.80	167	68.39
122	67.21	168	66.29
123	69.42	169	67.19
124	68.94	170	65.99
125	67.94	171	69.43
126	65.63	172	67.97
127	66.50	173	67.76
128	67.93	174	65.28
129	68.89	175	73.83
130	70.24	176	66.81
131	68.27	177	66.89
132	71.23	178	65.74
133	69.10	179	65.98
134	64.40	180	66.58
135	71.10	181	67.11
136	68.22	182	65.87
183	66.78	192	65.52
184	68.74	193	67.46
185	66.23	194	67.41
186	65.96	195	69.66
187	68.58	196	65.80
188	66.59	197	66.11
189	66.97	198	68.24
190	68.08	199	68.02
191	70.19	200	71.39

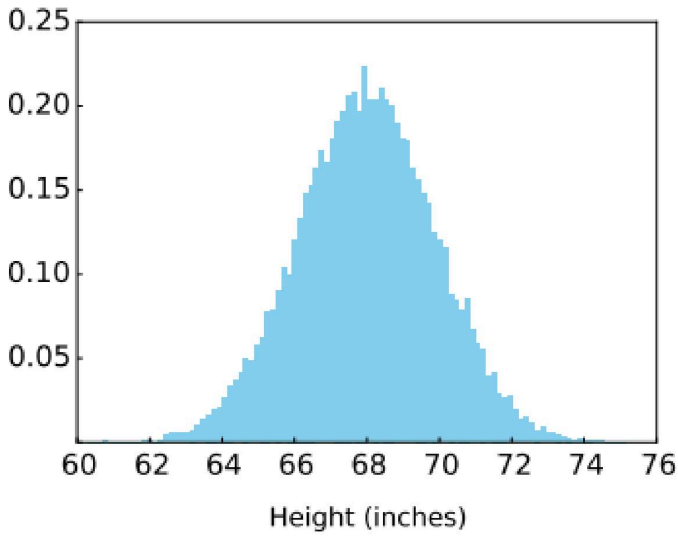


Figure :4.2 Histogram of the height of a group of 25000 people.

Total Cholesterol

Total Cholesterol $\rightarrow < 200$(N)

$\rightarrow$200 to 250 (Borderline high)

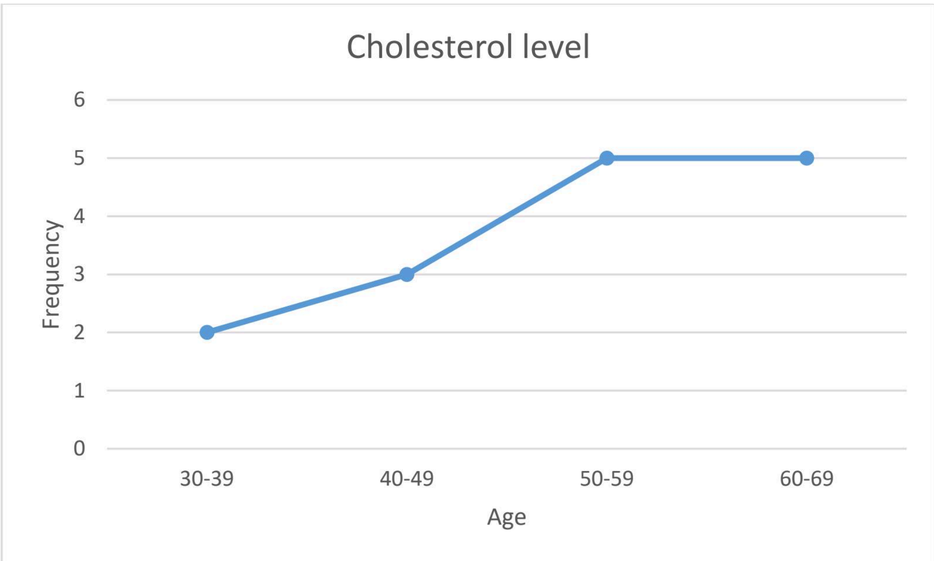
$\rightarrow > 200\ (high)$

Example 4.4.3.1

We have calculated total cholesterol level of 15 patients from the MGM hospital Chennai. We have categories based on age.

Age	Frequency
30-39	2
40-49	3
50-59	5
60-69	5

Solution :



Conclusion:

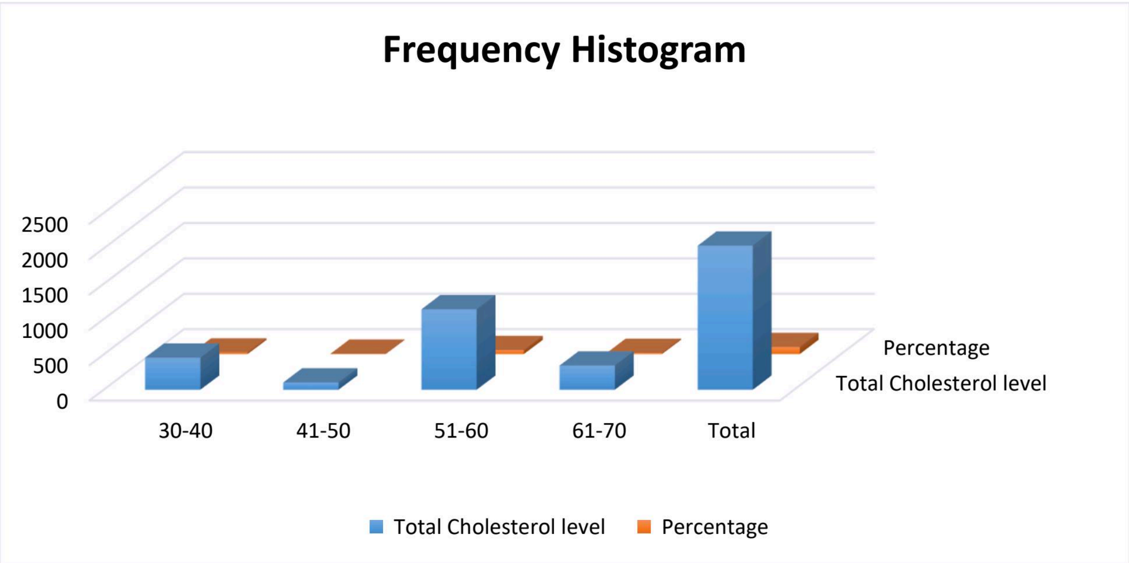
Thus the people above the age of 50 are **more than high** Cholesterol level.

Example 4.4.3.2

We have calculated total cholesterol level of the15 patients from the MGM hospital Chennai. We have categories based on age.

Age	No.of patient total cholesterol	Frequency Percentage
30-40	454.2	22.3
41-50	102.7	5.0
51-60	1140.1	55.9
61-70	343.1	16.8
Total	2040.1	100

Solution :



4.5 CONFIDENCE INTERVAL

Consistency implies that an estimator will be perfect if we acquire infinite data, but this is of course impossible in practice. It is therefore important to quantify the accuracy of an estimator for a fixed number of data. Confidence intervals allow to do this from a frequentist point of view. A confidence interval can be interpreted as a soft estimate of the deterministic quantity of interest, which guarantees that the true value will belong to the interval with a certain probability.

4.5.1 Definition (Confidence interval)

A $1-\alpha$ confidence interval $\mathcal{I}$ for $\gamma \in \mathbb{R}$ satisfies

$$p(\gamma \in \mathcal{I}) \geq 1-\alpha$$

where $0 < \alpha < 1$

Confidence intervals are usually of the form $[Y-c,\ Y+c]$ where Y is an estimator of the quantity of interest and c is a constant that depends on the number of data. The following theorem derives a confidence interval for the mean of an iid sequence. The confidence interval is centered at the sample mean.

4.5.2 Theorem (Central limit theorem with sample standard deviation).

Let $\widetilde{X}$ be an iid discrete random process with mean $\mu_{\widetilde{X}} := \mu$ such that its variance and fourth moment $E(\widetilde{X}(i^4))$ are bounded. The sequence

$$\frac{\sqrt{n}\left(\operatorname{av}\left(\widetilde{X}(1),\ldots,\widetilde{X}(n)\right)-\mu\right)}{\operatorname{std}\left(\widetilde{X}(1),\ldots,\widetilde{X}(n)\right)}$$

converges in distribution to a standard Gaussian random variable.

Recall that the cdf of a standard Gaussian does not have a closed-form expression. To simplify notation we express the confidence interval in terms of the Q function.

4.5.3 Definition (Q function).

Q(x) is the probability that a standard Gaussian random variable is greater than x for positive x,

$$Q(x) := \int_{u=x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du, \quad x > 0.$$

By symmetry, if U is a standard Gaussian random variable and y < 0

$$\mathrm{P}(U < y) = Q(-y).$$

4.5.4 Corollary (Approximate confidence interval for the mean).

Let $\tilde{X}$ be an iid sequence that satisfies the conditions of Theorem 4.5.2. For any $0 < \alpha < 1$

$$\mathcal{I}_n := \left[Y_n - \frac{S_n}{\sqrt{n}} Q^{-1}\left(\frac{\alpha}{2}\right), Y_n + \frac{S_n}{\sqrt{n}} Q^{-1}\left(\frac{\alpha}{2}\right)\right],$$
$$Y_n := \mathrm{av}\left(\tilde{X}(1), \tilde{X}(2), \ldots, \tilde{X}(n)\right),$$
$$S_n := \mathrm{std}\left(\tilde{X}(1), \tilde{X}(2), \ldots, \tilde{X}(n)\right),$$

is an approximate 1 -∝ confidence interval for μ, i.e.

$$P(\mu \in \mathcal{I}_n) \approx 1 - \alpha.$$

Proof :

By the central limit theorem, when n → ∞ $\tilde{X}_n$ is distributed as a Gaussian random variable with mean μ and variance σ^2. As a result

$$
\begin{aligned}
P(\mu \in \mathcal{I}_n) &= 1 - P\left(Y_n > \mu + \frac{S_n}{\sqrt{n}} Q^{-1}\left(\frac{\alpha}{2}\right)\right) - P\left(Y_n < \mu - \frac{S_n}{\sqrt{n}} Q^{-1}\left(\frac{\alpha}{2}\right)\right) \\
&= 1 - P\left(\frac{\sqrt{n}\,(Y_n - \mu)}{S_n} > Q^{-1}\left(\frac{\alpha}{2}\right)\right) - P\left(\frac{\sqrt{n}\,(Y_n - \mu)}{S_n} < -Q^{-1}\left(\frac{\alpha}{2}\right)\right) \\
&\approx 1 - 2Q\left(Q^{-1}\left(\frac{\alpha}{2}\right)\right) \\
&= 1 - \alpha.
\end{aligned}
$$

(by theorem 4.5.2)

It is important to stress that the result only provides an accurate confidence interval if n is large enough for the sample variance to converge to the true variance and for the CLT to take effect.

CHAPTER 5

Bayesian Statistics

5.1 INTRODUCTION

In the frequentist paradigm we model the data as realizations from a distribution that is fixed. In particular, if the model is parametric, the parameters are deterministic quantities. In contrast, in Bayesian parametric modeling the parameters are modeled as random variables. The goal is to have the flexibility to quantify our uncertainty about the underlying distribution beforehand, for example in order to integrate available prior information about the data.

5.2 BAYESIAN PARAMETRIC MODEL

In this section we describe how to fit a parametric model to a data set within a Bayesian framework. As we assume that the data are generated by sampling from known distributions with unknown parameters. The crucial difference is that we model the parameters as being random instead of deterministic. This requires selecting their prior distribution before fitting the data, which allows to quantify our uncertainty about the value of the parameters beforehand. A Bayesian parametric model is specified by:

5.2.1 Definition

The **prior** distribution is the distribution of $\vec{\theta}$, which encodes our uncertainty about the model before seeing the data.

5.2.2 Definition

The **likelihood** is the conditional distribution of $\vec{X}$ given $\vec{\theta}$, which Specifies how the data depend on the parameters. In contrast to the frequentist framework, the likelihood is not interpreted as a deterministic function of the parameters. Our goal when learning a Bayesian model is to compute the posterior distribution of the parameters θ given $\vec{X}$. Evaluating this posterior Distribution at the realization $\vec{x}$ allows to update our uncertainty about θ using the data.

5.3 CONJUGATE PRIORS

Both posterior distributions are beta distributions since the prior and the posterior belong to the same family, computing the posterior is equivalent to just updating the parameters. When the prior and posterior are guaranteed to belong to the same family of distributions for a particular likelihood, the distributions are called conjugate priors.

5.3.1 Definition

A conjugate family of distributions for a certain likelihood satisfies the following property: if the prior belongs to the family, then the posterior also belongs to the family. Beta distributions are conjugate priors when the likelihood is binomial.

5.3.2 Theorem (The beta distribution is conjugate to the binomial likelihood)

If the prior distribution of θ is a beta distribution with parameters a and b and the likelihood of the data X given θ is binomial with parameters n and x, then the posterior distribution of θ given X is a beta distribution with parameters x + a and n - x + b.

Proof:

$$\begin{aligned} f_{\theta|X}(\theta|x) &= \frac{f_\theta(\theta)p_{X|\theta}(x|\theta)}{pX(x)} \\ &= \frac{f_\theta(\theta)p_{X|\theta}(x|\theta)}{\int_u f_\theta(u)p_{X|\theta}(x|u)du} \\ &= \frac{\theta^{a-1}(1-\theta)^{b-1}\ \binom{n}{x}\theta^x(1-\theta)^{n-x}}{\int_u u^{a-1}(1-u)^{b-1}\ \binom{n}{x}u^x(1-u)^{n-x}du} \\ &= \frac{\theta^{x+a-1}(1-\theta)^{n-x+b-1}}{\int_u u^{x+a-1}(1-u)^{n-x+b-1}du} \\ &= f_\beta(\theta; x+a, n-x+b) \end{aligned}$$

5.4 BAYESIAN ESTIMATORS

The Bayesian approach to learning probabilistic models yields the whole posterior distribution of the parameters of interest. In this section we describe two alternatives for deriving a single estimate of the parameters from the posterior distribution.

5.4.1 Minimum mean-square-error estimation

The mean of the posterior distribution is the conditional expectation of the parameters given the data. Choosing the posterior mean as an estimator for the parameters $\vec{\theta}$ has a strong theoretical justification: it is guaranteed to achieve the minimum mean square error (MSE) among all possible estimators. Of course, this only holds if all of the assumptions hold, i.e. the parameters are generated according to the prior and the data are then generated according to the likelihood, which may not be the case for real data.

5.4.1.1 Theorem (The posterior mean minimizes the MSE).

The posterior mean is the minimum mean-square-error (MMSE) estimate of the parameter $\vec{\theta}$ given the data $\vec{X}$. To be more precise, let us define

$$\theta_{MMSE}(\vec{x}) := \mathrm{E}(\vec{\theta}|\vec{X} = \vec{x}).$$

For any arbitrary estimator $\theta_{other}(\vec{x})$,

$$\mathrm{E}\left((\theta_{other}(\vec{X}) - \vec{\theta})^2\right) \geq \mathrm{E}\left((\theta_{MMSE}(\vec{X}) - \vec{\theta})^2\right)$$

Proof:

We begin by computing the MSE of the arbitrary estimator conditioned on $\vec{X} = \vec{x}$ in terms of the conditional expectation of θ given $\vec{X}$,

$$\mathrm{E}\left((\theta_{other}(\vec{X}) - \vec{\theta})^2|\vec{X} = \vec{x}\right)$$

$$= \mathrm{E}\left((\theta_{other}(\vec{x}) - \theta_{MMSE}(\vec{X}) + \theta_{MMSE}(\vec{X}) - \vec{\theta})^2|\vec{X} = \vec{x}\right)$$

$$= (\theta_{other}(\vec{x}) - \theta_{MMSE}(\vec{x}))^2 + \mathrm{E}\left((\theta_{MMSE}(\vec{X}) - \vec{\theta})^2|\vec{X} = \vec{x}\right)$$

$$+2\,(\theta_{other}(\vec{x}) - \theta_{MMSE}(\vec{x}))E(\theta_{MMSE}(\vec{x}) - E(\vec{\theta}|\vec{X} = \vec{x}))$$

$$= (\theta_{other}(\vec{x}) - \theta_{MMSE}(\vec{x}))^2 + E((\theta_{MMSE}(\vec{X}) - \vec{\theta})^2 | \vec{X} = \vec{x}).$$

By iterated expectation,

$$E((\theta_{other}(\vec{X}) - \vec{\theta})^2) = E(E((\theta_{other}(\vec{X}) - \vec{\theta})^2 | \vec{X}))$$

$$= E((\theta_{other}(\vec{X}) - \theta_{MMSE}(\vec{X}))^2) + E(E((\theta_{MMSE}(\vec{X}) - \vec{\theta})^2 | \vec{X}))$$

$$= E((\theta_{0ther}(\vec{X}) - \vec{\theta}_{MMSE}(\vec{X}))^2) + E((\theta_{MMSE}(\vec{X}) - \vec{\theta})^2)$$

$$\geq E((\theta_{MMSE}(\vec{X}) - \vec{\theta})^2)$$

Since the expectation of a nonnegative quantity is nonnegative.

5.4.2 Maximum-a-posteriori estimation

An alternative to the posterior mean is the posterior mode, which is the maximum of the pdf or the pmf of the posterior distribution.

5.4.2.1 Definition (Maximum-a-posteriori estimator)

The maximum-a-posteriori (MAP) estimator of a parameter $\vec{\theta}$ given data $\vec{x}$ modeled as a realization of a random vector $\vec{X}$ is

$$\theta_{MAP}(\vec{x}) := \arg\max_{\vec{\theta}} p_{\vec{\theta}|\vec{X}}(\vec{\theta}|\vec{x})$$

if $\vec{\theta}$ is modeled as a discrete random variable and

$$\theta_{MAP}(\vec{x}) := \arg\max_{\vec{\theta}} f_{\vec{\theta}|\vec{X}}(\vec{\theta}|\vec{x})$$

if it is modeled as a continuous random variable.

5.4.2.2 Lemma

The maximum-likelihood estimator of a parameter θ is the mode (maximum value) of the pdf of the posterior distribution given the data $\vec{X}$ if its prior distribution is uniform.

Proof:

We prove the result when the model for the data and the parameters is continuous, if any or both of them are discrete the proof is identical (in that case the ML estimator is the mode of the pmf of the posterior).If the prior distribution of the parameters is uniform, then $f_{\vec{\theta}}(\vec{\theta})$ is constant for any $\vec{\theta}$, which implies

$$\underset{\vec{\theta}}{\arg\max}\, f_{\vec{\theta}|\vec{X}}(\vec{\theta}|\vec{x}) = \underset{\vec{\theta}}{\arg\max}\, \frac{f_{\vec{\theta}}(\vec{\theta})f_{\vec{X}|\vec{\theta}}(\vec{x}\,|\vec{\theta})}{\int_u f_{\vec{\theta}}(u) f_{\vec{X}|\vec{\theta}}(\vec{x}\,|\vec{u}\,)\,du}$$

$$= \underset{\vec{\theta}}{\arg\max}\, f_{\vec{X}|\vec{\theta}}(\vec{x}|\vec{\theta}) \qquad \text{(the rest of the terms do not depend on } \vec{\theta}\text{)}$$

$$=\underset{\vec{\theta}}{\arg\max}\, \mathcal{L}_{\vec{x}}(\vec{\theta}).$$

Note that uniform priors are only well defined in situations where the parameter is restricted to a bounded set.

We now describe a situation in which the MAP estimator is optimal. If the parameter $\vec{\theta}$ can only take a discrete set of values, then the MAP estimator minimizes the probability of making the wrong choice.

5.4.2.3 Theorem (MAP estimator minimizes the probability of error)

Let $\vec{\theta}$ be a discrete random variable vector and $\vec{X}$ be a random vector modeling the data. We define

$$\theta_{MAP}(\vec{x}) := \underset{\vec{\theta}}{\arg\max}\ p_{\vec{\theta}|\vec{X}}(\vec{\theta}|\vec{X} = \vec{x}).$$

For any arbitrary estimator $\theta_{other}(\vec{x})$,

$$\mathrm{P}(\theta_{other}(\vec{X}) \neq \vec{\theta}) \geq \mathrm{P}\ (\theta_{MAP}(\vec{X}) \neq \vec{\theta}).$$

In words, the MAP estimator minimizes the probability of error.

Proof:

We assume that $\vec{X}$ is a continuous random vector, but the same argument applies if it is discrete. We have

$$\begin{aligned}\mathrm{P}(\theta = \theta_{other}(\vec{X})) &= \int_{\vec{x}} f_{\vec{X}}(\vec{x}) P\ (\theta = \theta_{other}(\vec{x})|\ \vec{X} = \vec{x})\ \mathrm{d}\vec{x} \\ &= \int_{\vec{x}} f_{\vec{X}}(\vec{x}) p_{\vec{\theta}|\vec{X}}\ (\theta_{other}(\vec{x})|\vec{x})\ \mathrm{d}\vec{x} \\ &\leq \int_{\vec{x}} f_{\vec{X}}(\vec{x}) p_{\vec{\theta}|\vec{X}}\ (\theta_{MAP}(\vec{x})|\vec{x})\ \mathrm{d}\vec{x} \\ &= \mathrm{P}\ (\theta = \theta_{MAP}(\vec{X}))\end{aligned}$$

where $\leq \int_{\vec{x}} f_{\vec{X}}(\vec{x}) p_{\vec{\theta}|\vec{X}}\ (\theta_{MAP}(\vec{x})|\vec{x})\ \mathrm{d}\vec{x}$ follows from the definition the MAP estimator as the mode of the posterior.

CONCLUSION

Statistics is presented using a special assignment. The present data's of the patients affected by cholesterol are taken and medical biostatistics are made. Medical Biostatistics can be implemented by Kappa statistics, Frequentist statistics and Bayesian statistics. Statistics is concerned with scientific method for collecting and presenting, organizing and summarizing and analyzing data as well as deriving valid conclusions and making reasonable decisions on the basis of this analysis.

The use of statistics allows clinic researches to draw reasonable and accurate inferences from collected information and to make sound decisions in the presence of uncertainty .It can prevent numerous errors and biases in the medical research. Biostatistics helps researchers make sense of the datas collected to decide whether a treatment is working or to find factors that contribute to diseases. Medical statisticians design and analyse studies to identify the real causes of health issues as distinct from chance variation.

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HB COLOURING

Project Report submitted to

ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI

affiliated to

Manomaniam Sundaranar University, Tirunelveli

In partial fulfilment of the requirement for the award of degree of

Bachelor of Science in Mathematics

Submitted by

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(2022– 2023)

CERTIFICATE

We hereby declare that the project report entitled **"HB COLOURING"** being submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI** in partial fulfilment for the award of degree of **Bachelor of Science in Mathematics** and it is a record of work done during the year 2022 -2023 by the following students.

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DECLARATION

We hereby declare that the project report entitled "**HB COLOURING**", is our original work. It has not been submitted to any university for any degree or diploma.

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ACKNOWLEDGEMENT

First of all, we thank the Almighty for showering his blessing to undergo this project.

With immense pleasure, we register our deep sense of gratitude to our guide and head of the Department **Dr. V. L. Stella Arputha Mary M.Sc., M.Phil., B.Ed., Ph.D** for having suggested this project topic and imported necessary guidelines throughout the period of our investigation.

We thank our beloved Principal **Sr. Dr. A. S. J. Lucia Rose M.Sc., M. Phil., Ph.D., PGDCA.,** providing the help to carry out our project work successfully.

Finally, we thank all those who extended their helping hands regarding this project.

CONTENTS

INTRODUCTION

Colouring of graphs by **HB COLOUR MATRIX** algorithm method was proposed by **A A Bhange, H R Bhapkar** in the journal of Physics published under licence by **IOP Publishing Ltd** at International Conference On Mathematics & Data Science (ICMDS) 2020.

Graph Colouring is a chief element in graph theory with tremendous applicability in computer science like data mining, clustering, networking, image segmentation etc.

And a variety of implementations in aircraft scheduling, register allocation, sudoku, mobile networking, etc. Various algorithms were contrived for vertex colouring. This paper defines the HB colour matrix method and its kinds. There are three types of such matrices, **Vertex HB colour matrix** (VHBCM), **Edge HB Colour matrix** (EHBCM), and **Region HB colour matrix** (RHBCM). Also, the HB colour matrix algorithm is developed using a special assignment method, which gives the chromatic number of the given graph. Further, the algorithm is used to develop **the python program, giving time complexity O(n) and space complexity O(n^2).**

Also, the output of the python program for some standard graphs is calculated. The Similar algorithm can be developed for edge and face colouring of the graphs. Colouring of the graph further can be extended to perfect colouring

CHAPTER-1

CHAPTER-1
PRELIMINARIES

DEFINITION 1.1

A graph G consists of a pair (V(G), X(G)) where V(G) is a non-empty finite set whose elements are called **points** or **vertices** of the graph G and X(G) is a set of unordered pairs of distinct elements of V(G). The elements of X(G) are called **lines** or **edges** of the graph G.

DEFINITION 1.2

If two vertices in a graph are connected by an edge, we say that the vertices are **adjacent.**

DEFINITION 1.3

If two distinct lines x and y are incident with a common point then they are called **adjacent lines**.

DEFINITION 1.4

A graph in which any two distinct points are adjacent is called a **complete graph**. The complete graph with p points is denoted by $\boldsymbol{K_P}$.

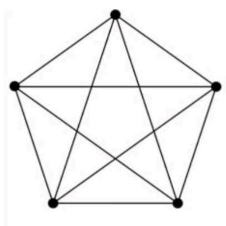


Figure 1.1: Complete graph

DEFINITION 1.5

A simple graph of n vertices $n \geq 3$ and n edges forming a cycle of length n is called as a **cycle graph**.

The graph consisting of a cycle of length n is denoted by $\boldsymbol{C_n}$.

DEFINITION 1.6

A graph that is drawn on the plane without intersecting edges is called a **plane graph**.

DEFINITION 1.7

A graph is called **planar** if it can be drawn on a plane without intersecting edges.

DEFINITION 1.8

A graph is called **maximal planar** if no line can be added to it without losing planarity. In a maximal planar graph, each face is a triangle and such a graph is sometimes called a **triangulated graph**.

DEFINITION 1.9

The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colours needed to colour G. A graph G is called, n-**colourable** if $\chi(G) \leq n$.

DEFINITION 1.10

An assignment of colours to the edges of a graph G so that no two adjacent edges get the same colour is called an **edge colouring** or **line colouring** of G.

DEFINITION 1.11

Let G = (V, E) be any graph **HB colour matrix** of a graph G is defined as

$C = (C_{ij})$, where

$C_{ij} = \infty$, if labels of row R_i and column C_j have different colours and

$C_{ij} = 0$, if labels of row R_i and column C_j have the same colour.

DEFINITION 1.12

Let G be a graph with n nodes or vertices say $v_1, v_2, v_3, \ldots, v_n$. A **Vertex HB Colour matrix** of a graph G is denoted as C(V) and is defined as C(V)= $(C_{ij})_{n\times n}$, where

$C_{ij} = \infty$, if v_i and v_j are adjoint vertices,

$C_{ij} = \infty$, if i is equal to j and

$C_{ij} = 0$, if v_i and v_j are not adjoint vertices.

DEFINITION 1.13

Let G = (V, E) be a graph with m number of edges say $e_1, e_2, e_3, \ldots e_m$. An **Edge HB colour matrix** of a graph G is denoted by C(E) and defined as C(E) = $(C_{ij})_{m\times m}$,where

$C_{ij} = \infty$, if e_i is adjoint to e_j in graph G.

$C_{ij} = \infty$, if i is equal to j and

$C_{ij} = 0$, if e_i is not adjoint to e_j in the graph G.

DEFINITION 1.14

Consider any planar graph H with regions or faces say $F_1, F_2, F_3, \ldots F_r$. A **Region HB Colour Matrix** of graph H is indicated by C(F) and defined as C(F) = $(C_{ij})_{r\times r}$ where

$C_{ij} = \infty$, if F_i is adjoint to F_j.,

$C_{ij} = \infty$, if $i = j$ and

$C_{ij} = 0$, if F_i is not adjoint to F_j in a graph H.

DEFINITION 1.15

A square matrix A = (a_{ij}) is said to be **symmetric** if $a_{ij} = a_{ji}$ for all i, j.

DEFINITION 1.16

Two **regions** are said to be **adjacent** if they have a common edge.

DEFINITION 1.17

A square matrix a_{ij} is called **upper triangular matrix** if all the entries below the principal diagonal are zero.

Hence $a_{ij} = 0$ whenever $i > j$ in an upper triangular matrix. For example:

$$\begin{bmatrix} 3 & 4 & 7 & 8 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Figure 1.2: Upper Triangular Matrix

DEFINITION 1.18

The definition of a graph does not allow more than one line joining two points. It also does not allow any line joining a point to itself. Such a line joining a point to itself is called a **loop**.

DEFINITION 1.19

Let G = (V, X) be a (p, q) graph, Let V= $\{v_1, v_2, v_3, \ldots, v_p\}$. The $p\times p$ matrix A = (a_{ij}) where

$$[a_{ij}]=\begin{cases} 1 \; if \; v_i, v_j \; are \; adjacent \\ 0 \qquad\qquad Otherwise \end{cases}$$

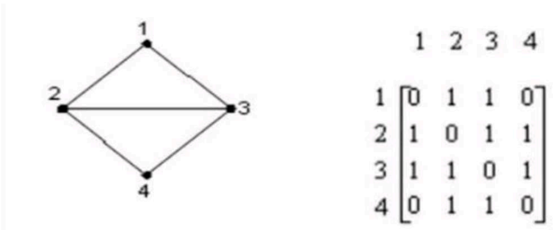


Figure 1.3: Adjacency Matrix

CHAPTER-2

CHAPTER-2

COLOURING OF GRAPHS BY HB COLOUR MATRIX ALGORITHM METHOD

2. HB colour matrix method

Let G = (V , E) be any graph HB colour matrix of a graph G is defined as $C = (C_{ij})$,where

$C_{ij} = \infty$, if labels of row R_i and column C_j have different colours and

$C_{ij} = 0$, if labels of row R_i and column C_j have the same colour.

The rows or columns of this matrix are labelled by using vertices or edges or region or any other property of the corresponding graph. HB colour matrix has only two elements, either 0 or ∞. There are different kinds of HB colour matrices which are given below.

2.1. Vertex HB colour matrix

Let G be a graph with n nodes or n vertices say $v_1, v_2, v_3, \dots v_n$. A Vertex HB Colour matrix of a graph G is denoted as C(V) and is defined as C(V)=$(C_{ij})_{n\times n}$, where

$C_{ij}=\infty$, if v_i and v_j are adjoint vertices,

$C_{ij}=\infty$, if i is equal to j and

$C_{ij}=0$, if v_i and v_j are not adjoint vertices.

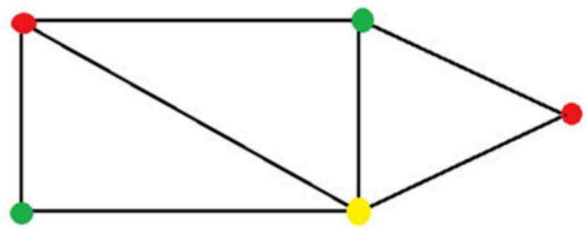


Figure 2.1: Vertex colouring of graph G

2.1.1. Properties of Vertex HB colour matrix

1. Every Vertex HB colour Matrix (VHBCM) is a symmetric matrix. All diagonal elements of this matrix are ∞.

2. The number of zeros in each column or row is equivalent to the number of vertices that are non-adjoint to the corresponding vertex.

3. The number of ∞ in every column or row is equal to the a+1, where a is the number of vertices adjoint to the corresponding vertex.

4. If all elements of a row are ∞, then the corresponding vertex or node is adjoint to all remaining vertices or nodes of that graph.

5. If all elements of a row are zeros except the diagonal element, then the corresponding vertex is not adjacent to all remaining vertices of that graph. Such vertex is either a null vertex or a vertex with loops only.

6. If a VHBCM with n vertices contains all zeros except diagonal elements then the corresponding graph is either a Null graph or a disconnected graph. Such a graph is one colourable.

2.1.2 Algorithm of HB colour matrix method for the vertex colouring of graphs

Let G be any graph with n vertices $v_1, v_2, v_3, \dots v_n$. The following is an algorithm for the vertex colouring of any graph.

Step1: Write the HB colour matrix C(V) of the given graph G. Make assignments only in the upper triangular form of C(V). So write the upper triangular form of C(V) and denote it by H .

.**Step2:** Select vertex v_1 i.e. the first row. Find $(v_1, v_j) = 0$, for the smallest j=2, 3, 4,...n. If the smallest j is the k then assign the same colour say C_1 to $v_1$and v_k.

i) If ∃ the smallest r such that $(v_{1,}v_{k+r}) = 0$ and $(v_k, v_{k+r}) = 0$, then assign the same colour to $v_{k\,,}$and v_{k+r} i.e. C_1 colour. If ∃ smallest s such that $(v_1, v_{k+r+s}) = 0$ then check labels of (v_k, v_{k+r+s}) and (v_{k+r}, v_{k+r+s}).

a) If one of them is ∞ then cross $(v_1, v_{k+r+s}) = 0$.

or

b) If all are zero then assign colour C_1 to v_{k+r+s} Continue in this way for all elements of row 1.

ii) If $(v_{1,}\ v_{k+r}) = \infty$ and $(v_k, v_{k+r}) = 0$ then cross zero or if $(v_1, v_{k+r}) = 0$ and $(v_k, v_{k+r}) = \infty$ then cross the zero of the corresponding place.

iii) If $(v_1, v_{k+r}) = \infty$ and $(v_k, v_{k+r}) = \infty$ then v_1, v_{k+r} should have different colours.

Step3: Apply the same procedure for the second row vertex $v_{2,}$ then $v_3, v_4, v_5, \dots v_n$.

Step4: If all zeros get assigned or crossed then check whether we colour all vertices or not. If vertices are remaining then assign a different colour to these vertices.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} v_1 & v_2 & \cdots & v_k & \cdots & v_{k+r} & \cdots & v_{k+r+s} & \cdots & v_n \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_k \\ \vdots \\ v_{k+r} \\ \vdots \\ v_{k+r+s} \\ \vdots \\ v_n \end{array} &
\begin{bmatrix}
- & - & \cdots & 0 & \cdots & - & \cdots & - & \cdots & - \\
- & - & \cdots & - & \cdots & - & \cdots & - & \cdots & - \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & - & \cdots & - & \cdots & - & \cdots & - & \cdots & - \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
- & - & \cdots & - & \cdots & - & \cdots & - & \cdots & - \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
- & - & \cdots & - & \cdots & - & \cdots & - & \cdots & - \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
- & - & \cdots & - & \cdots & - & \cdots & - & \cdots & -
\end{bmatrix}
\end{array}
$$

Figure 2.2: Vertex HB colouring matrix

2.1.3. Illustration of VHBCM algorithm

Consider a graph G with five vertices viz. 1, 2, 3, 4, 5,seven edges x ,y ,z ,p ,q ,r ,s and four regions A ,B ,C ,D (Figure2.3).

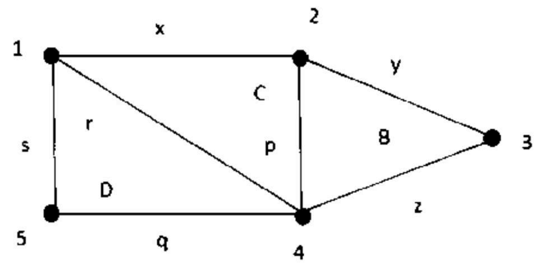


Figure 2.3: Graph G

The vertex HB colour matrix is created from graph G (figure 2.4(a)). The upper triangular of the HB matrix is formed as it's a symmetric matrix (figure 2.4(b)). Following the steps of HB colour matrix algorithm, an assignment is made at first zero of the first row i.e. cell (1, 3) and strike out column 3 (figure2.4(c)), hence vertex 1 and 3 will have the same colour.

$$
C(V)=\begin{array}{c}\\1\\2\\3\\4\\5\end{array}\!\!\begin{array}{c}\begin{array}{ccccc}1&2&3&4&5\end{array}\\\begin{bmatrix}\infty&\infty&0&\infty&\infty\\\infty&\infty&\infty&\infty&0\\0&\infty&\infty&\infty&0\\\infty&\infty&\infty&\infty&\infty\\\infty&0&0&\infty&\infty\end{bmatrix}\end{array};\;
H(V)=\begin{array}{c}\\1\\2\\3\\4\\5\end{array}\!\!\begin{array}{c}\begin{array}{ccccc}1&2&3&4&5\end{array}\\\begin{bmatrix}\infty&\infty&0&\infty&\infty\\-&\infty&\infty&\infty&0\\-&-&\infty&\infty&0\\-&-&-&\infty&\infty\\-&-&-&-&\infty\end{bmatrix}\end{array}
\;H(V)=\begin{array}{c}\\1\\2\\3\\4\\5\end{array}\!\!\begin{array}{c}\begin{array}{ccccc}1&2&3&4&5\end{array}\\\begin{bmatrix}\infty&\infty&\boxed{0}&\infty&\infty\\-&\infty&\infty&\infty&0\\-&-&\infty&\infty&0\\-&-&-&\infty&\infty\\-&-&-&-&\infty\end{bmatrix}\end{array};\;
H(V)=\begin{array}{c}\\1\\2\\3\\4\\5\end{array}\!\!\begin{array}{c}\begin{array}{ccccc}1&2&3&4&5\end{array}\\\begin{bmatrix}\infty&\infty&\boxed{0}&\infty&\infty\\-&\infty&\infty&\infty&\boxed{0}\\-&-&\infty&\infty&0\\-&-&-&\infty&\infty\\-&-&-&-&\infty\end{bmatrix}\end{array}
$$

Figure 2.4: Implementation of VHBCM algorithm to graph G

Further, there is no more zero in the first row so move to row 2 and search for the first zero, here it is at cell (2,5) so make assignment at (2,5) and strike out the column 5 (figure 2.4). This denotes vertex 2 and 5 will have the same colour. As row 4 is having all ∞ so assign the third colour to it. In VHBCM, colour assignments are colour C_1(Red) to vertices 1 and 3, colour C_2 (Green) to vertices 2 and 5 and colour C_3(Yellow) to vertex 4 (figure2.4).Hence the vertex chromatic number of graph G is 3.

THEOREM 2.1 If a VHBCM with n vertices contains all ∞ then the corresponding graph is the complete graph on n vertices (K_n).

PROOF:

Let G be a graph with n vertices whose VHBCM contains ∞ everywhere. Hence every vertex of graph G is adjacent to remaining all vertices. Therefore G is a complete graph on n vertices.

THEOREM 2.2 The VHBCM of a wheel graph with p vertices contains a row with all elements ∞.

PROOF:

Let G be a wheel graph with p vertices. Therefore there exist a vertex v in G which will be adjoint to the rest of the vertices, hence the corresponding row contains all ∞s

2.2 Edge HB colour matrix

Let G = (V, E) be a graph with m number of edges say $e_1, e_2, e_3, \ldots e_m$. An Edge HB colour matrix of a graph G is denoted by C(E) and defined as

C(E) = ($C_{ij})_{m\times m}$,where

C_{ij} = ∞, if e_i is adjoint to e_j in graph G.

C_{ij} = ∞, if i is equal to j and

C_{ij} = 0, if e_i is not adjoint to e_j in the graph G.

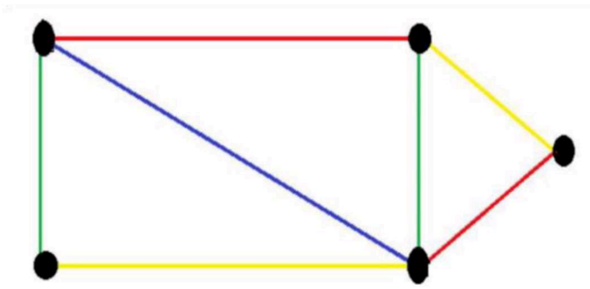


Figure 2.5: Edge colouring of graph G

2.2.1. Algorithm of HB colour matrix method for the edge colouring of graphs

Let H be any graph with m number of edges. An edge HB colour matrix of graph H is of order m. By using analogy of the vertex HB algorithm and replacing vertices by edges, resultant algorithm can be generated.

2.2.2. Illustration of an EHBCM algorithm

An edge chromatic number calculation of graph G (figure2.3) using EHBCM algorithm is shown in figure2.6. In EHBCM, colour assignments are, colour C_1(Red) to edges x and z, colour C_2(yellow) to edges y and q, colour C_3 (Green) to edges p and s, colour C_4 (blue) to edge r. Thus the edge chromatic number of G is 4.

$$\mathbf{C(E)} = \begin{array}{c} \\ x \\ y \\ z \\ p \\ q \\ r \\ s \end{array} \begin{array}{c} \begin{array}{ccccccc} x & y & z & p & q & r & s \end{array} \\ \begin{bmatrix} \infty & \infty & 0 & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & 0 & 0 \\ 0 & \infty & \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty & \infty & 0 \\ 0 & 0 & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty & \infty & \infty \end{bmatrix} \end{array} \quad \mathbf{H(E)} = \begin{array}{c} \\ x \\ y \\ z \\ p \\ q \\ r \\ s \end{array} \begin{array}{c} \begin{array}{ccccccc} x & y & z & p & q & r & s \end{array} \\ \begin{bmatrix} \infty & \infty & \boxed{0} & \infty & 0 & \infty & \infty \\ - & \infty & \infty & \infty & \boxed{0} & 0 & 0 \\ - & - & \infty & \infty & \infty & \infty & 0 \\ - & - & - & \infty & \infty & \infty & \boxed{0} \\ - & - & - & - & \infty & \infty & \infty \\ - & - & - & - & - & \infty & \infty \\ - & - & - & - & - & - & \infty \end{bmatrix} \end{array}$$

Figure 2.6: Implementation of EHBCM algorithm to graph G

Example for EHBCM:

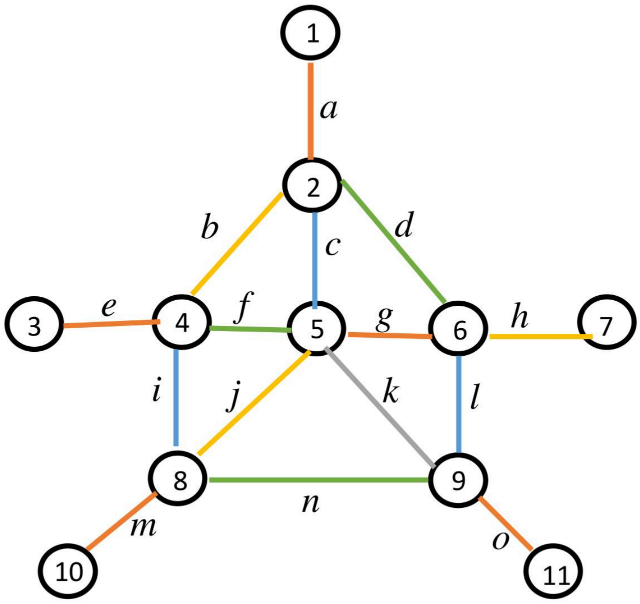


Figure 2.7: Edge colouring of Helm graph

$$
H(E) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \\ n \\ o \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o \end{array} \\
\begin{bmatrix}
\infty & \infty & \infty & \infty & \boxed{0} & 0 & \boxed{0} & 0 & 0 & 0 & 0 & 0 & \boxed{0} & 0 & \boxed{0} \\
- & \infty & \infty & \infty & \infty & \infty & 0 & \boxed{0} & \infty & \boxed{0} & 0 & 0 & 0 & 0 & 0 \\
- & - & \infty & \infty & 0 & \infty & \infty & 0 & \boxed{0} & \infty & \infty & \boxed{0} & 0 & 0 & 0 \\
- & - & - & \infty & 0 & \boxed{0} & \infty & \infty & 0 & 0 & \infty & \infty & 0 & \boxed{0} & 0 \\
- & - & - & - & \infty & \infty & 0 & 0 & \infty & 0 & \boxed{0} & 0 & 0 & 0 & 0 \\
- & - & - & - & - & \infty & \infty & 0 & \infty & \infty & \infty & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & \infty & \infty & 0 & \infty & \infty & \infty & 0 & 0 & 0 \\
- & - & - & - & - & - & - & \infty & 0 & 0 & 0 & \infty & 0 & 0 & 0 \\
- & - & - & - & - & - & - & - & \infty & \infty & 0 & 0 & \infty & \infty & 0 \\
- & - & - & - & - & - & - & - & - & \infty & \infty & 0 & \infty & \infty & 0 \\
- & - & - & - & - & - & - & - & - & - & \infty & \infty & 0 & \infty & \infty \\
- & - & - & - & - & - & - & - & - & - & - & \infty & 0 & \infty & \infty \\
- & - & - & - & - & - & - & - & - & - & - & - & \infty & \infty & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & \infty & \infty \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & \infty
\end{bmatrix}
\end{array}
$$

Figure 2.7: Edge colouring of Helm graph G

THEOREM 2.3 If an EHBCM is a diagonal matrix then all components of G are either K_1 or K_2 or vertex with loops.

PROOF:

If EHBCM is a diagonal matrix then all non-diagonal elements are zeros, which means either edges are not adjacent to each other or vertices are isolated. Therefore no two edges of G are adjacent. So all components of G are either K_1 or K_2 or vertex with loops only.

THEOREM 2.4 If all elements of C(E) are ∞s then the corresponding graph is the cycle graph C_3.

PROOF:

Consider an EHBCM of graph G with all elements ∞s. So every edge is adjacent to every other edge of graph G, Hence G is either a cycle graph C_3 .

2.3. Region HB colour matrix

Consider any planar graph H with regions or faces say $F_1, F_2, F_3, \dots F_r$. A Region HB Colour Matrix of graph H is indicated by C(F) and defined as C(F) = ($C_{ij})_{r\times r}$ where

$C_{ij} = \infty$, if F_i is adjoint to F_j

$C_{ij} = \infty$, if $i = j$ and

$C_{ij} = 0$, if F_i is not adjacent to F_j in a graph H.

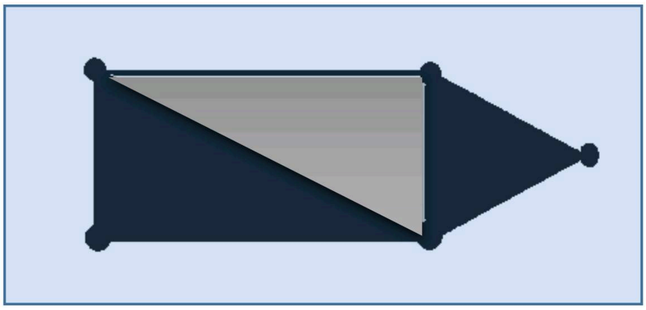


Figure 2.8: Region colouring of graph G

2.3.1. Properties of RHBCM

1. Every RHBCM is a symmetric matrix of size r.

2. If H is a null or zero graph then its RHBCM is of order 1.

3. RHBCM of non-planar graph does not exists.

2.3.2. Algorithm of HB colour matrix method for the region colouring of graphs

Let H be any graph with r edges $F_1, F_2, F_3, \dots . F_r$. The region HB colour matrix of H is of order r. Using analogy of the Vertex HB colour matrix algorithm and replacing vertices by regions will give HB colour matrix algorithm for the region colouring of graphs.

2.3.3. Illustration of RHBCM algorithm

Calculation of the RHBCM of graph G (figure2.3) is shown in figure2.8.

$$C(R) = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \end{bmatrix} \end{array} ; H(R) = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} \infty & \infty & \infty & \infty \\ - & \infty & \infty & \boxed{0} \\ - & - & \infty & \infty \\ - & - & - & \infty \end{bmatrix} \end{array}$$

Figure 2.9: Implementation of RHBCM algorithm to graph G

In RHBCM, assign colour C_1(Red) to regions B and D, colour C_2 (Blue) to region C, and colour C_3(Green) to region A (figure2.6). Hence the region chromatic number of G is 3.

Example for EHBCM

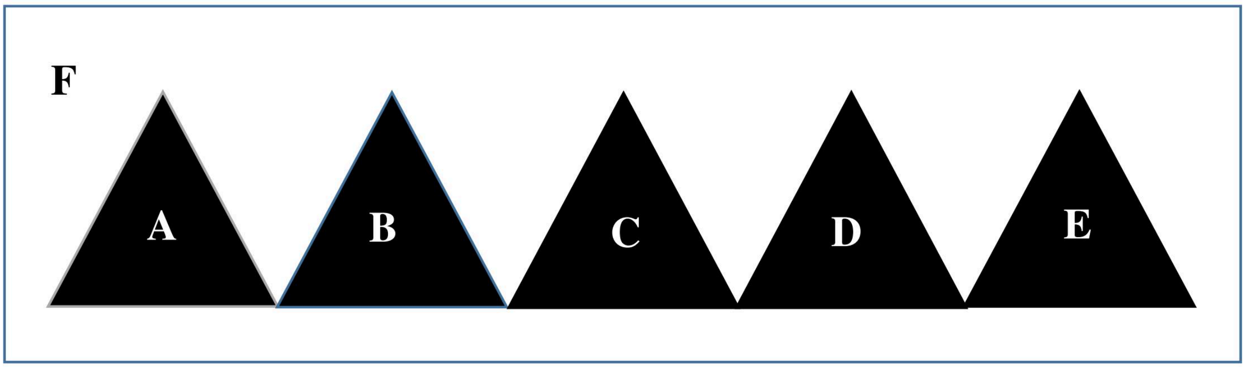


Figure 2.10: Region colouring of triangular snake graph

Calculation of the RHBCM of triangular snake graph

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} \infty & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \infty \\ - & \infty & 0 & 0 & 0 & \infty \\ - & - & \infty & 0 & 0 & \infty \\ - & - & - & \infty & 0 & \infty \\ - & - & - & - & \infty & \infty \\ - & - & - & - & - & \infty \end{bmatrix} \end{array}$$

THEOREM 2.5 If all elements of RHBCM of planar graph H are ∞ then H is the perfect HB map.

PROOF:

Let H is any planar graph with regions say $F_1, F_2, F_3, \ldots F_r$ where r $\leq$ 4. If all elements of RHBCM of graph H are ∞, then all regions of H are adjoint to each other. So each region of H is the pivot region. Hence H must be the perfect HB map.

THEOREM 2.6 If H is a planar graph then RHBCM of H does not involve 5 or more rows with all ∞s.

PROOF:

Let H be any planar graph. Assume that RHBCM of H involves 5 or more rows with all elements as ∞s. Therefore, there are 5 or more regions that are adjacent to each other, which contradicts the assumption that H is a planar graph. The contrapositive of this statement is ”If RHBCM of H involves 5 or more rows with all elements as ∞s then the corresponding graph is not planar”.

CHAPTER-3

CHAPTER-3
VERTEX COLOURING USING THE ADJACENCY MATRIX

3.1 Adjacency matrix

Let G = (V, E) be a simple graph where |V|=n and n > 1. Let $V = \{v_1, v_2, v_3, \dots, v_n\}$.

The $n \times n$ Adjacency matrix of graph G is denoted by A= (a_{ij}) where

$$a_{ij} = \begin{cases} 1 \text{ if } v_i \text{ is adjacent to } v_j \\ 0 \text{ otherwise} \end{cases}$$

Adjacency matrix is a simple symmetrical graph, so $a_{ij} = a_{ji}$

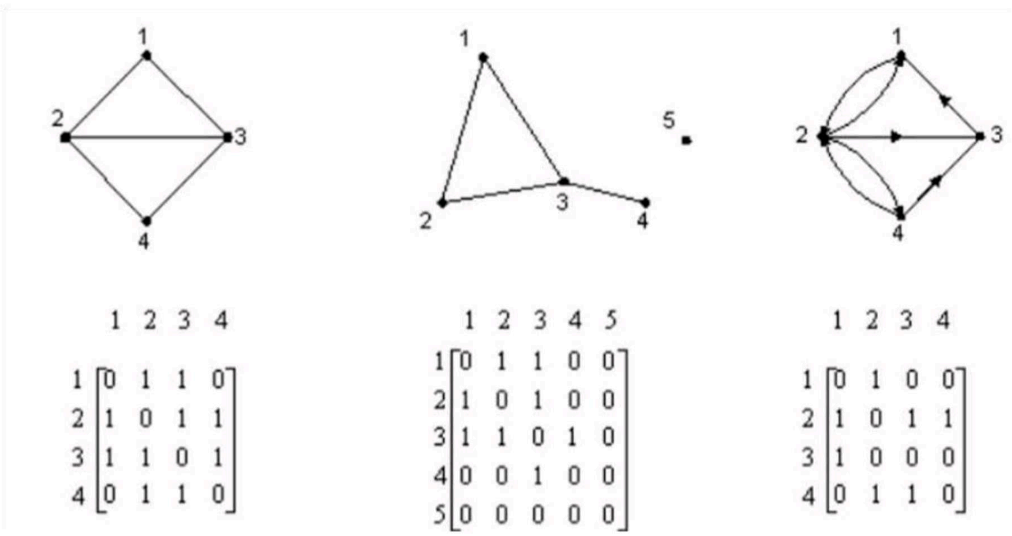


Figure 3.1: Example of graphs with it is adjacency matrix

3.2 Proposed algorithm

Now we will introduce a new colouring method, namely vertex colouring using an adjacency matrix.

The following is a colouring method that is proposed to colour the vertex in a graph using its adjacency matrix:

Step 1: Make an adjacency matrix of the graph which its vertex will be coloured.

Step 2: Sum the matrix elements in each row.

Step 3: Select the row matrix that has the biggest value.

Step 4: Strikethrough the selected matrix row and give the colour at its vertex.

Step 5: Strikethrough the row of matrix that corresponds to the column of the selected row that has a value 0.

Step 6: Give the colour at its vertex with the same colour of selected row.

Step 7: Select another row of matrix which has not been strikethrough and have the biggest row value (if the biggest row value is more than one, please choose one).

Step 8: Repeat step 4, give another colour at its vertex and so on until all the matrix row are strikethrough or all vertices have been coloured.

To understand this colouring method, an example is given to colour the vertex of graph using the adjacency matrix.

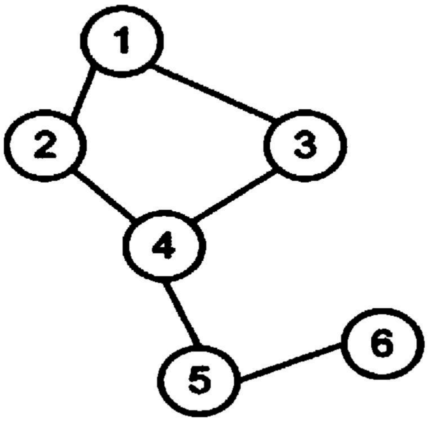


Figure 3.2: Graphs sample

First step, make adjacency matrices for graph base on figure3.2. The results of the adjacency matrix can be seen in figure3.3

	1	2	3	4	5	6
1	0	1	1	0	0	0
2	1	0	0	1	0	0
3	1	0	0	1	0	0
4	0	1	1	0	1	0
5	0	0	0	1	0	1
6	0	0	0	0	1	0

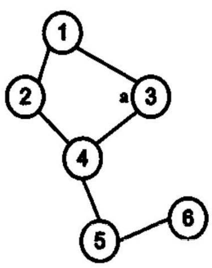
Figure 3.3: Matrix adjacency

The **second step** is sum the values of each matrix row. The results can be seen in figure3.4.

$$
\begin{array}{c|cccccc|l}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 0 & \rightarrow 2 \\
2 & 1 & 0 & 0 & 1 & 0 & 0 & \rightarrow 2 \\
3 & 1 & 0 & 0 & 1 & 0 & 0 & \rightarrow 2 \\
4 & 0 & 1 & 1 & 0 & 1 & 0 & \rightarrow 3 \\
5 & 0 & 0 & 0 & 1 & 0 & 1 & \rightarrow 2 \\
6 & 0 & 0 & 0 & 0 & 1 & 0 & \rightarrow 1
\end{array}
$$

Figure 3.4: Sum of the matrix row

The **third step** is choosing the biggest row value. In this graph the biggest value is 3 , that is at vertex 4.The fourth step strikethrough the selected matrix row and put colour to the selected vertex (e.g. a). The results of this step can be seen in figure3.5.



$$
\begin{array}{c|cccccc|l}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 0 & \rightarrow 2 \\
2 & 1 & 0 & 0 & 1 & 0 & 0 & \rightarrow 2 \\
3 & 1 & 0 & 0 & 1 & 0 & 0 & \rightarrow 2 \\
4 & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & \rightarrow 3a \\
5 & 0 & 0 & 0 & 1 & 0 & 1 & \rightarrow 2 \\
6 & 0 & 0 & 0 & 0 & 1 & 0 & \rightarrow 1
\end{array}
$$

Figure 3.5: Selected matrix row colouring

The **fifth step** 5, strikethrough the row of matrix that corresponds to the column of the selected matrix row and has a value 0. In this case, the selected rows are vertex 1, vertex 4 and vertex 6. The sixth steps, put the colour at its vertex by the same colour like selected row. The colouring results can be seen in figure3.6.

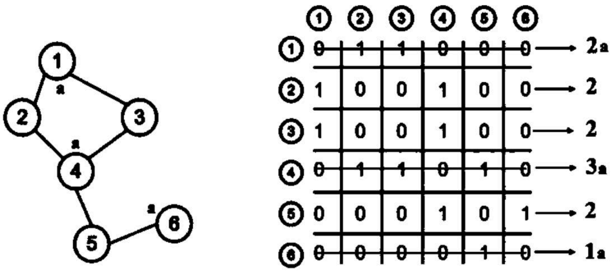


Figure 3.6: The colour of vertex that have same to the selected row

The **seven steps**, select a matrix row which has not been strikethrough and has the biggest row value (if more than one row have the biggest values, please choose one). In this case there are three vertices that have the biggest row values, those are vertex 2, vertex 3, and vertex 5. Suppose vertex 2 is selected as the selected row. The next step strikethrough the selected row and put a different colour to the selected vertex (eg: b). The result of this step can be seen in figure3.7.

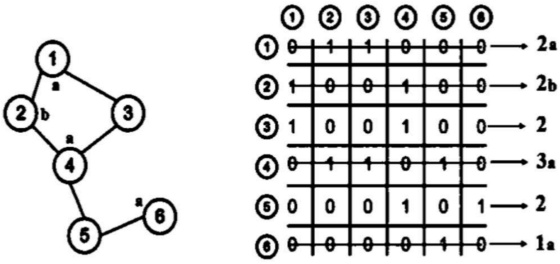


Figure 3.7: Another selected matrix row

The **eight steps**, strikethrough the row of the matrix that matches with the column of the selected row which have 0 value, those are vertex 2, 3, 5 and vertex 6. Vertex 6 is not chosen because it has been strikethrough or has been coloured. The results can be seen in figure3.8

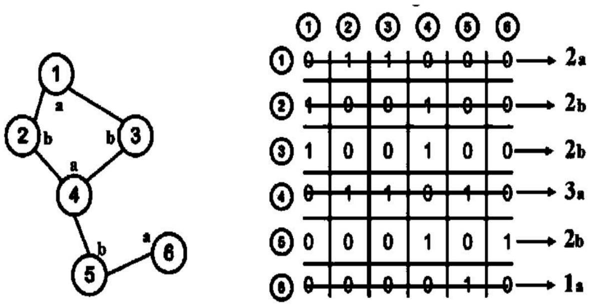


Figure 3.8: vertex colouring according to the selected row

Repeat step 7, give the other colour at vertex colouring using the adjacency matrix above is the minimum colour that used to colour the vertex (chromatic number) is 2.

CHAPTER-4

CHAPTER-4
NEW ALGORITHM FOR CHROMATIC NUMBER OF GRAPHS AND THEIR APPLICATIONS

4.1 Associated Adjacency Matrix

The associated adjacency matrix of a graph G, denoted by AA(G), is a matrix whose entry a_{ij}=1 if the vertices v_i and v_j are adjacent such that i≠j and a_{ij}=0 otherwise, where v_i and v_j are vertices in G.

Consider we have the graph G shown in Figure.4.1

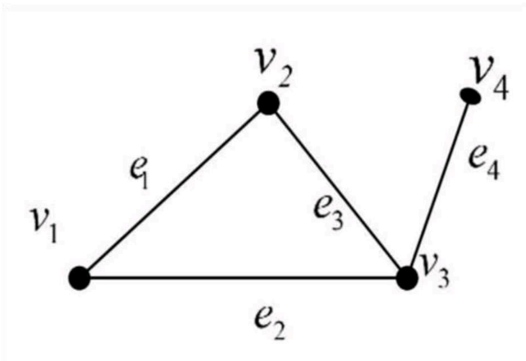


Figure 4.1

then the associated adjacency matrix of this graph is given by

$$AA(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4.2 R-colouring algorithm

Let G be a graph of order n with vertices as $v_1, v_2, v_3, \ldots., v_n$

Step1: Evaluate the associated adjacency matrix of the graph G

$$\text{AA(G)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Step 2: Confirm R-colouring matrix from the associated adjacency matrix as follows:

In the first row of R-colouring matrix, put $a^*{}_{11} = 1$. This means that the vertex v_1 is coloured by colour1. Then put

$$a^*{}_{12} = \begin{cases} 0 \ \text{ if } \mathrm{a}_{12} = 0 \\ 2 \ \text{ if } \mathrm{a}_{12} = 1 \end{cases}$$

After that put

$$a^*{}_{13} = \begin{cases} 0 \ if \ a_{13} = 0 \\ 2 \ if \ a_{13} = 1, a_{12} = 0 \\ 3 \ if \ a_{13} = a_{12} = a_{23} = 1 \end{cases}$$

In the same way, for entry $a^*{}_{1j}$, where 1< j ≤ n,

$$a^*{}_{ij} = \begin{cases} 0 & if \ a_{1j} = 0 \\ \min\{a^*{}_{1k^1}, a^*{}_{1k^2}, .., a^*{}_{1k^m}\} \ if \ a_{1j} = a_{1k^1} = 1, a_{k^1 j=0} \ \forall \ 1 \in Z^+, 1 \le l \le m, 1 < k^l < j & \\ j & if \ a_{12} = a_{13} = \ldots = a_{1j} = a_{2j} = a_{3j} = \ldots = a_{(j-1)j} = 1 \end{cases}$$

For any column I has the entry $a^*{}_{1i} = h$, we put

$$a^*{}_{ji} = \begin{cases} h \ \ if \ a_{ji} = 1, i \neq j \\ 0 \ \ if \ a_{ji} = 0, i \neq j \\ h \ \ if \ a_{ji} = 0, i = j \end{cases}$$

Where i, j ≤ n

(c) Now start from the row of the vertex which is coloured after v_1 and repeat steps (a) and (b).

(d) Again repeat step (c) to complete R-colouring matrix

$$\mathrm{RC(G)} = \begin{bmatrix} a^*{}_{11} & a^*{}_{12} & \cdots & a^*{}_{1n} \\ a^*{}_{21} & a^*{}_{22} & \cdots & a^*{}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a^*{}_{n1} & a^*{}_{n2} & \cdots & a^*{}_{nn} \end{bmatrix}$$

Step3: The greatest number in the diagonal of R-colouring matrix is the chromatic number of the graph G and the value of entry $a^*{}_{ii}$ is the colour of the vertex v_i.

EXAMPLE 1: Let G be a graph as shown in Figure 4.2

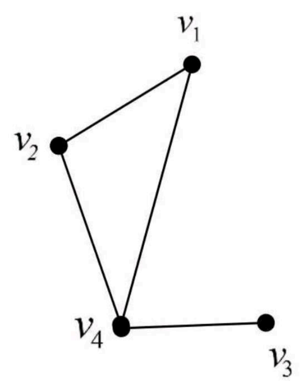


Figure 4.2

R-colouring algorithm is applied step by step as shown in the sequence of matrices below:

$$AA(G)=\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}\xrightarrow{Put\ a^*{}_{11}=1}\begin{bmatrix}1 & \square & \square & \square\\ \square & \square & \square & \square\\ \square & \square & \square & \square\\ \square & \square & \square & \square\end{bmatrix}$$

$$\xrightarrow{Since\ a_{12}=1\ Put\ a^*{}_{12}=2}\begin{bmatrix}1 & 2 & \square & \square\\ \square & \square & \square & \square\\ \square & \square & \square & \square\\ \square & \square & \square & \square\end{bmatrix}$$

$$\xrightarrow{Since\ a_{13}=0\ Put\ a^*{}_{13}=0}\begin{bmatrix}1 & 2 & 0 & \square\\ \square & \square & \square & \square\\ \square & \square & \square & \square\\ \square & \square & \square & \square\end{bmatrix}$$

$$\xrightarrow{Since\ a_{12}=a_{14}=a_{24}=1, Put\ a^*{}_{14}=3}\begin{bmatrix}1 & 2 & 0 & 3\\ \square & \square & \square & \square\\ \square & \square & \square & \square\\ \square & \square & \square & \square\end{bmatrix}$$

$$\xrightarrow{Coloumn\ 1\ Put\ a^*{}_{31}=0,\ a^*{}_{21}=a^*{}_{41}=1}\begin{bmatrix}1 & 2 & 0 & 3\\ 1 & \square & \square & \square\\ 0 & \square & \square & \square\\ 1 & \square & \square & \square\end{bmatrix}$$

$$\xrightarrow{Coloumn\ 2\ Put\ a^*{}_{32}=0, a^*{}_{22}=a^*{}_{42}=2}\begin{bmatrix}1 & 2 & 0 & 3\\ 1 & 2 & \square & \square\\ 0 & 0 & \square & \square\\ 1 & 2 & \square & \square\end{bmatrix}$$

$$\xrightarrow{Coloumn\ 4\ Put\ a^*{}_{24}=a^*{}_{34}=a^*{}_{44}=3}\begin{bmatrix}1 & 2 & 0 & 3\\ 1 & 2 & \square & 3\\ 0 & 0 & \square & 3\\ 1 & 2 & \square & 3\end{bmatrix}$$

$$\xrightarrow{2^{nd}\ row\ Since\ a_{23}=0\ Put\ a^*{}_{23}=0} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & \square & 3 \\ 1 & 2 & \square & 3 \end{bmatrix}$$

$$\xrightarrow{4^{th}\ row\ Since\ a_{43}=1 Put\ a^*{}_{43}=min\,\{a^*{}_{41},\ \ a^*{}_{42}\}=1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & \square & 3 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{Put\ a^*{}_{33}=1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

Then R- colouring matrix is

$$\text{RC(G)=} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

Hence the chromatic number of the graph G is 3, i.e. χ(G)=3, as shown from the diagonal of R-colouring matrix and the colouring matrix and the colour of v_1 is 1, v_2 is 2, v_3 is 1 and v_4 is 3

THEOREM 4.1

R-colouring algorithm calculates the exact value of chromatic number of a graph G.

Proof:

Suppose we have a graph G with more than one chromatic number calculated by R-colouring algorithm. This means that there exists more than one of R-colouring matrices with different greatest entry in their diagonals. Since R-colouring algorithm depends on the colour of the neighbour vertices of the coloured vertex coloured by the smallest colour is not used according to the order in the associated adjacency matrix which is fixed for the same graph (from property of matrices). Then, this is a contradiction and hence R-colouring algorithm calculates the exact value of chromatic number of the graph G.

RESULT 1

Let G be a graph of order n. Then $\chi(G) = 1$ if and only if $G \cong N_n$, where N_n is a null graph.

Proof:

Let G be a graph of order n, $\chi(G) = 1$ and $G \ncong N_n$. Since G is a graph of order n and $\chi(G)=1$ then every entry in the diagonal of R-colouring matrix equals 1 and the other entries equal 0, i.e.

$$RC(G)=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

This means the entries in the associated adjacency matrix of the graph G equal 0,i.e. each vertex in G does not connect to any vertex. This is a contradiction. Hence $G \cong N_n$.

To prove the converse, Let G be a graph of order n and $G \cong N_n$. Then the associated adjacency matrix of the graph G is given by

$$AA(G)=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence by using R-colouring algorithm, we find that R-colouring matrix is

$$RC(G)=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then the chromatic number of the graph G equals 1 from R-colouring matrix, i.e. χ(G)=1.

RESULT 2

Let G be a graph of order n ≥ 2. Then χ(G) = n if and only if G $\cong K_n$, where K_n is a complete graph of order n.

Proof:

Let G be a graph of order n ≥ 2 and χ(G) = n. Then R-colouring matrix of the graph G is $n \times n$ matrix and each entry in its diagonal of it takes number from 1 to n such that $a^*_{ii} \neq a^*_{jj} \forall\, i \neq j$

This means that, every vertex in G is connected to the others. Hence G $\cong K_n$.

To prove the converse, Let G be a graph of order n and G $\cong K_n$. Then the associated Adjacency matrix of the graph G by

$$AA(G) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

Hence by using R-colouring algorithm, we find that R-colouring matrix is

$$AA(G) = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

Then the chromatic number of the graph G equals n from R-colouring matrix,

i.e. $\chi(G) = n$.

Real-life application of colouring

We shall see the usage of colouring in traffic lights. Figure shows the intersection roads.

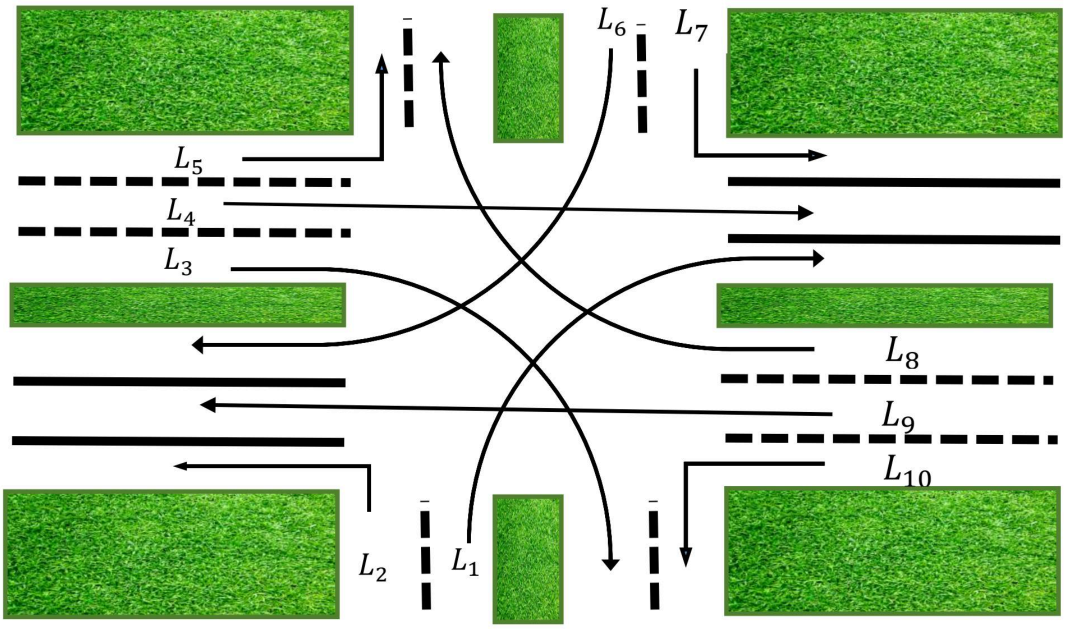


Figure 4.3

We find 10 traffic lanes L_1 to L_{10}. A traffic light system includes phases. At every phase, vehicles in lanes for which the light is green may proceed safely through the intersection. Now, we want to calculate a minimum number of phases needed for the traffic light system so that all vehicles may proceed safely through the intersection. 54

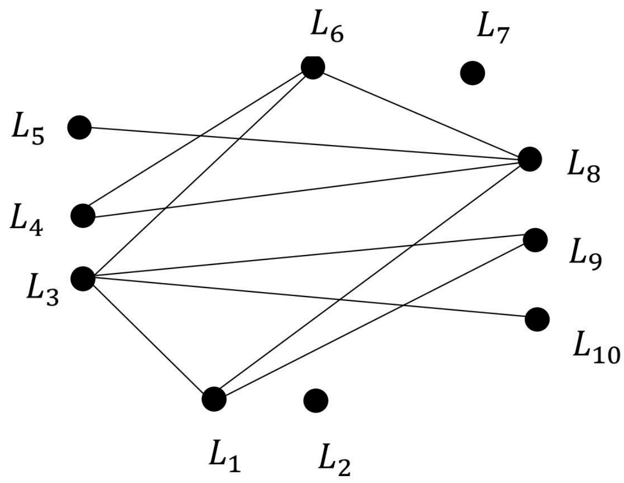


Figure 4.4

Let us solve this problem by drawing the graph G, shown in fig 4.4, to model the situation the associated adjacency matrix of the above graph is

$$
AA(G) = \begin{array}{c} \\ L_3 \\ L_1 \\ L_2 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \\ L_9 \\ L_{10} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} L_3 & L_1 & L_2 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

By applying R-colouring algorithm, we find that R-colouring matrix is

$$
RC(G) = \begin{array}{cc}
 & \begin{array}{cccccccccc} L_3 & L_1 & L_2 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} \end{array} \\
\begin{array}{c} L_3 \\ L_1 \\ L_2 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \\ L_9 \\ L_{10} \end{array} &
\begin{bmatrix}
1 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 2 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 3 & 2 & 2 & 0 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\end{array}
$$

Hence, three is the minimum number of phases in our problem.

Note:

Also, we can write associated adjacency matrix of our graph without drawing the graph, i.e. $a_{ij} = 1$ if L_i intersect L_j and $a_{ij} = 0$ if L_i does not intersect L_j.

CONCLUSION

The HB colour matrix algorithm is presented using a special assignment. The results of the HB colour matrix for some standard graphs are presented. This graph colouring method can be implemented for a total colouring of the graphs. Vertex colouring method uses an adjacency matrix can be applied to any graph without looking at the type or class of the graph. A new algorithm for calculating the exact value of the chromatic number of a graph has been designed and dubbed as R-colouring algorithm. Some basic results have been proved and the traffic light problem has been taken as a real life application.

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A STUDY ON Z-TRANSFORM

Project Report submitted to

ST. MARY'S COLLEGE (AUTONOMOUS), THOOTHUKUDI

Affiliated to

MANOMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI

In partial fulfilment of the requirement for the award of degree of

Bachelor of Science in Mathematics

Submitted by

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CERTIFICATE

We hereby declare that the project report entitled **Z- Transforms** being submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of degree of **Bachelor of Science in Mathematics** and it is a record of work done during the year 2022 -2023 by the following students:

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We hereby declare that the project entitled **A STUDY ON Z-TRANSFORM** submitted for the degree of Bachelor of Science is our work carried out under the guidance of **Dr. A. Punitha Tharani M.Sc., M.Phil., Ph.D.,** Associate Professor, Department of Mathematics, **St. Mary's College (Autonomous)**, Thoothukudi.

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ACKNOWLEDGEMENT

First of all, we thank Lord Almighty for showering his blessing to undergo this project.

With immense pleasure, we register our deep sense of gratitude to our guide **Dr. A. Punitha Tharani M.Sc., M.Phil., Ph.D.** for having suggested this project topic and imparted necessary guidelines throughout the period of our Investigation.

We thank our beloved Principal **Sr. A.S.J. Lucia Rose M.Sc., M.Phil., Ph.D., PGDCA.,** for Providing us help to carry out our project work successfully.

We thank our parents and friends who helped us a lot in finalizing this project within the limited time frame.

At last, we end up by thanking all those who extended their helping hands towards this project.

CONTENTS

Introduction

In mathematics and signal processing, the **Z-transform** convert a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain (**z-domain** or **z-plane**) representation.

Transformation is a very powerful mathematical tool so using it in mathematical treatment of problem is arising in many applications. The idea of Z-transform back to 1730 when De Moivre introduced the concept of “generating functions” to probability theory. In 1947 a transform of sampled signal or sequence defined by W. Hurewicz as a tractable way to solve linear constant-coefficients difference equations. The transformation named "Z-transform" by Ragazzini and Lotfi Zadeh in the sampled-data control group at Columbia University in 1952.

Z-transform is transformation for discrete data equivalent to the Laplace transform of continuous data and it’s a generalization of discrete Fourier transform. Z-transform is used in many areas of applied mathematics as digital signal processing, control theory, economics and some other fields.

In this thesis, we present Z-transform, the one-sided Z-transform and the two-dimensional Z-transform with their properties, finding their inverse and someexamples on them.

Chapter 1
Z-Transforms

1.1 Introduction

The Z-transform play an important role in communication theory in particular signals and systems. Many engineering areas like control systems. Signal processing, image processing and mobile communication. Z transformation is the process of standardize that allows for comparison of scores from disparate distributions using a distribution mean and standard deviation, Z transformations convert sperate distribution into a standardized distribution, allowing for the comparison of dissimilar matrices.

Laplace transform and Fourier transform play important roles in the study of continuous time signals, Z – transform which is the discrete time counterpart of the Laplace transform plays an engineering discipline of board scope. It is used not only in communication technology, but also in the fields of astronomy, oceanography, crystallography, bio-engineering, antenna design, system theory, computer sciences and in many other fields.

1.2 Definitions:

1.2.1 Definition 1:

Let {f(n)} be a sequence defined for n = 0, ± 1, ± 2, ….., then Z-transform is defined as

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}, \qquad (z \to a\ complex\ number)$$

$$= F(z)$$

This is called **two sided or bilateral Z-transform.**

1.1.1 Definition 2:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} \qquad (z \to a\ complex\ number)$$

$$= F(z)$$

This is called **one sided z-transforms.**

1.2.3. Definition 3:

If f (t) is defined for discrete values of 't' where $t = nT$, $n = 0,1,2,3,\ldots..T$ being the sampling period, then

$$Z\,[f\,(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$
$$= F\,(z)$$

Note:

1. In this chapter mostly we use one sided Z- transform.
2. If f(n) is given then keep 'n' as it is.
3. If f(t) is given then replace 't' by 'nT'
4. The symbols { }, [ ] or ( ) are used to denote a sequence.
5. F (z) and f (n) are called a Z- transform pair.

1.3 Important Results

$$(1-x)^{-1} = 1 + x + x^2 + \ldots\ldots + x^n + \ldots\ldots$$

$$(1+x)^{-1} = 1 - x + x^2 - \ldots\ldots. + (-1)^n x^n + \ldots\ldots.$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - \ldots\ldots. + (-1)^n (n+1)\, x^n + \ldots\ldots.$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots. + (n+1)\, x^n + \ldots\ldots.$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots. + \frac{x^n}{n!} + \ldots\ldots\ldots\ldots\ldots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots. + (-1)^n \frac{x^n}{n!} + \ldots$$

$$\mathrm{Log}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots.$$

$$\mathrm{Log}(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots\ldots.\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

1.4 Convolution Theorem

1.4.1 Definition:

The convolution of the two sequences {f(n)} and {g(n)} is defined as

(i) $\{f(n)*g(n)\} = \sum_{r=-\infty}^{\infty} f(r).\ g(n-r)$ if the sequences are non-causal and

(ii) $\{f(n)*g(n)\} = \sum_{r=0}^{n} f(r).\ g(n-r)$, if the sequences are causal.

The convolution of two functions f(t) and g(t) is defined as f(t)*g(t) = $\sum_{r=0}^{n} f(rT).\ g(n-r)T$, where T is the sampling period.

1.4.2 Statement of the theorem

If Z{f(n)} = $\bar{f}$(z) and Z{g(n)} = $\bar{g}$(z), then

Z{f(n)*g(n)} = $\bar{f}$(z) . $\bar{g}$(z)

Proof:

$$Z\{f(n)*g(n)\} = Z\left[\sum_{r=-\infty}^{\infty} f(r).g(n-r)\right]$$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{r=-\infty}^{\infty} f(r)g(n-r)\right]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} f(r)\left[\sum_{r=-\infty}^{\infty} g(n-r)z^{-n}\right]$$

By changing the order of summation,

$$= \sum_{n=-\infty}^{\infty} f(r)\left[\sum_{m=-\infty}^{\infty} g(m)z^{-(m+r)}\right]$$, by putting n-r = m

$$= \sum_{n=-\infty}^{\infty} f(r)z^{-r}\left[\sum_{m=-\infty}^{\infty} g(m)z^{-m}\right]$$

$$= \sum_{n=-\infty}^{\infty} f(r)z^{-r}.\ \bar{g}(z)$$

$$= \bar{g}(z).\ \sum_{n=-\infty}^{\infty} f(r)z^{-r}$$

Z{f(n)*g(n)} = $\bar{f}$(z) . $\bar{g}$(z)

Chapter-2

Properties of z-transform

2.1 Introduction

The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics.

2.2 Linearity

The Z-transform is linear

i.e. $Z\{a\, f(n)+b\, g(n)\} = a\, Z\{f(n)\} + b\, Z\{g(n)\}$

Proof:

$$Z\{a\, f(n) + b\, g(n)\} = \sum_{n=0}^{\infty}\{af(n) + bg(n)\}z^{-n}$$

$$= a\sum_{n=0}^{\infty} f(n)z^{-n} + b\sum_{n=0}^{\infty} g(n)z^{-n}$$

$$= a\, Z\{f(n)\} + b\, Z\{g(n)\}$$

Similarly, $Z\{a\, f(t) + b\, g(t)\} = a\, Z\{f(t)\} + b\, Z\{g(t)\}$

Problem:1

Find k using linearity.

Solution:

$$Z[k] = \sum_{n=0}^{\infty} kz^{-n}$$

$$= k\sum_{n=0}^{\infty} z^{-n}$$

$$= k\,\{1+z^{-1}+z^{-2}+\ldots\ldots.\}$$

$$= \left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \ldots\right)$$

$= k \left(1-\frac{1}{z}\right)^{-1}$ (since $(1-x)^{-1} = 1 + x + x^2 + \ldots\ldots + x^n + \ldots\ldots$)

$$Z[k] = \frac{kz}{z-1}$$

Corollary 1: $Z[1] = \frac{z}{z-1}$

Proof:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z(1) = \sum_{n=0}^{\infty} 1 \,.\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1-\frac{1}{z}\right)^{-1} \; for \; \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{1-\frac{1}{z}} \qquad \begin{bmatrix} (1-x)^{-1} = 1 + x + x^2 + \cdots if\ |x| < 1 \\ Here\ x = \frac{1}{z} \\ |x| < 1 \\ \left|\frac{1}{z}\right| < 1 \\ 1 < |z| \end{bmatrix}$$

$$= \frac{\frac{1}{z-1}}{z}$$

$$= \frac{z}{z-1}$$

$$z(1) = \frac{z}{z-1}$$

Corollary 2: $Z(n) = \frac{z}{(z-1)^2}$

Proof:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z(n) = \sum_{n=0}^{\infty} n\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right]$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)^2$$

$$= \frac{1}{z}\left(\frac{1}{\frac{z-1}{z}}\right)^2$$

$$= \frac{1}{z} \cdot \frac{z^2}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

$$Z(n) = \frac{z}{(z-1)^2}$$

Problem 2:

Find Z(n-2).

Solution:

Z(n-2) = Z(n) – Z(2)

$$= \frac{z}{(z-1)^2} - 2z(1)$$ (by Corollary 1)

$$= \frac{z}{(z-1)^2} - 2.\frac{z}{z-1}$$

$$= \frac{z-2z(z-1)}{(z-1)^2}$$

$$= \frac{z-2z^2+2z}{(z-1)^2}$$

$$Z(n\text{-}2) = \frac{3z-2z^2}{(z-1)^2}$$

2.3. Derivative Of Transforms

$$Z[nf(n)] = -z\frac{dF}{dz}$$

Proof:

We know that

$$F(z)=\sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\frac{dF}{dz} = \sum_{n=0}^{\infty} f(n)\,(-nz^{-n-1})$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty} nf(n)\, z^{-n}$$

$$-z\frac{dF}{dz} = Z[nf(n)]$$

$$\therefore\ Z[nf(n)] = -z\frac{dF}{dz}$$

Problem 4:

Find Z[n] using derivatie of Transform

Solution:

$Z[nf_n] = -z\frac{dF}{dz}$ (By derivative of Transform)

Z[n] $= z\,[n.1]$ (since z[f(n)] = F(z))

$$= -z\frac{d}{dz}Z[1]$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) \quad \text{(by Corollary 1)}$$

$$= -z\left[\frac{(z-1).1-z}{(z-1)^2}\right]$$

$$Z[n] = \frac{z}{(z-1)^2}$$

2.4. Scaling In Z-Domain

$$Z[a^n f(n)] = F\left(\frac{z}{a}\right), F(z) = Z[f(n)]$$

Proof:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z[a^n f(n)] = \sum_{n=0}^{\infty} a^n f(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\frac{z^{-n}}{a^{-n}}$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n}$$

$$= F\left(\frac{z}{a}\right)$$

$$\therefore Z[a^n f(n)] = F\left(\frac{z}{a}\right)$$

$$= \{F(z)\}_{z\to\frac{z}{a}}$$

Problem 5:

Find$Z[a^n \cos n\,\theta]$

Solution:

$$Z[a^n f(n)] = F\left(\frac{z}{a}\right), \qquad Z[f(n)] = F(z)$$

$$Z[a^n \cos n\,\theta] = Z[\cos n\,\theta]_{z\to\frac{z}{a}}$$

$$= \left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right]_{z\to\frac{z}{a}} \qquad \left(\because Z[\cos n\,\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right)$$

$$= \frac{\frac{z}{a}\left(\frac{z}{a}-\cos\theta\right)}{\frac{z^2}{a^2}-\frac{2z}{a}\cos\theta+1}$$

$$= \frac{\frac{z}{a}\left(\frac{z-a\cos\theta}{a}\right)}{\frac{z^2-2az\cos\theta+a^2}{a^2}}$$

$$= \frac{z(z-a\cos\theta)}{a^2} \times \frac{a^2}{z^2-2az\cos\theta+a^2}$$

$$Z[a^n \cos n\,\theta] = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$

Problem 6:

Find $Z(na^n)$

Solution:

$$Z(na^n) = Z(a^n.n)$$

$$= \{Z(n)\}_{z \to \frac{z}{a}} \qquad \text{(By corollary 2)}$$

$$= \left\{\frac{z}{(z-1)^2}\right\}_{z \to \frac{z}{a}}$$

$$= \frac{\frac{z}{a}}{(\frac{z}{a}-1)^2}$$

$$= \frac{\frac{z}{a}}{\frac{(z-a)^2}{a^2}}$$

$$= \frac{z}{a} \times \frac{a^2}{(z-a)^2}$$

$$= \frac{az}{(z-a)^2}$$

Chapter 3

Inverse Z-Transform

3.1 Introduction

The inverse Z-transform of $\bar{f}(z)$ has been already defined as $Z^{-1}\{\bar{f}(z)\} = f(n)$ when $Z\{f(n)\}=\bar{f}(z)$. $\bar{f}(z)$ can also be called as F(z).

Represents the integration around the circle of radius |z|=r in the counter clockwise direction. This is the direct method of finding inverse Z-transform. The direct method is quite tedious. Hence, indirect methods are used for finding the inverse Z-transform.

3.2 Types of Inverse Z-Transform

3.2.1 Partial Fraction Method

When $\bar{f}(z)$ is a rational function in which the denominator can be factorised, f(z) is resolved into partial fractions and then $Z^{-1}\{F(z)\}$ is derived as the sum of the inverse Z-transform of the partial fractions.

Corollary :

If $Z[a^n] = \frac{z}{z-a}$, then $z^{-1}\left[\frac{z}{z-a}\right] = a^n$

Proof:

$$Z[a^n]=\sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1+az^{-1}+a^2z^{-2}+\cdots$$

$$=1+\frac{a}{z}+\frac{a^2}{z^2}+\cdots$$

$$=\left(1-\frac{a}{z}\right)^{-1}$$

$$=\left(\frac{z-a}{z}\right)^{-1}$$

$$Z[a^n] = \frac{z}{z-a}$$

$$\therefore\ z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

Problem 1:

Find inverse Z-transform $Z^{-1}\left[\frac{10z}{z^2-3z+2}\right]$

Solution:

Let $F(z) = \frac{10z}{z^2-3z+2}$

$= \frac{10z}{(z-1)(z-2)}$

$\therefore \quad \frac{F(z)}{10z} = \frac{1}{(z-1)(z-2)} \qquad \dots (1)$

$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$1 = A(z-2)+B(z-1)$

Put z= 1	Put z=2
A= -1	B= 1

$\therefore \quad \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$

$\therefore \quad \frac{F(z)}{10z} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$

$F(z) = \frac{-10z}{(z-1)} + \frac{10z}{(z-2)}$

$Z^{-1}[F(z)] = Z^{-1}\left[\frac{-10z}{(z-1)}\right] + Z^{-1}\left[\frac{10z}{(z-2)}\right]$

$= -10Z^{-1}\left[\frac{z}{(z-1)}\right] + 10\,Z^{-1}\left[\frac{z}{(z-2)}\right]$ (By corollary)

$= -10(1)^n + 10(2)^n$

$\therefore \quad Z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = -10(1)^n + 10(2)^n, \quad n \geq 0$

Problem 2:

Find $Z^{-1}\left[\frac{z}{z^2+7z+10}\right]$

Solution:

$$z^2 + 7z + 10 = (z+2)(z+5)$$

$$\frac{z}{z^2+7z+10} = \frac{z}{(z+2)(z+5)}$$

$$\frac{z}{(z+2)(z+5)} = \frac{A}{z+2} + \frac{B}{z+5}$$

$$z = A(z+5) + B(z+2)$$

Put $z = -2$	Put $z = -5$
$-2 = 3A$	$-5 = -3B$
$\Rightarrow A = -\frac{2}{3}$	$\Rightarrow B = \frac{5}{3}$

$$\therefore \frac{z}{(z+2)(z+5)} = -\frac{2}{3}\frac{1}{z+2} + \frac{5}{3}\frac{1}{z+5}$$

$$Z^{-1}\left[\frac{z}{(z+2)(z+5)}\right] = -\frac{2}{3}Z^{-1}\left[\frac{1}{z+2}\right] + \frac{5}{3}Z^{-1}\left[\frac{1}{z+5}\right] \qquad \left(\because Z^{-1}\left[\frac{1}{z-a}\right] = (a)^{n-1}\right)$$

$$Z^{-1}\left[\frac{z}{z^2+7z+10}\right] = -\frac{2}{3}(-2)^{n-1} + \frac{5}{3}(-5)^{n-1}, \ n \geq 1$$

3.2.2 EXPANSION METHOD

If $\bar{f}(z)$ can be expanded in a series of ascending powers of z^{-1}, i.e. in the form $\sum_{n=0}^{\infty} f(n)\, z^{-n}$, by binomial, exponential and logarithmic theorems, the coefficient of z^{-n} in the expansion gives $Z^{-1}\{\bar{f}(z)\}$.

3.2.3 LONG DIVISION METHOD (POWER SERIES METHOD)

When the usual methods of expansion of $\bar{f}(z)$ fail and if $\bar{f}(z) = \frac{g(z^{-1})}{h(z^{-1})}$, then $g(z^{-1})$ is divided by $h(z^{-1})$ in the classical manner and hence the expansion $\sum_{n=0}^{\infty} f(n) z^{-1}$ is obtained in the quotient.

Problem :3

Find the inverse Z-transform of F(z) $=\frac{1}{1-az^{-1}}$, $|z|>|a|$ using power series method

Solution:

The long division is performed as follows to get power series expansion

$$
\begin{array}{r|l}
 & 1+ a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots\ldots \\
\hline
1 - a z^{-1} & 1 \\
 & 1- \ a z^{-1} \\
\hline
 & \quad a z^{-1} \\
 & \quad a z^{-1} - a^2 z^{-2} \\
\hline
 & \qquad\qquad a^2 z^{-2} \\
 & \qquad\qquad a^2 z^{-2} - a^3 z^{-3} \\
\hline
 & \qquad\qquad\qquad\qquad a^3 z^{-3} \\
 & \qquad\qquad\qquad\qquad a^3 z^{-3} - a^4 z^{-4} \\
\hline
 & \qquad\qquad\qquad\qquad\qquad\qquad a^4 z^{-4} - \ldots.
\end{array}
$$

Thus, we get the series,

$$F(Z) = 1+ a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots\ldots$$

$$= \frac{1}{1-az^{-1}}, \ |z|>|a|$$

Problem :4

Find the inverse Z-transform of $\frac{1+2z^{-1}}{1-z^{-1}}$, by the long division method.

Solution:

$$
\begin{array}{r|l}
 & 1+3z^{-1}+3z^{-2} \\
\hline
1-z^{-1} & 1+2z^{-1} \\
 & 1-z^{-1} \\
\hline
 & 3z^{-1} \\
 & 3z^{-1}-3z^{-2} \\
\hline
 & \qquad 3z^{-2} \\
 & \qquad 3z^{-2}-3z^{-3} \\
\hline
 & \qquad\qquad 3z^{-3}
\end{array}
$$

Thus $\frac{1+2z^{-1}}{1-z^{-1}} = \sum_{n=0}^{\infty} f(n)z^{-n}$

$= 1+3z^{-1}+3z^{-2}+\ldots+3z^{-n}+\ldots$

$\therefore \quad f(n) = \begin{cases} 1, & for\ n=0 \\ 3, & for\ n\geq 1 \end{cases}$

3.2.4 CAUCHY'S RESIDUE THEOREM

By using the relation between the Z-transform and Fourier transform of a sequence , it can be proved that

$$f(n) = \frac{1}{2\pi i}\int_C f(z)\, z^{n-1}dz$$

Where C is a circle whose center the origin and radius is sufficiently large to include all the isolated singularities of $\bar{f}(z)$. C may also be a closed contour including the origin and all the isolated singularities of $\bar{f}(z)$.

By Cauchy's residue theorem,

$\oint_C \bar{f}(z)\, z^{n-1} dz = 2\pi i \times$ sum of the residues of $\bar{f}(z) z^{n-1}$ at the isolated singularities.

$\therefore$ f(n) = Sum of the residues of $\bar{f}(z) z^{n-1}$ at the isolated singularities.

Problem : 5

Find $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

Solution:

Let $\frac{z}{(z-1)(z-2)} = F(z)$, $f(n) = Z^{-1}[F(z)]$

$$\therefore Z^{n-1} F(z) = \frac{z^{n-1} \cdot z}{(z-1)(z-2)}$$

$$= \frac{z^n}{(z-1)(z-2)}$$

The poles are z =1, z = 2 (simple poles).

$$\text{Res}\,\{Z^{n-1} F(z)\}_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{z^n}{\cancel{(z-1)}(z-2)}$$

$$= -(1)^n$$

$$\text{Res}\,\{Z^{n-1} F(z)\}_{z=2} = \lim_{z\to 1} \cancel{(z-2)} \cdot \frac{z^n}{(z-1)\cancel{(z-2)}}$$

$$= 2^n$$

$\therefore$ f(n) = Sum of the residues of Z^{n-1} F(z) at its poles

$$= 2^n - (1)^n, \; n \geq 0$$

$$Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = 2^n - (1)^n, \; n \geq 0$$

Problem 6:

Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Solution:

Let $\quad F(z) = \left[\frac{z^2}{(z-a)(z-b)}\right]$, $Z^{-1}[F(z)] = f(n)$

$\therefore \; Z^{n-1} F(z) = \frac{z^{n-1} \cdot z^2}{(z-a)(z-b)}$

$= \frac{z^{n+1}}{(z-a)(z-b)}$

The poles are z=a and z= b (simple pole)

$\text{Res}\,\{Z^{n-1} F(z)\}_{z=a} = \lim_{z\to a} \cancel{(z-a)} \frac{z^{n+1}}{\cancel{(z-a)}\,(z-b)}$

$= \frac{a^{n+1}}{a-b}$

$\text{Res}\,\{Z^{n-1} F(z)\}_{z=b} = \lim_{z\to b} \cancel{(z-b)} \frac{z^{n+1}}{(z-a)\,\cancel{(z-b)}}$

$= \frac{b^{n+1}}{b-a}$

$\therefore \quad f(n) = \text{Sum of the residues of } Z^{n-1} F(z) \text{ at its poles}$

$= \frac{1}{a-b}[a^{n+1} - b^{n+1}]$, n$\geq 0$

$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{1}{a-b}[a^{n+1} - b^{n+1}]$, n$\geq 0$

Chapter 4

Shifting Properties

4.1 Introduction

The first shifting theorem is a useful tool when faced with the challenge of taking the Laplace transform of the product of exponential function with another function. The Laplace transform is very useful in solving ordinary differential equations. The second shifting theorem is a useful tool when faced with the challenge of taking the Laplace transform of the product of a shifted unit step function with another shifted function.

4.2 First Shifting Theorem

If Z[f(t)] = F(z), then

$$Z[e^{-at}f(t)] = F[ze^{aT}]$$

Proof : We know that ,

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$Z[e^{-at}f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\, (e^{aT})^{-n}\ z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\, (ze^{aT})^{-n}$$

$$= F[ze^{aT}]$$

NOTE :

The above result can also be written as

$$Z[e^{-at}f(t)] = \{F(z)\}_{z \to ze^{aT}}$$

$$Z[e^{at}f(t)] = \{F(z)\}_{z \to ze^{-aT}}$$

Problem 1 :

Find $Z[e^{-at}t]$

Solution :

$$Z[e^{-at}f(t)] = \{F(z)\}_{z \to ze^{aT}} \qquad (\because z[f(t)] = F(z))$$

$$Z[e^{-at}t] = Z[t]_{z \to ze^{aT}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^{aT}}$$

$$= \frac{T.\ ze^{aT}}{(ze^{aT}-1)^2}$$

Problem 2 :

Find $Z(e^{-iat})$

Solution :

$$Z[e^{-iat}] = Z[e^{-iat}.1]$$

$$= \{Z(1)\}_{z \to ze^{iaT}} \quad \text{[By shifting property]}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{iaT}} \qquad \left[Result: z(1) = \frac{z}{z-1}\right]$$

$$= \frac{ze^{iaT}}{ze^{iaT}-1}$$

Problem :3

Find $Z[\cos h\ at \sin bt]$

Solution:

$$Z[\cos h\ at \sin bt] = Z\left[\left(\frac{e^{at}+e^{-at}}{2}\right) \sin bt\right]$$

$$= \frac{1}{2}\{ Z\,(e^{at} \sin bt\,) + Z(e^{-at} \sin bt) \}$$

$$= \frac{1}{2}\left\{Z(\sin bt\,)_{z\to ze^{-aT}} + Z(\sin bt\,)_{z\to ze^{aT}}\right\}$$

$$= \frac{1}{2}\left[\left\{\frac{z \sin bT}{z^2-2z\cos bT+1}\right\}_{z\to ze^{-aT}} + \left\{\frac{z \sin bT}{z^2-2z\cos bT+1}\right\}_{z\to ze^{aT}} \right]$$

[by shifting property]

$$=\frac{1}{2}\left[\frac{ze^{-aT}\sin bT}{z^2e^{-2aT}-2ze^{-aT}\ \cos bT+1} + \frac{ze^{aT}\sin bT}{z^2e^{2aT}-2ze^{aT}\ \cos bT+1}\right]$$

Problem 4:

Find $Z\,(e^{-at}cos\ \omega t)$

Solution:

$Z\,[e^{-at}f(t)] = \{ F(z) \}_{z\to ze^{aT}}$ $\qquad$ $Z\,[f(t)] = F(z)$

Here $f(t) = cos\ \omega t$

$Z(e^{-at}cos\ \omega t) = \{Z(cos\ \omega t)\}_{z\to ze^{aT}}$ $\qquad$ [By shifting property]

$$= \left\{\frac{z(z-cos\ \omega t\,)}{z^2-2zcos\ \omega t+1}\right\}_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}(ze^{aT}-cos\ \omega t\,)}{z^2e^{2aT}-2ze^{aT}cos\ \omega t+1}$$

Problem 5:

Find Z($e^{-at} sin\ \omega t$)

Solution :

Z [e^{-at}f(t)] = $\{ F(z) \}_{z \to ze^{aT}}$ Z [f(t)] = F(z)

Here f(t) = $sin\ \omega t$

Z($e^{-at} sin\ \omega t$) = $\{Z(sin\ \omega t)\}_{z \to ze^{aT}}$ [By shifting property]

$$= \left\{\frac{zsin\ \omega t}{z^2 - 2zcos\ \omega t + 1}\right\}_{z \to ze^{aT}}$$

$$= \frac{ze^{aT} sin\ \omega t}{z^2 e^{2aT} - 2ze^{aT} cos\ \omega t + 1}$$

4.3 Second Shifting Theorem

1. Z [f(n+1)] = z F (z) −z f(0)

Proof :

Z [f(n+1)] = $\sum_{n=0}^{\infty} f(n+1) z^{-n}$

$=\sum_{n=0}^{\infty} f(n+1)\ z.z^{-1}\ z^{-n}$

$=\ z\ \sum_{n=0}^{\infty} f(n+1) z^{-(n+1)}$

Put n+1 = m , we get

n = 0 → m = 1

n =∞ → $m = \infty$

Z [f(n+1)] = z $\sum_{m=1}^{\infty} f(m) z^{-m}$

=z[$\sum_{m=1}^{\infty} f(m) z^{-m} + f(0) z^{-0} - f(0) z^{-0}$]

$$= z[\sum_{m=0}^{\infty} f(m)z^{-m} - f(0)]$$

$$= z\,F(z) - zf(0)$$

2. Prove that if Z[f(t)]=F(z) then Z[f(t+T)] = z F(z)$-zf(0)$

Proof:

$$Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T)z^{-n}$$

$$= \sum_{n=0}^{\infty} f[(n+1)T]z^{-n}$$

$$= \sum_{n=0}^{\infty} f[(n+1)T]z.z^{-1}z^{-n}$$

$$= z\sum_{n=0}^{\infty} f[(n+1)T]z^{-(n+1)}$$

Put n+1 = m , we get

$$\begin{array}{|l} n = 0 \rightarrow m = 1 \\ n = \infty \rightarrow m = \infty \end{array}$$

$$Z[f(t+T)] = z\sum_{m=1}^{\infty} f(mT)z^{-m}$$

$$= z[\sum_{m=1}^{\infty} f(mT)z^{-m} + f(0T)z^{-0} - f(0)z^{-0}]$$

$$= z[\sum_{m=0}^{\infty} f(mT)z^{-m} - f(0)]$$

$$= z\,F(z) - zf(0)$$

Problem 1

Find $Z[a^{k-1}]$

Solution

$$Z\,[f(k-n)] = z^{-n} Z[f(k)] \qquad \text{[Shifting property]}$$

$$Z[a^{k-1}] = z^{-1} Z(a^k) \qquad \text{[Here } n = 1\text{]}$$

$$= z^{-1} \frac{z}{z-a} \qquad \left[z(a^k) = \frac{z}{z-a}\right]$$

$$= \frac{1}{z} \frac{z}{z-a}$$

$$Z[a^{k-1}] = \frac{1}{z-a}$$

Problem 2 :

Find $Z[(k-1)a^{k-1}]$

Solution :

We know that

$$Z[(k-1)a^{k-1}] = z^{-1} Z[ka^k] \qquad [\text{ Using } Z[f(k-n)] = z^{-1}F(z);\ Here\ n = 1]$$

$$Z[(k-1)a^{k-1}] = z^{-1} \frac{az}{(z-a)^2}$$

$$= \frac{1}{z} \frac{az}{(z-a)^2}$$

$$= \frac{a}{(z-a)^2} \qquad \left[\text{since } Z[ka^k] = \frac{az}{(z-a)^2}\right]$$

Problem 3 :

Find $Z[\cos(t+T)]$

Solution :

$Z[f(t+T)] = zF(z) - zf(0)$

Here $f(t) = \cos t$, $f(0) = \cos 0 = 1$

$$Z[\cos(t+T)] = z\,Z[\cos t] - zf(0)$$

$$= z\frac{z\,(z-\cos T)}{z^2-2z\cos T+1} - z \qquad [since\ f(0) = 1]$$

$$= \frac{z^2\,(z-\cos T)}{z^2-2z\cos T+1} - z$$

$$= \frac{z^3-z^2\cos T - z(z^2-2z\cos T+1)}{z^2-2z\cos T+1}$$

$$= \frac{z^3-z^2\cos T-z^3+2z^2\cos T-z}{z^2-2z\cos T+1}$$

$$= \frac{z^2\cos T-z}{z^2-2z\cos T+1}$$

Problem 4:

Find $Z\,[e^{-3(t+T)}]$

Solution:

$Z\,[f(t+T)] = zF(z) - zf(0)$

Given $f(t) = e^{-3t}$

$\therefore\ f(0) = e^0 = 1$

$$Z[f(t)] = Z\,[e^{-3t}] = \frac{z}{z-e^{-3T}} = F(z)$$

$$Z\,[e^{-3(t+T)}] = z\,F(z) - z\,f(0)$$

$$= z\frac{z}{z-e^{-3T}} - z \qquad [\text{Since } f(0) = 1]$$

$$= z\left[\frac{z}{z-e^{-3T}} - 1\right]$$

Problem 5:

Find Z [(t+T)$e^{-(t+T)}$]

Solution:

Z [f(t+T)] = z F(z) $-z\,f(0)$ ....(1)

Here f(t) = $t\,e^{-t}$, f(0) = 0 ....(2)

$$F(z) = Z[te^{-t}]$$

$$= Z[t]_{z \to ze^{T}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^{T}}$$

$$= \frac{Tze^{T}}{(ze^{T}-1)^2} \qquad \text{......(3)}$$

Substituting (2) and (3) in (1)

$$\therefore Z[f(t+T)] = Z\left[(t+T)e^{-(t+T)}\right]$$

$$= z\ \frac{Tze^{T}}{(ze^{T}-1)^2} - z\,.0$$

$$= \frac{z^2Te^{T}}{(ze^{T}-1)^2}$$

Chapter 5

Theorems

5.1 Introduction:

In mathematical analysis, the **Initial Value Theorem** is a theorem used to relate frequency domain expressions to the time domain behavior as time approaches zero.[1]

Let

$$\boldsymbol{F(S) = \int_0^\infty f(t)e^{-st}\,dt}$$

be the (one-sided) Laplace transform of $f(t)$.If f is bounded on $(0,\infty)$ and $\lim\limits_{t\to 0+\_} f(t)$ exists then the initial value theorem says

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

The **Final Value Theorem** (FVT) is one of several similar theorems used to relate frequency domain expression to the time domain behavior as time approaches infinity. Mathematically, if $f(t)$

in continuous time has Laplace transform $F(S)$, then a final value theorem establishes conditions under which

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

5.2 Initial value theorem

If $Z[f(n)] = F(z), then \ \lim\limits_{z\to\infty} F(z) = f(0)$

Proof :

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$= f(0) + f(1)\, z^{-1} + f(2)z^{-2} + \cdots$$

$$F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots$$

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots\right]$$

$$= f(0) \qquad \begin{bmatrix} Result: \\ \quad \lim_{z\to\infty} \frac{1}{z} = 0 \end{bmatrix}$$

Note:

If $Z[f(t)] = F(z)$

then $\lim_{z\to\infty} F(z) = f(0)$

(or)

$$\lim_{z\to\infty} F(z) = \lim_{t\to 0} f(t)$$

Example 1 :

$If\ F(z) = \frac{5z}{(z-2)(z-3)}\ find\ f(0).$

Solution:

By initial value theorem

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{5z}{(z-2)(z-3)}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}(5z)}{\frac{d}{dz}(z^2-5z+6)} \qquad [L'Hopital's\ Rule]$$

$$= \lim_{z\to\infty} \frac{5}{2z-5}$$

$$= 0 \qquad (\text{since } \frac{x}{\infty} = 0)$$

Example 2 :

$If\ F(z) = \frac{2z^2+5z+14}{(z-1)^4}\ find\ f(0)$

Solution:

By initial value theorem

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{2z^2+5z+14}{(z-1)^4}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}(2z^2+5z+14)}{\frac{d}{dz}[(z-1)^4]} \qquad [L'Hopital's\ Rule]$$

$$= \lim_{z\to\infty} \frac{4z+5}{4(z-1)^3}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}(4z+5)}{\frac{d}{dz}[4(z-1)^3]}$$

$$= \lim_{z\to\infty} \frac{4}{12(z-1)^2}$$

$$= 0$$

Example 3 :

If $\mathrm{F(z)} = \frac{\mathrm{z(z-cos\ aT)}}{\mathrm{z^2-2zcos\ aT + 1}}$ find f(0).

Solution:

By initial value theorem:

$$\mathrm{f(0)} = \lim_{\mathrm{z}\to\infty} \mathrm{F(z)}$$

$$= \lim_{\mathrm{z}\to\infty} \frac{\mathrm{z(z-cos\ aT)}}{\mathrm{z^2-2zcos\ aT + 1}}$$

$$= \lim_{\mathrm{z}\to\infty} \frac{\mathrm{z^2-zcos\ aT}}{\mathrm{z^2-2zcos\ aT + 1}}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}[z^2-z\cos at]}{\frac{d}{dz}[z^2-2z\cos at+1]} \qquad [L'Hopital's\ Rule]$$

$$= \lim_{z\to\infty} \frac{2z-\cos aT}{2z-2\cos aT}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}[2z-\cos aT]}{\frac{d}{dz}[2z-2cosaT]} \qquad [L'Hopital's\ Rule]$$

$$= \lim_{z\to\infty} \frac{2}{2}$$

f(0) = 1

5.3 Final value theorem

$$If \quad Z[f(t)] \ = F(Z)$$

$$then \quad \lim_{t\to\infty} f(t) \ = \ \lim_{z\to 1}\ (z-1)F(z)$$

Proof :

We know that

$$Z[f(t+T)] \ = z\,[F(z)-f(0)] \quad \text{(using Second Shifting Theorem)}$$

Now $\quad Z[f(t+T)-f(t)] = \sum_{n=0}^{\infty}\{f(nT+T)-f(nT)\}z^{-n}$

$$Z[f(t+T)]-Z[f(t)] \ = \ \sum_{n=0}^{\infty}\{f(nT+T)-f(nT)\}z^{-n}$$

$$zF(z)-zf(0)-F(z) \ = \ \sum_{n=0}^{\infty}\{f(nT+T)-f(nT)\}z^{-n}$$

$$\lim_{z\to 1}[(z-1)F(z)+f(0)] = \lim_{z\to 1}\sum_{n=0}^{\infty}\{f(nT+T)-f(nT)\}z^{-n}$$

$$= \ \sum_{n=0}^{\infty}\{f(nT+T)-f(nT)\}\ \lim_{z\to 1} z^{-n}$$

$$= \ \sum_{n=0}^{\infty}\{f(nT+T)-f(nT)\} \qquad [FORMULA\ \lim_{z\to 1} z^{-n} = 1]$$

$$= \lim_{n\to\infty}[f(T)-f(0)+f(2T)-f(T)$$

$$+f(3T)-f(2T)+\cdots+f(n+1)T-f(nT)]$$

$$= \lim_{n\to\infty}[\,f(n+1)T-f(0)]$$

$$= f(\infty)-f(0)$$

$$\lim_{z\to 1}\{(z-1)F(z)\} - \ f(0) = \ f(\infty)-f(0)$$

$$\lim_{z\to 1}(z-1)F(z) = f(\infty) = \ \lim_{t\to\infty} f(t)$$

Example 1 :

$If\ F(z) = \frac{5z}{(z-2)(z-3)}\ find \lim_{t\to\infty} f(t).$

Solution :

By final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)F(z)$$

$$= \lim_{z\to 1} (z-1)\frac{5z}{(z-2)(z-3)}$$

$$= \frac{(1-1)\times 5}{(1-2)(1-3)}$$

$$= 0$$

Example 2 : *Find the final value of the function*

$$F(z) = \frac{1+z^{-1}}{1-0.25\,z^{-2}}\,.$$

Solution :

By final value theorem,

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)\,F(z)$$

$$= \lim_{z\to 1} (z-1)\,.\frac{1+z^{-1}}{1-0.25\,z^{-2}}$$

$$= \lim_{z\to 1} (z-1)\;\frac{1+\frac{1}{z}}{1-\frac{0.25}{z^2}}$$

$$= \frac{(1-1)\left(1+\frac{1}{1}\right)}{1-\frac{0.25}{1}} = 0$$

$$f(\infty) = 0$$

Example 3 :

If $F(z) = \frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$ *find* $\lim_{t\to\infty} f(t)$.

Solution:

By final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)F(z)$$

$$= \lim_{z\to 1} (z-1)\frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$$

$$= \lim_{z \to 1} \frac{(z-1)\,(z^2 - z\cos aT)}{z^2 - 2z\cos aT + 1}$$

$$= 0.$$

Table of Z-Transforms

S.No	f_n	$Z[f_n] = F(z)$
1.	1	$\frac{z}{z-1}$
2.	$(-1)^n$	$\frac{z}{z+1}$
3.	n	$\frac{z}{(z-1)^2}$
4.	a^n	$\frac{z}{z-a}$
5.	n^2	$\frac{z(z+1)}{(z-1)^3}$
6.	na^n	$\frac{az}{(z-a)^2}$
7.	$\cos n\theta$	$\frac{z(z-cos\theta)}{z^2-2zcos\theta+1}$
8.	$\sin n\theta$	$\frac{zsin\theta}{z^2-2zcos\theta+1}$
9.	$a^n cos\, n\theta$	$\frac{z(z-acos\theta)}{z^2-2azcos\theta+a^2}$
10.	$a^n sin\, n\theta$	$\frac{azsin\theta}{z^2-2azcos\theta+a^2}$

Conclusion

The Z-Transform proves a useful, more general form of the Discrete Time Fourier Transform. It applies equally well to describing systems as well as signals using the eigen function method, and proves extremely useful in digital filter design. For finding the inverse of Z-Transform Three methods are used. It’s an efficient method for solving linear difference equations with constant coefficients and Volterra equations of convolution type. Also, it has many important applications in digital signal processing as analysis of linear shift-invariant systems, implementation of Finite Impulse Response (FIR) and (IIR filters and design of IIR filters from analog filters. Chirp Z- transform algorithm is an important algorithm that overcomes the limits of fast Fourier transform and has many applications such as enhancement of poles and high resolution, narrow band frequency analysis.

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A STUDY ON VITAL STATISTICS

Project report submitted to

ST. MARY'S COLLEGE (AUTONOMOUS), THOOTHUKUDI

Affiliated to

MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI

In partial fulfilment of the requirement for the award of degree of

Bachelor of Science in Mathematics

Submitted by

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ST. MARY'S COLLEGE (Autonomous), THOOTHUKUDI.

(2022– 2023)

CERTIFICATE

We hereby declare that the project report entitled **"A STUDY ON VITAL STATISTICS"** being submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of degree of **Bachelor of Science in Mathematics** and it is a record of work done during the year 2022 -2023 by the following students:

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DECLARATION

We hereby declare that the project reported entitled "A STUDY ON VITAL STATISTICS", is our original work. It has not been submitted to any university for any degree or diploma.

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ACKNOWLEDGEMENT

First of all, we thank the Almighty for showering his blessing to undergo this project.

With immense pleasure, we register our deep sense of gratitude to our guide

Dr. G. Priscilla Pacifica M.Sc., B.Ed., M. Phill., Ph.D., SET. and the Head of the Department, **Dr. V. L. Stella Arputha Mary M.Sc., B.Ed., M. Phill., Ph.D.** for having imparted necessary guidelines throughout the period of our studies.

We thank our beloved Principal, **Rev. Dr. Sr. A.S.J. Lucia Rose M.Sc., M.Phil., Ph.D., PGDCA.,** for Providing us to help to carry out our project work successfully.

Finally, we thank all those who extended their helping hands regarding this project.

A STUDY ON

VITAL STATISTICS

CONTENTS

INTRODUCTION

Statistics is a branch of applied mathematics that involves the collection, description, analysis, and inference of conclusions from quantitative data. Statistics are used in virtually all scientific disciplines such as the physical and social sciences, as well as in business, the humanities, government, and manufacturing.

Demography is the science, which deals with human populations.The study of demography also involves dynamic processes of the population such as fertility, mortality, marriage and migration. There are two main aspects of the behaviour of any population: the composition of the aggregate and the changes that occur during some kind of observation. The composition of a population is described by the distribution of people among certain more or less standard categories. Changes on the other hand, are the result of "events" which either add some members in the existing population or take away some members of the population. Such events which add members in the population is mainly births, but marriages, migrations and social mobility also add members whereas mainly through deaths, members are taken away from the population. However, migrations, marriages are also responsible for decrease in the population size. For this reason, births and deaths are called "vital events".

Vital Statistics is a branch of Demography, which is the science applied to the analysis and interpretation of numerical facts regarding vital events occurring in human population such as births, deaths, marriages, divorces, migration *etc*. The most common way of collecting information on these events is through civil registration, an administrative system used by governments to record vital events which occur in their populations. In this project we see about the statistical form of vital events such as fertility rate, reproduction rate and mortality rate.

PRELIMINARIES

STATISTICS:

Statistics is the science of measurement of social organisms, regarded as a whole in all its manifestation - Bowley

VITAL STATISTICS:

Vital Statistics is a branch of Demography, which is the science applied to the analysis and interpretation of numerical facts regarding vital events occurring in human population such as births, deaths, marriages, divorces, migration *etc*.

BIRTH RATE AND DEATH RATE:

- Birth rate refers to the number of births that occur per thousand people, per year.

- Death rate refers to the number of deaths per thousand people in the population of a country over a period of time. It is also known as mortality rate, to reflect the fact of mortality of human beings.

- Births and deaths continue to take place at all places around the world. The difference between total births and total deaths in a population over a period of time decides whether the population growth of the place is positive or negative.

- If the birth rate is higher than the death rate in a country, it means that the population will register a growth. On the other hand, birth rate less than the death rate indicates a shrinking population.

FERTILITY RATE:

Fertility Rate is defined as the average number of children that would be born to a woman over her lifetime if the woman were to survive from birth through the end of her reproductive life. It is expressed as children per woman.

FERTILITY AND FECUNDITY:

Fertility refers to the births occurring to women of child bearing age (15 – 49 years)

Fecundity refers to the capacity of a woman to bear children

REPRODUCTION RATE:

The reproduction rate measures the conditions under which generations are replaced. This calculation can be made by taking account of mortality (net reproduction rate) or without mortality (gross reproduction rate).

INFANT MORTALITY RATE:

Infant mortality rates serve as one of the best indices to the general "healthiness" of a society. It is similar to age specific death rate for infants under 1 year of age. It is defined as:

$$\text{Infant mortality rate} = \frac{\begin{array}{c}\text{Number of deaths under 1 year of age which}\\ \text{occured among the population of a given}\\ \text{geographic area during a given year}\end{array}}{\begin{array}{c}\text{Number of live births which occurred}\\ \text{among the population of the given}\\ \text{geographic area during the same year}\end{array}} \times 1000$$

Still births are not included in the infant deaths. The infant mortality rate varies considerably according to time and place.

NEO-NATAL MORTALITY RATE:

The neo-natal mortality rate, like the infant mortality rate, is similar to an age specific rate. It is a rate used to measure the risk of death during the first month of life. This rate is defined as:

$$\text{Annual Neo- natal Mortality rate} = \frac{\begin{array}{c}\text{Annual deaths of infants under the age}\\ \text{of 1 month among the population of a}\\ \text{given geographic area}\end{array}}{\begin{array}{c}\text{Number of live births which occurred}\\ \text{among the population of a given geographic}\\ \text{area during the same year}\end{array}}$$

MATERNAL MORTALITY RATE:

The risk of dying from causes associated with child-birth is measured by the maternal mortality rate. For this purpose the deaths used in the numerator are those arising from puerperal causes. i.e., deliveries and complications of pregnancy, child-birth and the puerperium. The numbers exposed to the risk of dying from puerperal causes are women have been pregnant during the period. Their number being unknown the number of live births is used as the conventional base for computing comparable maternal mortality rates.

The formula is:

$$\text{Annual maternal mortality} = \frac{\begin{array}{c}\text{Number of deaths from puerperal causes}\\ \text{occurred among the female population of a}\\ \text{given geographic area during a given year}\end{array}}{\begin{array}{c}\text{Number of live births which occurred}\\ \text{among the population of the given geographic}\\ \text{area during the same year}\end{array}}$$

NATURAL INCREASE RATE:

Rates of natural increase or decrease, that is, rates computed on the balance of births and deaths, give some measure of the overall gain or loss in a population through the addition of the births and the subtraction of deaths.

The annual rate of a natural increase can be computed simply by subtracting the crude death rate from the crude birth rate. The formula for such a rate is as follows:

$$\text{Annual natural increase rate} = \frac{\begin{array}{c}\text{Number of births which occurred among}\\ \text{a population of a given geographic area during}\\ \text{a given year minus the corresponding}\\ \text{number of deaths}\end{array}}{\begin{array}{c}\text{Mid}-\text{year population of the given geographic}\\ \text{area during the same year}\end{array}}$$

NET MIGRATION RATE:

The annual rate of net migration is defined:

$$\text{Net Migration Rate} = \frac{\text{Annual net migration}}{\text{Annual mean population}} \times 100$$

$$= \frac{\text{Overseas arrivals} - \text{Overseas departures}}{\text{Annual mean population}} \times 1000$$

The rate of net migration for a given year tells us at what rate net migration has augmented the population over the course of the year.

VITAL INDEX:

Vital Index is the index of two vital statistics, i.e., births and deaths. Generally, it is more than one though for a specific region or of period it may even be less than 1. It throws light on the likely growth population. The Index is obtained as follows:

$$\text{Vital Index} = \frac{\text{Total live births}}{\text{Total deaths}}$$

It may be noted that the index is actually a short-term measure of the registration of vital statistics indicating the degree of error in registration of either the births or the deaths or both.

COHORT:

Cohort means a group of individuals who are born at the same time and who experience the same mortality conditions.

CENSUS IN STATISTICS:

A census is a survey that studies every member of a population. It doesn't have to be "official."

SOMEWHAT RESTRICTED DEFINITIONS OF VITAL STATISTICS ARE:

"Vital Statistics are conventionally numerical records of marriages births, sickness and deaths by which the health and growth of a community may be studied". -B. Benjamin

"That branch of biometry which deals with data and the laws of human mortality, morbidity and demography". -Arthur Newsholme

"It deals with mankind in the aggregate. It is the science of numbers applied to the life history of communities and nation". - Arthur Newsholme

1. MEASUREMENT OF FERTILITY

Introduction

In order to study the speed at which the population is increasing, fertility rates are used which are of various types. Important among these are

1.Crude birth rate,

2.Specific fertility rate,

3.General fertility rate,

4.Total fertility rate.

§ 1.1 Crude Birth Rate

It is the simplest method of measuring fertility. It acts as an index of the relative speed at which additions are being made to the population through child-birth. In this method the number of births are related to the total population.

The annual crude birth rate is defined as:

$$\text{Crude birth rate} = \frac{\text{Annual births}}{\text{Annual mean population}} \times 1000$$

§ 1.2 Specific Fertility Rate

The concept of specific fertility arises out of the fact that fertility is affected by several factors such as age, marriage, State or region, urban-rural characteristics, etc.The fertility of women differs from age to age and, therefore, the grouping of women of different ages is essential.

The specific fertility rate is defined as:

$$\text{S.F.R.} = \frac{\begin{array}{c}\text{Number of live births which occurred to females}\\ \text{of a specified agegroup of the population of a}\\ \text{given geographic area during a given year}\end{array}}{\begin{array}{c}\text{Mid year female population of the specified}\\ \text{age group in the given geographic area during}\\ \text{the same year}\end{array}} \times 1000$$

§ 1.3 General Fertility Rate

This rate refers to the proportion of the number of children born per 1,000 of females, the reproductive or child-bearing age. Thus, the numerator of this rate would remain the same as the crude rate, but the denominator would be limited to the age-sex group of population able to contribute to the birth rate.

The formula for general fertility rate is

$$\text{G. F. R.} = \frac{\begin{array}{c}\text{Number of live births which occurred}\\ \text{among the population of a given geographic area}\\ \text{during a given year}\end{array}}{\begin{array}{c}\text{Mid year female population of ages 15 to 49}\\ \text{in the given geographic area during the same year}\end{array}} \times 1000$$

§ 1.4 Total Fertility Rate

Total fertility rate is the sum of the age-specific fertility rates from a given age to the last point of child-bearing age of a female. In practice, we can shorten this procedure by working in quinquennial age groups. We define the specific fertility rate for group x years and under (x + 5) as:

$$\text{Specific Fertility Rate} = \frac{\begin{array}{c}\text{Annual births to females aged x}\\ \text{and under (x + 5)}\end{array}}{\begin{array}{c}\text{Mean number of females aged x}\\ \text{and under (x + 5)}\end{array}} \times 1000$$

Such a specific fertility rate is the rate per 1,000 per annum at which the females in the age-group produce off spring. If we add the quinquennial specific fertility rates and multiply by 5, we shall have the total number of children which 1,000 females aged 15 will bear over their lifetimes. A calculation based on quinquennial age-groups involves only one-fifth of the arithmetic of one based on single age groups and is very nearly as accurate,

Symbolically,

$$\text{T. F. R.} = \sum \text{S. F. R} \times t$$

where t= the magnitude of the age class.

§ 1.5 Illustrations:

Illustration 1.5.1

Compute the specific fertility rate, general fertility rate and total fertility rate from the data given below:

Age groups (years)	No. of women ('000)	No. of live births
15-19	25	800
20-24	20	2400
25-29	18	2000
30-34	15	1500
35-39	12	500
40-44	6	120
45-49	4	10
Total	100	7330

Solution.

Computation of specific fertility rate:

$$\text{S.F.R.}\,(\text{ for age } 15-19 \text{ years}) = \frac{800}{25000} \times 1000$$

$$= 32.00$$

$$\text{S.F.R.}\,(\text{ for age } 20-24 \text{ years}) = \frac{2400}{20000} \times 1000$$

$$= 120.00$$

$$\text{S.F.R.}\,(\text{ for age } 25-29 \text{ years}) = \frac{2000}{18000} \times 1000$$

$$= 111.11$$

$$\text{S. F. R. (for age } 30-34 \text{ years)} = \frac{1500}{15000} \times 1000$$

$$= 100.00$$

$$\text{S. F. R. (for age } 35-39 \text{ years)} = \frac{500}{12000} \times 1000$$

$$= 41.67$$

$$\text{S. F. R. (for age } 40-44 \text{ years)} = \frac{120}{6000} \times 1000$$

$$= 20.00$$

$$\text{S. F. R. (for age } 45-49 \text{ years)} = \frac{10}{6000} \times 1000$$

$$= 1.67$$

$$\text{G. F. R.} = \frac{\text{No. of live births}}{\text{No. of women of } 15-49 \text{ years}} \times 1000$$

$$= \frac{7330}{100000} \times 1000$$

$$\text{G. F. R.} = 73.3$$

It is clear from above that for 15-19 age group S.F.R. is 32. Accordingly, 1,000 females exactly aged 15 would by the time they reached 20 have borne 32 x 5 = 160 children. It is necessary to multiply by 5 since the specific fertility rate is a rate per annum and by the time the females reach the age of 20 they will have spent 5 years in the age group 15-19.

In the table below is shown the number of births which 1,000 females will have borne by the time they reach certain ages.

Exact age (years)	S.F.R. × 5	Total births per 1000 females aged 15 by stated ages
15	0	0
20	32 x 5 = 160.00	160.00
25	120 x 5 = 600.00	760.00
30	111:11 x 5 = 555.55	1315.55
35	100 x 5 = 500.00	1815.55
40	41.67 x 5 = 208.35	2023.90
45	20 x 5 = 100.00	2123.90
50	1.67 x 5 = 8.35	2132.25

$$\text{Total fertility rate} = \frac{\sum \text{S.F.R}}{1000}$$

$$= \frac{2132.25}{1000}$$

$$= 2.13225$$

T.F.R = 2.13225

Illustration 1.5.2

From the data given below calculate the General Fertility Rate and Total Fertility Rate:

Age group	No. of women	Specific fertility rate (per 1000)
15-20	100	15
20-25	120	100

25-30	110	120
30-35	105	140
35-40	100	80
40-45	80	50
45-50	70	10

Solution.

Age group	No. of women	Specific fertility rate (per1000)	No. of children born
15-20	100	15	$\frac{100 \times 15}{1000} = 1.5$
20-25	120	100	$\frac{120 \times 100}{1000} = 12.0$
25-30	110	120	$\frac{110 \times 120}{1000} = 13.2$
30-35	105	140	$\frac{105 \times 140}{1000} = 14.7$
35-40	100	80	$\frac{100 \times 80}{1000} = 8.0$
40-45	80	50	$\frac{80 \times 50}{1000} = 4.0$
45-50	70	10	$\frac{70 \times 10}{1000} = 0.7$
Total	685	515	54.1

$$\text{G. F. R.} = \frac{\text{Total No. of children born}}{\text{Total No. of womens}} \times 1000$$

$$= \frac{54.1}{685} \times 1000$$

G.F.R = 78.98 per thousand

$$\text{Total Fertility Rate (T. F. R.)} = \sum \text{S. F. R} \times t$$

$$= 515 \times 5$$

T.F.R. = 2575 per thousand

Illustration 1.5.3

Compute General Fertility Rate and Total Fertility Rate from the given data below

Age (years)	Number of woman ('000)	Total births
15-19	16.0	260
20-24	16.4	2244
25-29	15.8	1894
30-34	15.2	1320
35-39	14.8	916
40-44	15.0	280
45-49	14.5	145

Solution

$$\text{G. F. R.} = \frac{\text{No.of births}}{\text{No.of women of } 15-19 \text{ years}} \times 1000$$

Total births = 260 + 2244 + 1894 + 1320 + 910 + 280 + 145

Total births = 7059

No. of women = (16.0 + 16.4 + 15.8 + 15.2 + 14.8 +15.0 + 14.5) × 1000

=107.7 × 1000

No. of women = 107700

$$\text{G. F. R} = \frac{7059}{107700} \times 1000$$

$$G.F.R = 65.5$$

$$\text{Total fertility rate} = \frac{\sum \text{S. F. R} \times 5}{1000}$$

$$\text{S. F. R. (for age } 15-19 \text{ years)} = \frac{260}{16000} \times 1000$$

$$= 16.25$$

$$\text{S. F. R. (for age } 20-24 \text{ years)} = \frac{2244}{16400} \times 1000$$

$$= 136.83$$

$$\text{S. F. R. (for age } 25-29 \text{ years)} = \frac{1894}{15800} \times 1000$$

$$= 119.87$$

$$\text{S. F. R. (for age } 30-34 \text{ years)} = \frac{1320}{15200} \times 1000$$

$$= 86.84$$

$$\text{S. F. R. (for age } 35-39 \text{ years)} = \frac{916}{14800} \times 1000$$

$$= 61.89$$

$$\text{S.F.R.}(\text{ for age } 40-44 \text{ years}) = \frac{280}{15000} \times 1000$$

$$= 18.67$$

$$\text{S.F.R.}(\text{ for age } 45-49 \text{ years}) = \frac{145}{14500} \times 1000$$

$$= 10.00$$

$$\sum \text{S.F.R} = 450.35$$

$$\text{Total fertility rate} = \frac{\sum \text{S.F.R} \times 5}{1000}$$

$$\text{T.F.R.} = \frac{450.35 \times 5}{1000}$$

T.F.R. = 2.25175

2. REPRODUCTION RATES

Introduction:

The fertility rates are unsuitable for giving an idea of rate of population growth because they ignore the sex of the newly born children and their morality. If the majority of births are those of boys the population is bound to decrease while the reverse will be the case if the majority of births are girls. Similarly, if mortality is ignored a correct idea of the rate of growth of population cannot be formed because it is possible a number of female children may die before reaching the child-bearing age. For measuring the rate of growth of population we calculate the Reproduction rates.

§ 2.1 Types of Reproduction Rate:

Reproduction rates are of two types:

- ❖ Gross Reproduction Rate, and
- ❖ Net Reproduction Rate

2.1.1 Gross Reproduction Rate (GRR):

Under this total fertility rates given above are corrected by the ratio of female births to the total births. Thus

$$\text{GRR} = \frac{\text{No.of female births}}{\text{Total births}} \times \text{Total fertility rate}$$

(Sum of age -specific fertility rates can also be substituted for total fertility rate)

If the gross reproduction rate of a population is exactly 1. It indicates that the sex under consideration is exactly replacing itself; if it is less than 1, the population would decline, no matter how low the death rate may be and if it is more than 1, the population would increase, no matter how low the death rate may be.

$$\text{Also, G.R.R} = \frac{\text{No of female children born to 1000 women}}{1000}$$

2.1.2 Net Reproduction Rate (NRR):

Under this measure, the total fertility is corrected for the rate of survival of female children in child bearing age as follows:

$$\text{N.R.R.} = \frac{\sum(\text{No.of female births} \times \text{Survival Rate})}{100}$$

NRR can never exceed GRR. If NRR =1, we may conclude that if the current fertility and female morality rate prevail in future, a group of new born girls will exactly replace itself in the next generation.

§ 2.2 Illustrations:

Illustration 2.2.1 Calculate the gross and net reproduction rates from the data given below

Number of women in age groups and number of female children born in one year.

Age group	Female population ('000)	Female births	Survival rate
15-19	1,600	19,000	0.921
20-24	1,000	70,200	0.901
25-29	1,685	90,600	0.885
30-34	1,730	62,400	0.862
35-39	1,725	32,500	0.850
40-44	1,620	11,000	0.832
45-49	1,510	800	0.812

Solution:

$$\text{S.F.R. per women} = \frac{\text{Female births}}{\text{Female population}}$$

The calculations are shown below:

Age group	Female population	Female births	Specific fertility rate per women S.F.R.
15-19	16,00,000	19,000	$\frac{19,000}{16,00,000} = 0.0119$
20-24	10,00,000	70,200	$\frac{70,200}{10,00,000} = 0.0702$
25-29	16,85,000	90,600	$\frac{90,600}{16,85,000} = 0.0538$
30-34	17,30,000	62,400	$\frac{62,400}{17,30,000} = 0.0361$
35-39	17,25,000	32,500	$\frac{32,500}{17,25,000} = 0.0188$
40-44	16,20,000	11,000	$\frac{11,000}{16,20,000} = 0.0068$
45-49	15,10,000	800	$\frac{800}{15,10,000} = 0.0005$
Total			$\sum$ S.F.R = 0.1981

$$\text{G.R.R} = \sum [\text{S.F.R}] \times 5$$

$$= 0.1981$$

$$= 0.9905$$

Net reproduction rate:

Net reproduction rate is the number of female children surviving till their reproductive ages born to one woman as she passes through child bearing age. Thus, it is the sum of the specific fertility rate per women for all ages ×Survival rates. For data expressed in 5-yearly age group, it is the sum of the specific fertility rate per women for various group × Survival rates × 5.

The calculations are shown below:

Age group	Female population	Female births	Specific fertility rate per women S.F.R	Survival rate S	S.F.R X S
15-19	16,00,000	19,000	0.0119	0.921	0.0110
20-24	10,00,000	70,200	0.0702	0.901	0.0633
25-29	16,85,000	90,600	0.0538	0.885	0.0476
30-34	17,30,000	62,400	0.0361	0.862	0.0311
35-39	17,25,000	32,500	0.0188	0.850	0.0160
40-44	16,20,000	11,000	0.0068	0.832	0.0057
45-49	15,10,000	800	0.0005	0.812	0.0004
				$\sum [S.F.R \times S]$ =	0.1751

$$N.R.R = \sum [S.F.R. \times S] \times 5$$

$$= 0.1751 \times 5$$

$$= 0.8755$$

Illustration 2.2.2

From the following table calculate the female gross reproduction rate if the ratio of male and female children be 48:52

Age group	No. of children born to 1,000 women	Age group	No. of children born to 1,000 women
15-19	50	35-39	80
20-24	180	40-44	40
25-29	200	45-49	10
30-34	150		

Solution: CALCULATION OF GROSS REPRODUCTION RATE

Age group	No. of children born to 1,000 women	No. of female children born
15-19	50	$\frac{50 \times 52}{100} = 26.0$
20-24	180	$\frac{180 \times 52}{100} = 93.6$
25-29	200	$\frac{200 \times 52}{100} = 104.0$
30-34	150	$\frac{150 \times 52}{100} = 78.0$
35-39	80	$\frac{80 \times 52}{100} 41.6$
40-44	40	$\frac{40 \times 52}{100} = 20.8$
45-49	10	$\frac{10 \times 52}{100} = 5.2$
Total	710	369.2

$$\text{G.R.R} = \frac{Total\ births\ to\ 1{,}000\ women}{1{,}000}$$

$$= \frac{369.2}{1{,}000}$$

= 0.3692 per woman

Illustration 2.2.3

Calculate the net reproduction rates from the data given below :

Age group of child-bearing females	No. of children born to 1,000 women passing through each age group	No. of survivors out of each 1,000 female children
15-20	50	850
20-25	180	800
25-30	450	750
30-35	500	700
35-40	300	650
40-45	100	600
45-50	40	500

Solution:

CALCULATION OF NET REPRODUCTION RATE

Age group	No. of children born to 1,000 women	No. of survivors	No. of survivors which replaced present women
15-20	50	850	$\frac{50 \times 850}{1,000} = 42.5$
20-25	180	800	$\frac{180 \times 800}{1,000} = 144.0$
25-30	450	750	$\frac{450 \times 750}{1,000} = 337.5$
30-35	500	700	$\frac{500 \times 700}{1,000} = 350.0$
35-40	300	650	$\frac{300 \times 650}{1,000} = 195.0$

40-45	100	600	$\frac{100 \times 600}{1,000} = 60.0$
45-50	40	500	$\frac{40 \times 500}{1,000} = 20.0$

$$\text{N.R.R} = \frac{\sum(\text{No.of female births} \times \text{Survival Rate})}{1,000}$$

$$= \frac{1149.0}{1,000}$$

$$\text{N.R.R} = 1.149.$$

Illustration 2.2.4

From the following data calculate the gross reproduction rate and net reproduction rates:

Age group	No. of children born to 1,000 women passing through each age group	Mortality rate
16-20	150	120
21-25	1,500	180
26-30	2,000	150
31-35	800	200
36-40	500	220
41-45	200	230
46-50	100	250

Sex ratio being males: females 52:48

Solution: CALCULATION OF G.R.R. and N.R.R.

Age group	No. of children born to 1,000 women passing through the age group	No. of female children (48%)	Survival rate (1,000-Mortality rate)	No. of female children survived
16-20	150	$\frac{150 \times 48}{100} = 72$	880	$\frac{72 \times 880}{1,000}$ $= 63.36$
21-25	1,500	$\frac{1,500 \times 48}{100} = 720$	820	$\frac{720 \times 820}{1,000}$ $= 590.40$
26-30	2,000	$\frac{2,000 \times 48}{100} = 960$	850	$\frac{960 \times 850}{1,000}$ $= 816.00$
31-35	800	$\frac{800 \times 48}{100} = 384$	800	$\frac{384 \times 800}{1,000}$ $= 307.20$
36-40	500	$\frac{500 \times 48}{100} = 240$	780	$\frac{240 \times 780}{1,000}$ $= 187.20$
41-45	200	$\frac{200 \times 48}{100} = 96$	770	$\frac{96 \times 770}{1,000}$ $= 73.92$
46-50	100	$\frac{100 \times 48}{100} = 48$	750	$\frac{48 \times 750}{1,000}$ $= 36.00$
Total		= 2,520		=2,074.08

$$\text{G.R.R} = \frac{\text{Total No.of female children born}}{1{,}000}$$

$$= \frac{2{,}520}{1{,}000}$$

$$= 2.52 \text{ per woman}$$

$$\text{N.R.R} = \frac{\text{No.of female children born and survived to 1,000 woman}}{1{,}000}$$

$$= \frac{2{,}074.08}{1{,}000}$$

$$= 2.074 \text{ per woman}$$

Therefore,

G.R.R = 2.52 per woman and

N.R.R. = 2.074 per woman

Illustration 2.2.5 Compute general fertility rate and gross reproduction rate from the data given below:

Age group of bearing females	15-19	20-24	25-29	30-34	35-39	40-44	45-49
No. of women ('000)	16.0	16.4	15.8	15.2	14.8	15.0	14.5
Total births	260	2244	1894	1320	916	280	145

Assume that the proportion of female birth is 46.2 per cent.

Solution:

$$\text{G.F.R.} = \frac{\text{No. of births}}{\text{No. of women of } 15-49 \text{ years}} \times 1000$$

Total births = 7,059,

No. of women = 1,07,700

G.F.R. = $\frac{7{,}059}{1{,}07{,}700} \times 1000 = 65.5$

G.R.R. = $\frac{\text{No. of female births}}{\text{Total births}} \times \text{Total fertility rate}$

$$Total\ fertility\ rate = \frac{\sum S.F.R.\ \times\ 5}{1000}$$

S.F.R.(for age 15-19 years) = $\frac{260}{16000} \times 1000$

= 16.25

S.F.R.(for age 20-24 years) = $\frac{2244}{16400} \times 1000$

= 136.83

S.F.R.(for age 25-29 years) = $\frac{1894}{15800} \times 1000$

= 119.87

S.F.R.(for age 30-34 years) = $\frac{1320}{15200} \times 1000$

= 86.84

S.F.R.(for age 35-39 years) = $\frac{916}{14800} \times 1000$

= 61.89

S.F.R.(for age 40-44 years) = $\frac{280}{15000} \times 1000$

= 18.67

S.F.R.(for age 45-49 years) = $\frac{145}{14500} \times 1000$

= 10.00

$\sum S.F.R. \times 5 = 450.35 \times 5$

$= 2251.75$

T.F.R. $= \frac{2251.75}{1,000}$

$= 2.25175$ or 2.252

No. of. Female births = 7059×0.462

$=3261.26$

G.R.R. $= \frac{3261.26}{7059} \times 2.252$

$= 1.04$

3. MEASUREMENT OF MORTALITY

Introduction:

Mortality rate refers to various types of death rates. In this chapter, we deal with crude death rate, specific death rate and standardized death rate.

§ 3.1 Crude Death Rate:

This is the simplest measure of mortality and is defined as the number of deaths from all causes in a given period per thousand in a given community or region.

$$\text{CDR} = \frac{\begin{array}{c}\text{No.of deaths in a given region}\\ \text{during a given period}\end{array}}{\text{Total population}} \times 1000$$

$$= \frac{\Sigma D_x}{P} \times 1000$$

Since the entire population of a region is exposed to risk of mortality, CDR is the probability that a person belonging to the given population will die in the given period.

§ 3.2 Specific Death Rate:

Death rate computed for a particular specific section of the population is termed as specific death rate (SDR). This rate for a given region during a given period is defined as

$$\text{SDR} = \left[\frac{\text{Total number of deaths in a specific group}}{\text{Total population of the group}}\right] \times 1000$$

§ 3.3 Standardized death rate:

The crude death rate is not suitable for comparing the mortality prevailing in different regions because the composition of different age groups may differ markedly. Although specific death rates may be used to compare the mortality for two communities or places at different ages, they cannot provide a comparison of the overall mortality for these two populations, since the specific death rates for one community may be higher at certain ages and lower at other ages compared with a second community. So, it is necessary to obtained a single figure which is a weighted average of age specific death rates for each community, so that the figures are directly comparable. The process so adopted is called standardized or correction of CDR and the resulting death rate is known as standardized or corrected death rate.

For computing the standardized death rate, age group composition of any one of the populations (usually population of a larger community) is taken as 'standard population'. Then the standardized death rate of any population is defined as the weighted arithmetic mean of the age specific death rates of the population, using the figures of a given 'standard population' as weights. Thus

$$\text{Standardized Death Rate} = \frac{\Sigma p_x^s\, m_x}{\Sigma p_x^s} \times 1000$$

Where p_x^s = standard population at age x

m_x = specific death rate of community at age x.

§ 3.4 Illustrations

Illustration 3.4.1:

Compute the crude and standardized death rates of the two populations A and B from the following data:

Age-group (years)	A		B	
	Population	Deaths	Population	Deaths
Below 5	15,000	360	40,000	1,000
5-30	20,000	400	52,000	1,040
Above 30	10,000	280	8,000	240
Total	45,000	1,040	1,00,000	2,280

Solution:

$$\text{Crude Death Rate} = \frac{N}{P} \times 1000\ ,$$

where

N = No. of deaths,

P = Populations

C.D.R. for town A = $\frac{1,040}{45,000} \times 1000$

$= 23.11$

C.D.R. for town B = $\frac{2,280}{1,00,000} \times 1000$

$= 22.80$

Standardized death rate taking population of town A as standard population

Age-group (years)	A			B		
	Population	Deaths	Death Rate per thousand	Population	Deaths	Death Rate per thousand
Below 5 5-30 Above 30	15,000 20,000 10,000	360 400 280	24 20 28	40,000 52,000 8,000	1,000 1,040 240	25 20 30
Total	45,000	1,040		1,00,000	2,280	

Standardized Death Rate (town A)

$$= \frac{(15000 \times 24) + (20000 \times 20) + (10000 \times 28)}{15000 + 20000 + 10000}$$

$$= \frac{360000 + 400000 + 280000}{45000}$$

$$= \frac{1040000}{45000}$$

= 23.11

Standardized Death Rate (town B)

$$= \frac{(15000\times25)+(20000\times20)+(10000\times30)}{15000+20000+10000}$$

$$= \frac{375000+400000+300000}{45000}$$

$$= \frac{1075000}{45000}$$

$$= 23.89$$

We can now say that the death rate in town B is higher than in town A.

Illustration 3.4.2:

Compute the crude and standardized death rates of the two cities from the following data and find out which population is healthier:

Age	City A		City B	
	Population	Deaths	Population	Deaths
Under 5	16,000	176	5,000	130
5-40	50,000	250	27,000	162
40-75	1,20,000	840	62,000	527
Above 75	14,000	910	6,000	420

Solution:

COMPUTATION OF STANDARDIZED DEATH RATE

Age	City A			City B		
	Population	Deaths	Death Rate	Population	Deaths	Death Rate
Under 5	16,000	176	11	5,000	130	26
5-40	50,000	250	5	27,000	162	6

40-75 Above 75	1,20,000 14,000	840 910	7 65	62,000 6,000	527 420	8.5 70
	2,00,000	2,176		1,00,000	1,239	

C.D.R of locality A $= \frac{N}{P} \times 1000$

$= \frac{2{,}176}{2{,}00{,}000} \times 1000$

$= 10.88$

This is also the standardized death rate of population A as population A has been taken as standard.

S.D.R of locality B $= \frac{(16000 \times 26)+(50000 \times 6)+(120000 \times 8.5)+(14000 \times 70)}{200000}$

$= \frac{416000+300000+1020000+980000}{200000}$

$= \frac{2716000}{200000}$

$= 13.58$

Since the standardized death rate of Locality A is less than Locality B it can be concluded that Locality A is healthier.

Illustration 3.4.3: Estimate the standardized death rates from the following data:

Age group (in years)	Death rates per 1000		Standardized population (in lakh)
	Country I	Country II	
Below 5	20.00	5.0	100
5-14	1.00	0.5	200
15-24	1.40	1.0	190
25-34	2.00	1.0	180
35-44	3.30	2.0	120
45-54	7.00	5.0	100

55-64	15.00	12.0	70
65-74	40.00	35.0	30
75 and above	120.00	110.0	10

Solution:

Age group (in years)	Death rates per 1000		Standardized population W	WX_1	WX_2
	Country I X_1	Country II X_2			
Below 5	20.00	5.0	100	2,000	500
5-14	1.00	0.5	200	200	100
15-24	1.40	1.0	190	266	190
25-34	2.00	1.0	180	360	180
35-44	3.30	2.0	120	396	240
45-54	7.00	5.0	100	700	500
55-64	15.00	12.0	70	1,050	840
65-74	40.00	35.0	30	1,200	1,050
75 and above	120.00	110.0	10	1,200	1,100

$\sum W = 1000$ $\sum WX_1 = 7372$ $\sum WX_2 = 4,700$

(Country I): S.D.R. $= \frac{\sum WX_1}{\sum W}$

$= \frac{7372}{1000}$

$= 7.372$;

(Country II): S.D.R. $= \frac{\sum WX_2}{\sum W}$

$= \frac{4700}{1000}$

$= 4.7$

Illustration 3.4.4:

From given data, make a comparative study of crude and standardized death rates for town A and town B.

Age group	Town A		Town B		Standard Population (in '000)
	Population (in '000)	Specific Death Rate	Population (in '000)	Specific Death Rate	
0-5	6	40	7	40	4
5-25	30	18	40	18	34
25-45	24	12	20	13	25
45-60	12	25	8	20	11
Above 60	8	34	5	44	6

Solution:

COMPUTATION OF CDR

Age Group	Town A			Town B		
	Population P_x^a	Specific Rate m_x^a	No. of Deaths D_x^a	Population P_x^b	Specific Rate m_x^b	No. of Deaths D_x^b
0-5	6,000	40	240	7,000	40	280
5-25	30,000	18	540	40,000	18	720
25-45	24,000	12	288	20,000	13	260
45-60	12,000	25	300	8,000	20	160
Above 60	8,000	34	272	5,000	44	220
Total	80,000		1,640	80,000		1,640

CDR for town A $= \frac{1,640}{80,000} \times 1,000$

$= 20.5$

CDR for town B = $\frac{1,640}{80,000} \times 1,000$

= 20.5

COMPUTATION OF STANDARDIZED DEATH RATE (SDR)

Standard Population P_x^s	Town A		Town B	
	Specific Death Rate m_x^a	$P_x^s \times m_x^a$	Specific Death Rate m_x^b	$P_x^s \times m_x^b$
4,000	40	1,60,000	40	1,60,000
34,000	18	6,12,000	18	6,12,000
25,000	12	3,00,000	13	3,25,000
11,000	25	2,75,000	20	2,20,000
6,000	34	2,04,000	44	2,64,000
80,000		15,51,000		15,81,000

SDR for town A =$\frac{15,51,000}{80,000}$

= 19.39

SDR for town B = $\frac{15,51,000}{80,000}$

= 19.76

The crude death rates for both town is same. Since the proportions of population of the two towns in different age groups are different, the various age-groups exhibit different specific death rates for the two towns, crude death rates are not suitable for comparing the health conditions in the two towns. For this purpose, we have to use standardized death rate.

SDR of town A (19.39) is less than that of town B (19.76). Hence the population of town A is healthier than of town B.

Illustration 3.4.5:

Which of the two localities A and B is healthier?

Age group	Locality A		Locality B	
	Population	Deaths per 1000	Population	Deaths per 1000
0-10	600	30	400	40
10-20	1000	5	1500	4
20-60	3000	8	2400	10
60 and above	400	50	700	30

(Take locality A as standard)

Solution:

Standardized death rate of locality **A** is

$$= \frac{600\times30+1000\times5+3000\times8+400\times50}{600+1000+3000+400}$$

$$= \frac{67{,}000}{5{,}000}$$

$$= 13.4$$

Standardized death rate of locality **B** is

$$= \frac{600\times40+1000\times4+3000\times10+400\times30}{5{,}000}$$

$$= \frac{70{,}000}{5{,}000}$$

$$= 14$$

Since the SDR of the locality A is less than that of the locality B, the locality A is healthier.

Illustration 3.4.6:

From the following table compare the death rates in two towns A and B. Which town is healthier?

Age group	Town A		Town B		Standard Population	
	Population	No. of deaths	Population	No. of deaths	Population	No. of deaths
0-10	4000	36	3000	30	2000	60
10-25	12000	48	20000	100	8000	8
25-60	6000	60	4000	48	6000	4
60 and over	8000	152	3000	60	4000	50

Solution:

COMPARING DEATH RATES IN TOWNS A AND B

Age group	Town A			Town B			Standard Population		
	Popul -ation	No. of deaths	Death Rate	Popul -ation	No. of deaths	Death Rate	Popul -ation	No. of deaths	Death Rate
0-10	4000	36	9	3000	30	10	2000	60	30
10-25	12000	48	4	20000	100	5	8000	8	1
25-60	6000	60	10	4000	48	12	6000	4	0.67
≥60	8000	152	19	3000	60	20	4000	50	12.5

Standardized Death Rate (town A)

$$= \frac{(2000\times9)+(8000\times4)+(6000\times10)+ (4000\times19)}{20000}$$

$$= \frac{18000+32000+60000+ 76000}{20000}$$

$$= \frac{186000}{20000} = 9.3$$

Standardized Death Rate (town B)

$$= \frac{(2000\times10)+(8000\times5)+(6000\times12)+ (4000\times20)}{20000}$$

$$= \frac{20000+40000+72000+ 80000}{20000}$$

$$= \frac{212000}{20000}$$

$$= 10.6$$

Death rate is lower than town A as compared to B, hence the town A is healthier.

Since the S.D.R of the town A is less than that of town B, hence town A is healthier.

4. LIFE TABLE

Definition

"The life table is life-history of a hypothetical group or cohort of people as it is diminished gradually by deaths. The record begins at birth of each member and continues until all have dead."

§ 4.1 Assumption

The following are a few simplified assumptions which are used in the construction of life tables:

1. The cohort is closed for emigration or immigration. In other words, there is no change in the census except the losses due to deaths.
2. Individuals die at each age according to pre-determined schedule which is fixed and does not change.
3. The cohort originates from some standard number of births, say 10,000 or 1,00,000 which is called radix of the table.
4. The deaths are distributed uniformly over the period (x, x+1) for each x (except for first few years). In other words, deaths are uniformly distributed between one birthday and the next.

§ 4.2 Description of a life table

A typical life table has generally the following columns:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
x	l_x	d_x	q_x	p_x	m_x	L_x	T_x	e_x^0

§ 4.3 Life Table

Age x	Living of age x l_x	Dying between ages x and x+1 d_x	Mortality Rate q_x	Survival Rate p_x	Living between age x and x+1 L_x	Living above age x T_x	Mean after life time at age x e_x^0
1	2	3	4	5	6	7	8
0	1,42,759	27,124	.19000	.81000	1,23,143	46,32,557	32.45
1	1,15,635	7,059	.061104	.93896	1,11,899	45,09,414	39.00
2	1,08,576	3,900	.03648	.96352	1,06,591	43,97,525	40.50
3	1,04,616	2,610	.02495	.97505	1,03,392	42,90,934	41.02
4	1,02,006	2,006	.09167	.98033	1,01,121	41,87,542	41.05
5	1,00,000	1,710	.01710	.98290	99,145	40,86,420	40.86
6	98,298	1,591	.11620	.98380	97,494	39,87,275	40.67
7	96,698	1,483	.01534	.98466	95,957	38,89,781	40.23
8	95,215	1,383	.01452	.98548	94,523	37,93,824	39.84
9	93,832	1,293	.01378	.98622	93,186	36,99,301	39.42
10	92,539	1,210	.01308	.98692	91,934	36,06,115	38.97
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
95	21	9	.40957	.59043	16	32	1.52
96	12	6	.42932	.57068	9	16	1.34
97	6	3	.44964	.55036	5	7	1.17
98	3	1	.47046	.52954	2	2	0.64
99	1	1	.49176	.50823	..	..	..

1. The first column denoted by 'x' gives the exact years of age starting from 0,1,2, 3, ... up to 99
2. The second column denoted by l_x gives the number of persons who attain (or rather are expected to attain) exact age x out of an assumed number of births l_x (called the Cohort or radix of the life table). Thus, the number 1,42,759 in the l_x column against '0' year age indicates the number that began their life together and are running the first year of their life. The number 1,15,635 against one-year age indicates the number who have completed first year of their age and are running the second, and so on.
3. The third column d_x gives the number of the persons among the l_x persons reaching age x who die before reaching x+1. Thus the

$$d_x = l_x - l_{x+1}$$

Thus, for the life table given above, corresponding to x = 0

$$d_x = (142759 – 1.15635) = 27124; \text{ and}$$

for x = 1, d_x = (115635 – 108576) = 7059.

4. The fourth column entitled 'q_x' gives the mortality rates to which the population groups would be exposed throughout their lives. It should be noted that this mortality is not the same as the age-specific death rates obtained from death registration records. For the life table death rate, the denominator is the number of people alive at exact age x and the numerator the number, dying between age x and x+1. It follows that $q_x = \frac{d_x}{l_x}$

Thus, for the life table given above corresponding to x = 0

$q_0 = \frac{27124}{142759} = .19000$; for x = 1, $q_1 = \frac{7059}{115635} = .06104$. etc.

5. The fifth column entitled 'p_x' gives the probability that a person of precise age x will survive till his next birthday. Since a person must either live or die in a particular year of life $q_x + p_x = 1$ and $p_x = (1- q_x)$. Thus, for the table given for x = 0, p_0 = (1- .19000) = 0.81; for x = 1, p_1 = (1- .06104) = .93896. etc.
6. The sixth column entitled 'L_x' gives the number of years lived in the aggregate by the cohort of l_x persons between age x and x+1. The L_x column gives the distribution of the life table stationary population. For finding L_x the following is used:

$$L_x = l_x - \frac{1}{2} dx$$

Thus, for age 5. $L_x = 100000 - \frac{1}{2} \times 1710 = 99145$.

This is on the assumption that deaths for each year of age are evenly spread throughout the year.

L_x can also be calculated as follows: $\frac{l_x + l_{x+1}}{2}$

7. The seventh column entitles T_x gives the number of the years lived by the group from the age x until all of them die. Thus

$$T_x = L_x + L_{x+1} + L_{x+2} + \ldots\ldots L_n \text{ or } T_0 = L_0 + L_1 + L_2$$

Thus, for the specimen table

$T_0 = 123143 + 111889 + 10659 + 1\ldots\ldots16 + 9 + 5 + 2 = 4632557$.

8. The last column entitled e_x^0 measures the average number of years a person of a given age x can be expected to live under the prevailing mortality conditions. The expectation of life at age x is obtained from the following relation:

$$e_x^0 = \frac{T_x}{l_x}$$

Thus, for x = 5, $e_5^0 = \frac{4086420}{100000} = 40.86$

For x = 95, $e_{95}^0 = \frac{32}{21} = 1.52$ etc.

§ 4.4 Illustrations

Illustration 4.4.1

Fill in the blanks of the following skeleton life table which are marked with question marks:

Age	l_x	d_x	q_x	p_x	L_x	T_x	e_x^0
9	93832	1293	?	?	?	3699301	30.42
10	?	1210	-	-	-	?	?

Solution: $l_{10} = l_9 - d_9 = 93832 - 1293 = 92539$;

$q_x = \frac{d_x}{l_x}$ $\quad q_9 = \frac{d_9}{l_9} = \frac{1293}{93832} = 0.01378$;

$p_x = 1 - q_x$ $\quad p_9 = 1 - 0.01378 = 0.98622$;

$L_x = \frac{l_x + l_{x+1}}{2}$ $\quad L_9 = \frac{l_9 + l_{10}}{2} = \frac{93832 + 92539}{2} = 93185.5$;

$T_{x+1} = T_x - L_x$ $\quad$ $T_{10} = T_9 - L_9 = 3699301 - 93185.5. = 3606115.5;$

$e_x^0 = \frac{T_x}{l_x}$ $\quad$ $e_9^0 = \frac{3699301}{93832} = 39.42;$

$$e_{10}^0 = \frac{T_{10}}{l_{10}} = \frac{3603115.5}{92539} = 38.97$$

Thus, the table after completing the figures is as follows:

Age	l_x	d_x	q_x	p_x	L_x	T_x	e_x^0
9	93832	1293	.0137	93622	93185.5	3699301	30.42
10	92539	1210	-	-	-	3606115.5	38.97

Illustration 4.4.2

Find the blanks of the following skeleton life table which are marked with question marks:

Age	l_x	d_x	q_x	p_x	L_x	T_x	e_x^0
20	693435	?	?	?	?	35081126	?
21	690673	-	-	-	-	-	?

Solution:

$d_x = l_x - l_{x+1}$ $\quad$ $d_{20} = 693435 - 690673 = 2762$

$q_x = \frac{d_x}{l_x}$ $\quad$ $q_{20} = \frac{2762}{693435} = 0.0398;$

$p_x = 1 - q_x$ $\quad$ $p_{20} = 1 - 0.00398 = 0.99602;$

$L_x = \frac{l_x + l_{x+1}}{2}$ $\quad$ $L_{20} = \frac{693435+690673}{2} = 692054;$

$T_{x+1} = T_x - L_x$ $\quad$ $T_{20} = 35081126 - 692054 = 34389072;$

$e_x^0 = \frac{T_x}{l_x}$ $\quad$ $e_{20}^0 = \frac{T_{20}}{l_{20}} = \frac{35081126}{693435} = 50.59;$

$$e_{21}^0 = \frac{T_{21}}{l_{21}} = \frac{34389072}{690673} = 49.79$$

Illustration 4.4.3

Find the blanks of the following skeleton life table which are marked with question marks:

Age	l_x	d_x	q_x	p_x	L_x	T_x	e_x^0
30	762227	?	?	?	?	27296732	?
31	758580	-	-	-	-	?	?

Solution:

$d_x = l_x - l_{x+1}$ $\quad d_{20} = 762227 - 758580 = 3647$

$q_x = \frac{d_x}{l_x}$ $\quad q_{20} = \frac{3647}{762227} = 0.00478$

$p_x = 1 - q_x$ $\quad p_{20} = 1 - 0.00478 = 0.99522$

$L_x = \frac{l_x + l_{x+1}}{2}$ $\quad L_{20} = \frac{762227 + 758580}{2} = 760403.5$

$T_{x+1} = T_x - L_x$ $\quad T_{21} = 27296732 - 760403.5 = 26536328.5$

$e_x^0 = \frac{T_x}{l_x}$ $\quad e_{30}^0 = \frac{T_{20}}{l_{20}} = \frac{27296732}{762227} = 35.81$

$e_{31}^0 = \frac{T_{21}}{l_{21}} = \frac{26536328.5}{758580} = 34.98$

Illustration 4.4.4

The table below gives the life table for rabbits.

Find d_2, q_0, p_1, L_3.

x	0	1	2	3	4	5	6
l_x	100	90	80	75	60	30	0

Solution:

$d_x = l_x - l_{x+1}$

$d_2 = 80 - 75$

$d_2 = 5$

$$q_x = \frac{d_x}{l_x} = \frac{lx - lx+1}{lx}$$

$$q_0 = \frac{100-90}{100} = \frac{10}{100}$$

$$q_0 = 0.1$$

$$p_x = 1 - q_x = 1 - \frac{d_x}{l_x}$$

$$p_1 = 1 - \frac{d_1}{l_1} = 1 - \frac{l1 - l2}{l} = 1 - \frac{90-80}{90}$$

$$p_1 = 0.89$$

$$L_x = l_x - \frac{d_x}{2}$$

$$L_3 = l_3 - \frac{d_3}{2} = l_3 - \frac{l3 - l3+1}{l3} = 75 - \frac{75-60}{2}$$

$$L_3 = 67.5$$

§ 4.5 Data of Vital Rates

4.5.1 Total Fertility Rate of Tamil Nadu and India on the Year Range from 2005 – 2019:

Related Indicator	Last	Frequency	Range
Total Fertility Rate: Tamil Nadu: Urban (NA)	1.500	yearly	2005 - 2019
Total Fertility Rate: India: Urban (NA)	1.700	yearly	2005 - 2019
Total Fertility Rate: India: Rural (NA)	2.200	yearly	2005 - 2019
Total Fertility Rate: India (NA)	2.100	yearly	2005 - 2019

4.5.2 Mid Year Estimated Population(M), Births and Deaths of all districts in 2020:

No.	District	M	Births	Deaths	Birth rate	Death rate
1.	Chennai	8472503	95716	73243	11.3	8.6
2.	Kancheepuram	3209499	40362	31233	12.6	9.7
3.	Thiruvallur	2330597	18790	17858	8.1	7.7
4.	Cuddalore	2657722	30264	20722	11.4	7.8
5.	Villupuram	3207007	38636	28227	12.0	8.8
6.	Vellore	4003068	61778	34831	15.4	8.7
7.	Tiruvannamalai	2398745	25535	21026	10.6	8.8
8.	Salem	3750157	49173	33337	13.1	8.9
9.	Namakkal	1797796	17273	15145	9.6	8.4
10.	Dharmapuri	1452408	26863	11957	18.5	8.2
11.	Krishnagiri	1846495	25181	14391	13.6	7.8
12.	Erode	2428717	28345	23336	11.7	9.6
13.	Coimbatore	4012535	45588	39959	11.4	10.0
14.	Tiruppur	2756703	23140	20867	8.4	7.6
15.	The Nilgiris	812518	6097	5219	7.5	6.4
16.	Tiruchirappalli	2915416	42298	29442	14.5	10.1
17.	Karur	1110182	12592	10410	11.3	9.4
18.	Perambalur	544565	8051	5687	14.8	10.4
19.	Ariyalur	711846	9146	7078	12.8	9.9
20.	Pudukkottai	1572034	18162	14520	11.6	9.2
21.	Thanjavur	2465195	40311	24797	16.4	10.1
22.	Nagapattinam	1550938	16308	12760	10.5	8.2
23.	Thiruvarur	1231681	13865	11761	11.3	9.5
24.	Madurai	3372671	45737	35936	13.6	10.7
25.	Theni	1353830	16640	12776	12.3	9.4
26.	Dindigul	2227739	25693	19948	11.5	9.0
27.	Ramanathapuram	1363862	16979	11423	12.4	8.4
28.	Virudhunagar	2088680	26752	19666	12.8	9.4
29.	Sivagangai	1351592	18871	13159	14.0	9.7
30.	Tirunelveli	3078645	43546	32441	14.1	10.5
31.	Thoothukkudi	1879911	22629	16341	12.0	8.7
32.	Kanniyakumari	2211741	23698	17716	10.7	8.0

5. APPLICATIONS OF VITAL STATISTICS

The followings are some uses of vital statistics:

1. A highly useful statistical information is built up on important aspects of demography in the country. This is necessary for planning, evaluation and analysis of various economic and social policies.

2. Once a systematic beginning has been made and laws of population is framed, future projections can be made by the use of various scientific techniques.

3. Such information particularly on fertility, mortality, maternity, urban density is indispensable for planning and evaluation of the schemes of health, family planning and other civic amenities.

4. Regional population statistics throws light on regional disparities and helps in analysing various aspects of society which depends on population characteristics. e.g., health, longevity, labour force, widows, orphans, divorces, etc.

5. Data on life expectancy at various age levels is helpful in actuarial calculations of risk on life policies.

6. Vital statistics is very handy for authentication of some vital events of births, deaths and marriages.

7. The ‘life table' called the 'biometer' of public health and longivity helps in fixing the life insurance premium rates at various age levels.

6. CONCLUSION

This project, dealt with several vital events such as birth rate, death rate, fertility rate and reproduction rate with statistical methods. And the reader gets to know how to find the rate of such vital events at any particular year with adequate data. Other than rates, it also showed how to calculate the number of births or deaths that were occurred in the particular year. This project also dealt with life table and applications of vital statistics in real life. All of which gives the reader, a brief knowledge about vital statistics which comes under the branch of Demography.

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A Study on Fuchsian Groups

Project report submitted to

ST. MARY'S COLLEGE (AUTONOMOUS), THOOTHUKUDI

Affiliated to

MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI

In partial fulfilment of the requirement for the award of degree of

Bachelor of Science in Mathematics

Submitted by

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(2022– 2023)

CERTIFICATE

We hereby declare that the project report entitled **A Study on Fuchsian Groups** being submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfilment for the award of degree of **Bachelor of Science in Mathematics** and it is a record of work done during the year 2020 -2023 by the following students:

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Declaration

We hereby declare that the project entitled "A **STUDY ON FUCHSIAN GROUPS** " is our original work. It has not been submitted to any University for any degree or diploma.

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ACKNOWLEDGEMENT

First of all, we thank the Almighty for showering his blessing to undergo this project.

With immense pleasure, we register our deep sense of gratitude to our guide **Dr. G. Priscilla Pacifica M.Sc., B.Ed., M. Phill., Ph.D., SET.** and the Head of the Department, **Dr. V. L. Stella Arputha Mary M.Sc., B.Ed., M. Phill., Ph.D.** for having imparted necessary guidelines throughout the period of our studies.

We thank our beloved Principal, **Rev. Dr. Sr. A.S.J. Lucia Rose M.Sc., M.Phil., Ph.D., PGDCA.,** for Providing us to help to carry out our project work successfully.

Finally, we thank all those who extended their helping hands regarding this project.

Table of Contents

Introduction

Fuchsian groups are discrete subgroups of isometries of the hyperbolic plane. This project will primarily work with the upper half-plane model, though we will provide an example in the disk model.

We will define Fuchsian groups and examine their properties geometrically and algebraically. We will also discuss the relationships between fundamental regions and Dirichlet regions.

Möbius transformations are mappings from the complex plane to itself of the form

$$T\ (z) = \frac{\mathrm{az+b}}{\mathrm{cz+d}}$$

where a,b,c,d $\epsilon\ \mathbb{C}$, and ad - bc $\neq 0$. We restrict to a,b,c,d $\epsilon\ \mathbb{R}$, then T preserves $\mathbb{H}^2$. We will use associated matrices of these transformations to help us discover more about the geometric properties hidden in these mappings.

We will see the trace of a matrix will determine which transformation is either hyperbolic, elliptic or parabolic. These elements have special properties for Fuchsian groups and their geometries will differ.

To begin, we will give some background on hyperbolic geometry, then define what Fuchsian groups are. This will lead us to consider the geometry of fundamental regions and Dirichlet regions.

Chapter 1

Preliminaries

Necessary definitions are given in this section which are needed for the following chapters.

Definition 1.1.

The upper half plane model of the ***hyperbolic plane*** is the metric space consisting of the open half plane

$$\mathbb{H}^2 = \{ (x,y) \in \mathbb{R}^2 \,;\, y > 0 \} = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}.$$

Definition 1.2.

In the upper half plane model, the ***hyperbolic length*** of a curve, which is parametrized by a differentiable vector valued function

$$t \mapsto (x(t), y(t)), a \le t \le b$$

to be

$$h(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

Definition 1.3.

The ***hyperbolic distance*** between two points z_1 and z_2 is the infimum of the hyperbolic lengths of all piecewise differentiable curves going from z_1 to z_2. It is denoted by

$$d(z_1, z_2) = inf\{ h(\gamma) \,;\, \gamma \; goes \; from \; z_1 \; to \; z_2 \}.$$

Definition 1.4.

The set of linear fractional transformations of $\mathbb{H}^2$, also known as ***Möbius transformations*** is of the form

$$\{z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1\}$$

In addition to preserving circles, angles, and symmetry, these mappings are oneto-one and onto. For each w there is one and only one z that maps to w.

Definition 1.5.

The ***special linear group***, SL(2,$\mathbb{R}$) is the group of 2 $\times$2 with determinant 1.

Definition 1.6.

PSL(2, $\mathbb{R}$) is defined as the ***projective special linear group*** of degree two over the field of real numbers

In other words ,it is defined as (2, $\mathbb{R}$)/ { $\pm I_2$ },

where I_2 is the identity matrix.

Definition 1.7.

Let f be a bijective mapping between two topological spaces. We say if f and its inverse f^{-1} are continuous, then f is said to be a ***homeomorphism***.

Definition 1.8.

A transformation of $\mathbb{H}^2$ onto itself is called an ***isometry*** if it preserves the hyperbolic distance.

Definition 1.9.

A ***geodesic*** between two points in $\mathbb{H}^2$is a path of minimal length between them.

Definition 1.10

A transformation of $\mathbb{H}^2$ is called ***conformal***, if it preserves angles, and ***anti-conformal***, if it preserves the absolute values of angles, but changes the signs.

Definition 1.11

For a subset A ⊂ $\mathbb{H}^2$, we define μ(A) as the ***hyperbolic area*** of A by

$$\mu\,(A) = \int_A \frac{dxdy}{y^2}$$

if this integral exists.

Definition 1.12

Consider the matrix g = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ∈ PSL(2, ℝ).

Then, tr(g) = |a + d| is defined to be ***trace*** of g.

Definition 1.13

A geodesic in $\mathbb{H}^2$ joining the two fixed points of the hyperbolic transformation T is called the ***axis*** of T, and we denote it C(T).

Definition 1.14

A discrete subgroup of Isom(ℍ) is called a ***Fuchsian Group***, if it consists of orientation preserving transformations. A Fuchsian group is a discrete subgroup of PSL(2,ℝ).

Definition 1.15

If G is any group and g ∈ G, then the ***centralizer*** of g in G is defined by

$$C_G(g) = \{h \in G \mid hg = gh\}$$

Definition 1.16

A closed region $F \subset X$ is said to be a ***fundamental region*** for a group G if the following conditions hold:

1. $\bigcup_{T \in G} (F) = X$

2. $F^{\circ} \cap T(F^{\circ}) = \emptyset$

where F is the closure of a non-empty open set, F° called the **interior** of F.

Definition 1.17

Let Γ be an arbitrary Fuchsian Group and let $p \in \mathbb{H}^2$ be not fixed by any element of $\Gamma - \{Id\}$. We will define the ***Dirichlet region*** for Γ centered at p to be the set

$$D_p(\Gamma) = \{ z \in \mathbb{H}^2 \mid d(z,p) \leq d(z,T(p)) \,|\text{for all } T \in \Gamma\}$$

Definition 1.18

A ***perpendicular bisector*** of the geodesic segment $[z_1, z_2]$ is the unique geodesic through w, the midpoint of $[z_1, z_2]$, orthogonal to $[z_1, z_2]$.

Definition 1.19

The ***unit disk*** is defined to be

$$\mathbb{B}^2 = \{ z \in \mathbb{C} \,||z| < 1\}.$$

The map

$$f(z) = \frac{zi+1}{z+i}$$

is a 1-1 map of $\mathbb{H}^2$ and provides an isometry onto $\mathbb{B}^2$.

Chapter 2

Hyperbolic Geometry

This chapter deals with hyperbolic geometry. We will see how the hyperbolic length and distance share similarities to Euclidean space. We will also see how isometries and geodesics play an important role in our study. Finally, we will explore hyperbolic area.

§ 2.1 The Hyperbolic Metric

The hyperbolic plane is a less familiar metric space than the Euclidean plane. An introduction to some of the basic properties will be and find it has similarities with the Euclidean plane. There is a one-to-one correspondence between points in $\mathbb{R}^2$ and the complex plane $\mathbb{C}$. The notations for the real and imaginary parts of the complex number

$$z = x + iy \in \mathbb{C},$$

to be $Re(z) = x$ and $Im(z) = y$. The **conjugate** of z is defined to be

$$\bar{z} = x - iy.$$

Definition 2.1.1.

The upper half plane model of the **hyperbolic plane** is the metric space consisting of the open half plane

$$\mathbb{H}^2 = \{ (x,y) \in \mathbb{R}^2 ; y > 0 \} = \{z \in \mathbb{C} ; Im(z) > 0\}.$$

Definition 2.1.2.

In the upper half plane model, the **hyperbolic length** of a curve, which is parametrized by a differentiable vector valued function

$$t \mapsto (x(t), y(t)), a \le t \le b$$

to be

$$h(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

This metric is a way of finding the lengths of curves and we illustrate this with an example.

Example 2.1.3. Suppose $P_0 = (x,y_0)$ and $P_1 = (x,y_1)$. To compute the hyperbolic length of the line segment denoted by $[P_0,P_1]$, we will parametrize this segment by,

$t \mapsto (x,t),\ y_0 < t < y_1.$

Then,

$$h(\gamma) = \int_{y_0}^{y_1} \frac{\sqrt{0^2 + 1^2}}{t} dt.$$

$$= \int_{y_0}^{y_1} \frac{1}{t}\, dt$$

$$= \ln \frac{y_1}{y_0}.$$

Definition 2.1.4.

The **hyperbolic distance** between two points z_1 and z_2 is the infimum of the hyperbolic lengths of all piecewise differentiable curves going from z_1 to z_2. It is denoted by

$$d(z_1, z_2) = inf\{h(\gamma) ;\ \gamma \text{ goes from } z_1 \text{ to } z_2\}.$$

Definition 2.1.5. The set of linear fractional transformations of $\mathbb{H}^2$, also known as **Möbius transformations** is of the form

$$\{z \mapsto \frac{az+b}{cz+d} \ |a, b, c, d \in \mathbb{R}, ad - bc = 1\}$$

In addition to preserving circles, angles, and symmetry, these mappings are one-to-one and onto. For each w there is one and only one z that maps to w. This leads us to consider finding the inverse of such mappings. Before we do that, we want to have a simple way to view these transformations. We will uses matrices to help give us an algebraic point of view, so we can use it to discover geometric properties.

Definition 2.1.6. Let, $T(z) = \frac{az+b}{cz+d}$ where a , b ,c ,d $\in \mathbb{R}$ and ad −bc = 1. Then,

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the matrix associated with T.

Square brackets are used to indicate the matrix is identified with its negative. Since ad − bc = 1, every Möbius transformations is invertible. The inverse of a Möbius transformations is the associated inverse matrix of T, which is

$$T^{-1} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

These transformations form a group. To convince ourselves, we can show the transformations meet all the requirements to be a group. Composition of Möbius transformations is another Möbius transformations . These transformations have inverses because the determinant is 1 and the identity transformation is simply the associated identity matrix. Finally, the associative property follows since composition of maps is always associative.

§ 2.2. Special linear group

Definition 2.2.1. The **special linear group**, SL(2, ℝ) is the group of 2×2 with determinant 1.

Definition 2.2.2. PSL(2, ℝ) is defined as the **projective special linear group** of degree two over the field of real numbers

In other words ,it is defined as (2, ℝ)/ $\{\pm I_2\}$, where I_2 is the identity matrix.

We will now give examples of the associated matrices of Möbius transformations

Each example will have determinant 1.

Example 2.2.3.

$$T\,(z) = \frac{z+1}{z+2} \;\Leftrightarrow\; \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} z$$

Example 2.2.4.

$$T\,(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}} \Leftrightarrow \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} z.$$

Example 2.2.5.

$$T\,(z) = z+1 \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z$$

Example 2.2.6.

Let L be a Euclidean circle or a straight line orthogonal to the real axis, which meets the real axis at some finite point α. We would like to show the transformation

$$T\,(z) = -\,(\,z - \,\alpha\,)^{-1} \,+\, \beta$$

is a Möbius transformations, and for a suitable β maps L to imaginary axis. What this amounts to is showing T is of the form

$$T\,(z) = \frac{az+b}{cz+d}$$

where ad−bc = 1 and a ,b ,c ,d ϵ ℝ. Rearranging T we find that it is of the form

$$T\,(z) = \frac{-1}{z-\alpha} + \beta = \frac{\beta z+(-\alpha\beta-1)}{z-\alpha}$$

and its determinant is 1. Hence, T is a Möbius transformations.

Definition 2.2.7. Let f be a bijective mapping between two topological spaces. We say if f and its inverse f^{-1} are continuous, then f is said to be a **homeomorphism**.

Theorem 2.2.8. Every Möbius transformation ϕ in PSL(2, ℝ) is a homeomorphism of ℍ

Proof : First show that ϕ maps ℍ into ℍ. Let $\phi\,(z) = \frac{az+b}{cz+d}$ where ad − bc = 1 and

z ∈ ℍ.

$$\phi(z) = \frac{(az+b)(c\bar{z}+d)}{(cz+d)(c\bar{z}+d)} = \frac{ac|z|^{2}2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

$$\mathrm{Im}\big(\phi\,(z)\big) = \frac{\phi\,(z) - \overline{\phi\,(z)}}{2i} = \frac{adz + bc\bar{z} - ad\bar{z} - bcz}{2i|cz+d|^2}$$

$$= \frac{(ad-bc)(z-\bar{z}\,)}{2i|cz+d|^2}$$

$$= \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

The imaginary component of z is greater than 0, so clearly $\mathrm{Im}(\phi(z)) > 0$, which implies that $\phi(z)$ is in $\mathbb{H}$.

Because ϕ is a rational function with a nonzero denominator, continuity is clear. The existence of an inverse follows from the fact that ϕ^{-1} is also an element of $\mathrm{PSL}(2, \mathbb{R})$.By the argument above, it is continuous and maps into $\mathbb{H}$. Therefore, ϕ is a homeomorphism of $\mathbb{H}$**.**

Definition 2.2.9. A transformation of $\mathbb{H}^2$ onto itself is called an **isometry** if it preserves the hyperbolic distance.

Theorem 2.2.10. $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Isom}(\mathbb{H})$.

Proof.

Let $\gamma : [0, 1] \to \mathbb{H}$ be a piecewise differentiable path in $\mathbb{H}$.

Let γ be given by

$z(t) = (x(t), y(t))$ and $w(t) = T(z(t)) = u(t) + iv(t)$ By the quotient rule,

$$\frac{dw}{dt} = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{acz - caz + ad - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2}$$

Since $\mathrm{Im}(T(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2} = v$ and $y = \mathrm{Im}(z)$, $\left|\frac{dw}{dt}\right| = \frac{v}{y}$.

From the definition of hyperbolic length, we obtain the following equation.

$$h(T(\gamma)) = \int_0^1 \frac{\left|\frac{dw}{dt}\right|}{v(t)} dt$$

$$= \int_0^1 \frac{\left|\frac{dw}{dz}\frac{dz}{dt}\right|}{v(t)} dt$$

$$= \int_0^1 \frac{\left|\frac{v(t)}{y(t)}\frac{dz}{dt}\right|}{v(t)} dt$$

$$= \int_0^1 \frac{\left|\frac{dz}{dt}\right|}{y(t)} dt = h(\gamma).$$

The hyperbolic distance $\rho(j,k)$ is the infimum of the differentiable paths γ between j and k, so since each γ is invariant under T, $\rho(j,k)$ is invariant under T. Thus, $T \in PSL(2,\mathbb{R})$ is an isometry. There is one further property that classifies $PSL(2,\mathbb{R})$ as a subset of Isom($\mathbb{H}$), which is that $PSL(2,\mathbb{R})$ is the set of all orientation preserving isometries of $\mathbb{H}$. A linear operator on a vector space is orientation preserving if its determinant is positive.

§ 2.3. Geodesics

In Euclidean geometry, the shortest curve joining two points is the line segment with those two points as endpoints. This subsection defines and describes the shortest curves of the hyperbolic plane, which are also known as geodesics.

Definition 2.3.1. A **geodesic** between two points in $\mathbb{H}^2$is a path of minimal length between them.

Proposition 2.3.2 Two points in $\mathbb{H}^2$ can be joined by a unique geodesic and the hyperbolic distance between those points is equal to the hyperbolic length of the unique geodesic segment connecting them. This will be denoted by $[z_1,z_2]$, where z_1 and z_2 are in $\mathbb{H}^2$.

Proposition 2.3.3. The geodesics in $\mathbb{H}^2$ are semicircles and straight lines orthogonal to the real axis.([1] or [4]).

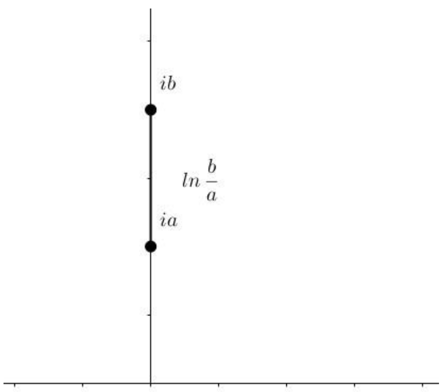

Figure 2.1: The hyperbolic length of a segment

Example 2.3.4. Let L ={ $z = e^{i\theta}$| $0 < \theta < \pi$}. Then, L is a geodesic orthogonal to the real axis

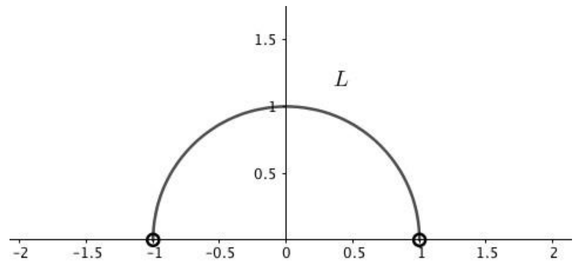

Figure 2.2: The semicircle L is a geodesic in $\mathbb{H}^2$

Example 2.3.5. Let L ={ x = 1 | y > 0} . Then, L is a geodesic and is clearly orthogonal to the real axis

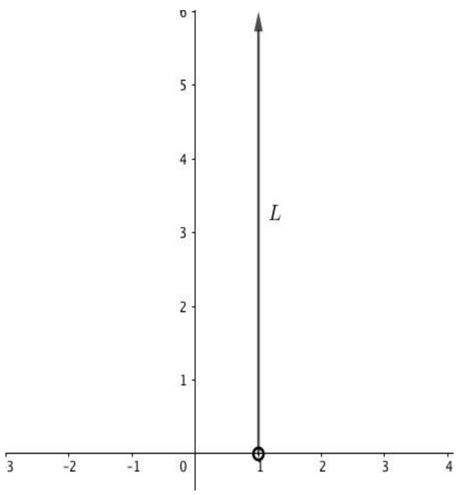

Figure 2.3: Vertical line L is a geodesic in $\mathbb{H}^2$

Corollary 2.3.6. If z_1 and z_2 are two distinct points in $\mathbb{H}^2$, then $d(z_1,z_2) = d(z_1,z_3) + d(z_3,z_2)$ if and only if $z_3 \in [z_1,z_2]$. This is a consequence of the triangle inequality.

Example 2.3.7. Take the example seen in figure 2.1. Suppose there is a point between ia and ib and call it ik, where a < k < b. Then, the corollary applies.

§ 2.4 Isometries

The hyperbolic plane has many symmetries and find that it is as symmetric as the Euclidean plane. In this subsection, we will define and describe the isometries of $\mathbb{H}^2$. In our discussion, we will explain that all isometries of H^2 are exactly of the form

$$T(z) = \frac{az+b}{cz+d}$$

Or

$$\phi(z) = \frac{-a\bar{z}+b}{-c\bar{z}+d}$$

where ad – bc = 1 and a,b,c,d $\in \mathbb{R}$

Example 2.4.1. Let ϕ: $\mathbb{H}^2 \longmapsto \mathbb{H}^2$defined by

$$\phi(x,y) = (kx,ky)$$

This isometry is known as the **homotheties transformation**, which is also known as **dilations**.

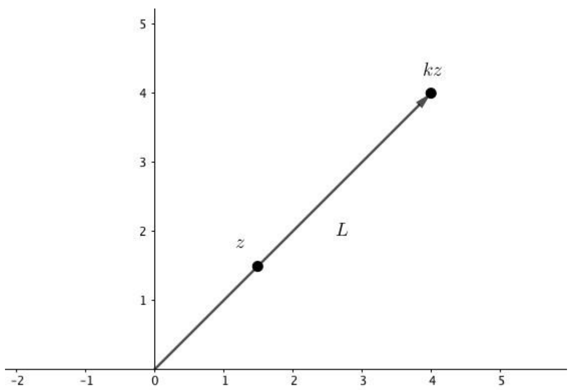

Figure 2.4: Dilation transformation

Example 2.4.2. Let ϕ: $\mathbb{H}^2 \longmapsto \mathbb{H}^2$ defined by ϕ $(x,y) = (x + x_0 ,y)$, for some $x_0 \in \mathbb{R}$. This is known as the **horizontal translations transformation**, which is also another isometry.

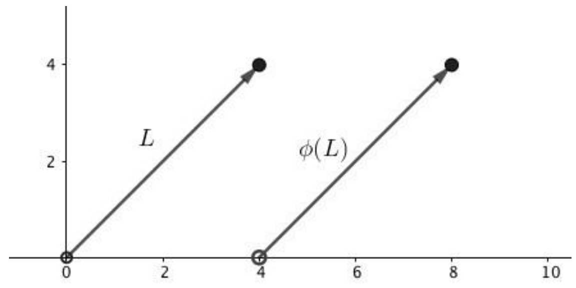

Figure 2.5: Translation transformation

Example 2.4.3. Let ϕ: $\mathbb{H}^2 \longmapsto \mathbb{H}^2$defined by $z \longmapsto -\overline{z}$. The transformation is **reflection across the** y-**axis**, which is another isometry of $\mathbb{H}^2$.

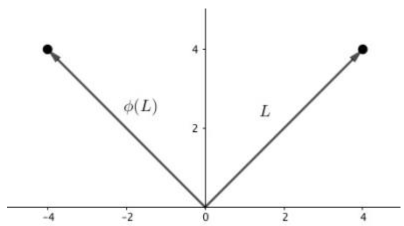

Figure 2.6: Reflection across the y axis

Example 2.4.4. The **standard inversion or simply inversion** ,across the unitb circle,which is defined by $\phi(x,y) = (\frac{x}{x^2+y^2} ,\frac{y}{x^2+y^2})$ is another isometry, In general,inversion across an arbitrary circle can also be defined as given any point P not the center,the point P' is the inverse to P if

1.P′ lies on a ray from O to P, and

2.OP · OP′ = r^2 where O is the center of the circle and r is the radius of the circle.

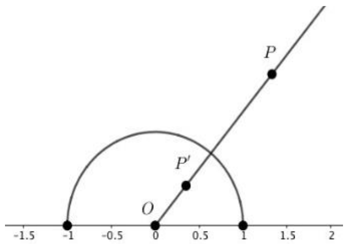

Figure 2.7: Inversion across the unit circle

Definition 2.4.5. A transformation of $\mathbb{H}^2$ is called **conformal**, if it preserves angles, and **anti-conformal**, if it preserves the absolute values of angles, but changes the signs.

Example 2.4.6. Homotheties and horizontal translations are conformal transformations because they preserve the angles.

Example 2.4.7. Inversion is an anti-conformal transformation.

§ 2.5 Hyperbolic Area and Gauss-Bonnet

Definition 2.5.1. For a subset A ⊂ $\mathbb{H}^2$, we define μ(A) as the **hyperbolic area** of A by

$$\mu\,(A) = \int_A \frac{dxdy}{y^2}$$

if this integral exists.

Example 2.5.2. Let's find the hyperbolic area by calculating the integral over the region above the semicircle

$$y = \sqrt{1 - x^2}$$

and between x = -1 to x = 1.Then,

$$\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dxdy}{y^2} = \int_{-1}^{1} -\frac{1}{y}\Big|_{\sqrt{1-x^2}}^{\infty} dx$$

Once we evaluate the integrand, we have

$$\left[\left[\lim_{y \to \infty} -\frac{1}{y}\right] - \left(-\frac{1}{y}\right)\right]\Big|_{\sqrt{1-x^2}}^{\infty} = \frac{1}{\sqrt{1-x^2}}$$

Now, we evaluate the second integral

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx$$

The function has discontinuities at x = -1 to x = 1. Consider the two integrals

$$\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}} dx$$

and

$$\int_{0}^{1} \frac{1}{\sqrt{1-x^2}} dx$$

We will sum up these integrals to get the area of the region. We take advantage of symmetry and just evaluate one. We see that

$$\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}(x) + C$$

where C is a constant. Therefore,

$$sin^{-1}(x)|_0^1 = sin^{-1}(1) - sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Then, by symmetry we also have $\frac{\pi}{2}$ for the other integral and therefore the area of the region is π

A hyperbolic n-sided polygon is a closed set of $\mathbb{H}^2$bounded by n hyperbolic geodesic segments. If two line segments intersect, then the point of intersection is called a **vertex** of the polygon. If these vertices are at 1 or on the real axis, these vertices are known as **ideal points**. If a polygon has vertices only in $\mathbb{R} \cup \{\infty\}$, we say that the polygon is an **ideal polygon**. There are four types of hyperbolic triangles, which depends on how many vertices belong to $\mathbb{R} \cup \{\infty\}$

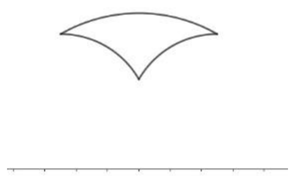

Figure 2.8: Hyperbolic triangle with 0 ideal points

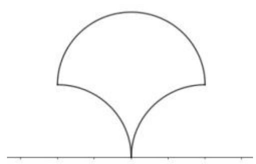

Figure 2.9: Hyperbolic triangle with 1 ideal point

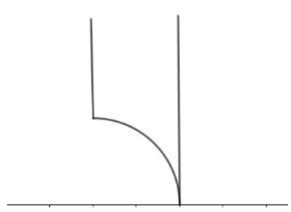

Figure 2.10: Hyperbolic triangle with 2 ideal points

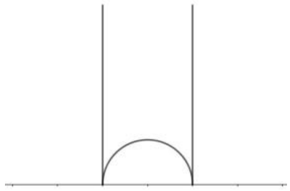

Figure 2.11: Hyperbolic triangle with 3 ideal points

Theorem 2.5.3 (Gauss-Bonnet) Let Δ be the hyperbolic triangle with angles α, β, γ. Then

$$\mu(A) = \pi - \alpha - \beta - \gamma.$$

Before we see some examples, it is important to note that if one of the vertices belongs to $\mathbb{R} \cup \{\infty\}$, then the angle at this vertex will be zero.

Example 2.5.4. Let T be a hyperbolic triangle with vertices at -1,0 and ∞ and . The geodesics are the vertical lines -1 and 0 joining to ∞ and the other is the semicircle with 0 joining- 1. Then, all these angles are zero, since all vertices are ideal. This triangle is similar to that of figure 2.11. Then, by the Gauss-Bonnet theorem, we have

$$\mu(T) = \pi - 0 - 0 - 0 = \pi.$$

Since all vertices were in $\mathbb{R} \cup \{\infty\}$, T is an ideal triangle. Example 2.4.1 is an example of an ideal triangle.

Example 2.5.5. Let T be a hyperbolic triangle with 2 vertices in $\mathbb{R} \cup \{\infty\}$. Let us assume one vertex is ∞ and the other is on the real axis. Let the third vertex be at an angle of $\frac{\pi}{2}$. This triangle is similar to that of figure 2.10. Then, by the Gauss-Bonnet theorem

$$\mu\ (T) = \pi - \left[\frac{\pi}{2} - 0 - 0\right] = \frac{2\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}$$

.

Example 2.5.6. Let T be a hyperbolic triangle with 1 vertex in $\mathbb{R} \cup \{\infty\}$. Let us assume the vertex is ∞ . Let the other two vertices have angles of $\frac{\pi}{4}$ and $\frac{\pi}{3}$respectively. This triangle is similar to that of figure 2.9. Then, by the Gauss-Bonnet theorem

$$\mu\ (T) = \pi - \left[\frac{\pi}{4} - \frac{\pi}{3} - 0\right] = \frac{12\pi}{12} - \frac{3\pi}{12} - \frac{4\pi}{12} - 0 = \frac{5\pi}{12}.$$

Chapter 3

Fuchsian Groups

In this chapter, we will define what a Fuchsian group is and what properties these types of groups have. We will also distinguish the elements of PSL(2, ℝ) by the value of its trace. Furthermore, we will discuss what it means for a Fuchsian group to be discrete and properly discontinuous. Finally, we will discuss algebraic properties of Fuchsian groups.

§ 3.1 The Group PSL(2, ℝ)

There are 3 types of elements in PSL(2, ℝ) and by the value of its trace we can distinguish which type of transformation it is.

1. If $|Tr(T)| < 2,$ then T is an elliptic transformation.

2. If $|Tr(T)| = 2,$ then T is a parabolic transformation.

3. If $|Tr(T)| > 2,$ then T is a hyperbolic transformation.

Definition 3.1.1. Consider the matrix g = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\boldsymbol{\epsilon}$ PSL(2, ℝ). Then, $tr(g) = |a + d|$ is defined to be **trace** of g. A geometrical meaning of the trace function allows us to identify what type of transformation we are working with and immediately know how it acts in the upper half plane

Example 3.1.2. Let $T(z) = \frac{z+1}{z-2}$ .Based on the value of its trace,T is hyperbolic.

Example 3.1.3. Let $T(z) = z + 1$. Based on the value of its trace, T is parabolic.

Example 3.1.4. Let $T(z) = \frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}$. Based on the value of its trace, T is elliptic.

We will now consider finding the fixed points of these transformations.The fixed points are found by solving

$$z = \frac{az+b}{cz+d}$$

with a,b,c,d $\in \mathbb{R}$ and ad $-$ bc = 1. The hyperbolic transformation has two fixed points in $\mathbb{R} \cup \{\infty\}$, one repulsive and one attractive, a parabolic transformation has one fixed point in $\mathbb{R} \cup \{\infty\}$. An elliptic transformation has a pair of conjugate fixed points and therefore, one fixed point in $\mathbb{H}^2$. We will illustrate this with a few examples.

Example 3.1.5. Let us find the fixed points of the transformation

$$T(z) = az + b$$

where a,b $\in \mathbb{R}$. So,

$$z = az + b \Longrightarrow z - az = b \Longrightarrow z(1-a) = b \Longrightarrow z = \frac{b}{1-a}$$

Hence, if a = 1, then T is parabolic and the only fixed point is ∞ . If a > 1, then by the value of the trace, T is hyperbolic and the second fixed point is $\frac{b}{1-a}$.

Example 3.1.6. Let us find the fixed points of the elliptic transformation

$$T(z) = \frac{z\cos\frac{\pi}{2} + \sin\frac{\pi}{2}}{-z\sin\frac{\pi}{2} + \cos\frac{\pi}{2}}$$

Setting $T(z) = z$ and simplifying T, we find

$$T(z) = \frac{z\cos\frac{\pi}{2} + \sin\frac{\pi}{2}}{-z\sin\frac{\pi}{2} + \cos\frac{\pi}{2}}$$

$$= \frac{1}{-z} = -\frac{1}{z}.$$

Isolating z and solving for z, we find that

$$z^2 = -1 \quad \Longrightarrow \quad z = \pm\sqrt{-1} \quad \Longrightarrow z = \pm i.$$

This shows the transformation has a pair of conjugate fixed points and one of them is in $\mathbb{H}^2$.

Definition 3.1.7.

A geodesic in $\mathbb{H}^2$joining the two fixed points of the hyperbolic transformation T is called the **axis** of T, and we denote it $C(T)$.

Example 3.1.8. Let us find the fixed points of a hyperbolic transformation

$$T(z) = \frac{z+1}{z+2} .$$

By setting T(z) = z, we have

$$\frac{z+1}{z+2} = z \Longrightarrow z^2 + 2z = z + 1 \Longrightarrow z^2 + z - 1 = 0.$$

By the quadratic formula, we find that the fixed points of T are z = $-\frac{1}{2} + \frac{\sqrt{5}}{2}$ and $-\frac{1}{2} - \frac{\sqrt{5}}{2}$.

Therefore, the geodesic connecting these fixed points is the axis of the transformation

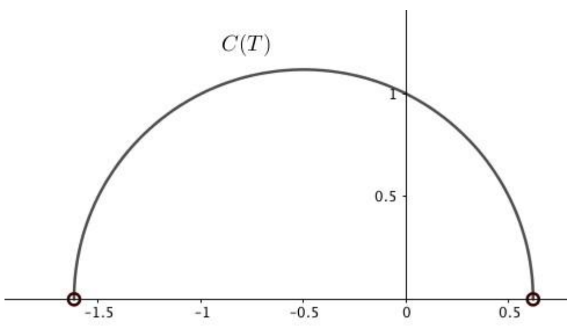

Figure 3.1: The axis of the transformation T

§ 3.2 Discrete and Properly Discontinuous Groups

In this subsection, we define what a Fuchsian group is and describe what it means to be locally finite and properly discontinuous. We also discuss the orbit and stablizers of Fuchsian groups.

Definition 3.2.1 A discrete subgroup of Isom($\mathbb{H}$) is called a **Fuchsian Group**, if it consists of orientation preserving transformations. A Fuchsian group is a discrete subgroup of PSL(2,$\mathbb{R}$).

Example 3.2.2. The modular group PSL(2,$\mathbb{Z}$) is a discrete subgroup of PSL(2, $\mathbb{R}$) and hence is a Fuchsian group.

Example 3.2.3. The group PSL(2,$\mathbb{Q}$) is a subgroup of PSL(2, $\mathbb{R}$), but it is not discrete, therefore is not a Fuchsian group.

Example 3.2.4. The set of integer translations $\{T(z) = z + n \mid n \in \mathbb{N}\}$ is a Fuchsian group.

Example 3.2.5. The set of all translations $\{T(z) = z + b \mid b \in \mathbb{R}\}$ is not a Fuchsian group as it is not discrete.

§ 3.3 Algebraic Properties of Fuchsian Groups

In this subsection, we will take an algebraic point of view to describe Fuchsian groups. We will look at centralizers of parabolic, elliptic and hyperbolic elements of PSL(2, $\mathbb{R}$) and examine their properties.

Definition 3.3.1. If G is any group and g ∈ G, then the **centralizer** of g in G is defined by

$$C_G(g) = \{h \in G \mid hg = gh\}$$

Lemma 3.3.2. If $ST = TS$, then S maps the fixed point set of T to itself.

Proof : Suppose that T fixes p, that is, $T(p) = p$. Then

$$S(p) = ST(p) = TS(p)$$

so that $S(p)$ is also fixed by T.

We now will look at centralizers of parabolic, elliptic and hyperbolic elements of PSL(2, ℝ).

Example 3.3.3. For a parabolic centralizer, let us consider $T(z) = z + 1$. We would like to find a $S \in$ PSL(2, ℝ) such that $ST = TS$. By the previous lemma, we know S will map the fixed points of T to itself. Since T is parabolic, this means $S(\infty) = \infty$. Hence, S is of the form

$$S(z) = az + b$$

and $ST = TS$ gives us a = 1. Therefore, the S we desire is $S(z) = z + k$, where

k ∈ ℝ .

Example 3.3.4. For a hyperbolic centralizer, let us consider $T(z) = 2z$. Observe $T(0) = 0$ and $T(\infty) = \infty$. We would like to find $S \in PSL(2, \mathbb{R})$ such that $ST = TS$. Since $S(\infty) = \infty$, S must have the form

$$S(z) = az + b$$

where a > 1. Consider

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

which implies, $\begin{bmatrix} 2a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 1 \end{bmatrix}$

which means this equality is true, if and only if b = 2b. Now, this is only possible if b = 0. Therefore, the S we desire is $S(z) = (2a)z$, where is a > 1.

Example 3.3.5. Let us find an elliptic centralizer for the following transformation. Suppose

$$T(z) = \frac{0z + 1}{-z + 0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z$$

This transformation is elliptic since its trace is less than 2. We would like to find a $S \in PSL(2, \mathbb{R})$ such that $ST = TS$. Consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -b & a \\ -d & c \end{bmatrix}$$

and
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}$$

In order for these matrices to commute, a = d and -b = c. Therefore, the centralizer is

$$\begin{bmatrix} a & -c \\ c & -a \end{bmatrix} z$$

or $S(z) = \frac{az-c}{cz-a}$.

Example 3.3.6. Let $T(z) = z + 2$ and $S(z) = z - 1$. We know both transformations are parabolic and fix ∞. Let us consider their corresponding matrices and show they commute. Consider

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Therefore, by direct calculation, we see these transformations commute.

Example 3.3.7. Let $T(z) = 2z$ and $S(z) = 3z$. We know both transformations are hyperbolic and fix both ∞ and 0. Let us consider their corresponding matrices and show they commute.Consider

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Since T and S have the same fixed point set, by direct calculation T and S commute.

Example 3.3.8. Let $T(z) = \frac{\frac{\sqrt{3}}{2}z + \frac{1}{2}}{-\frac{1}{2}z + \frac{\sqrt{3}}{2}}$ and $S(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}$ We know both transformations are elliptic and fix i. Let us consider their corresponding matrices and show they commute.

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\sqrt{3}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{3}\sqrt{2} \end{bmatrix}$$

and the other $$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\sqrt{3}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{3}\sqrt{2} \end{bmatrix}$$

Therefore, we have shown the transformations commute, so it must be the case they have the same fixed point set.

Theorem 3.3.9 The centralizer in PSL(2, ℝ) of a hyperbolic, parabolic, elliptic element of PSL(2, ℝ) consists of all hyperbolic, parabolic, elliptic elements with the same fixed point set, together with the identity.

Corollary 3.3.10. Two hyperbolic elements in PSL(2, ℝ) commute if and only if they have the same axes.

Example 3.3.11. Consider again the hyperbolic transformation $T(z) = 2z$. We saw that in example 3.3.2 $S(z) = 3z$ shares the same fix point set as $T(z) = 2z$ and they commute. This means the geodesic connecting the fix points is the vertical line from 0 to ∞. The vertical line is the axis for both transformations.

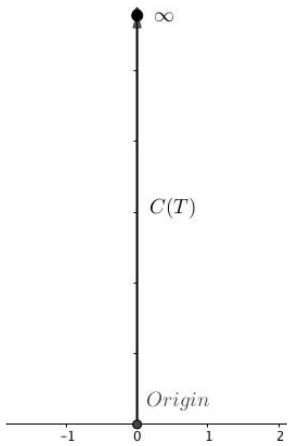

Figure 3.4: The axis of S and T

Chapter 4

Fundamental Regions

In this chapter we examine the properties and geometries of fundamental regions, Dirichlet regions and the Ford region. We will also examine the idea of isometric circles, which can be used to construct these regions.

§ 4.1 Definition of Fundamental Region

Fundamental regions can be useful to visualize a group structure. These regions can also tell us if an action is discontinuous. The regions also determine the geometry of the quotient space X/Γ. For our purposes, theses regions form a tessellation on $\mathbb{H}^2$.In other words, these regions can be viewed as a partition of the upper-half plane.

Definition 4.1.1. A closed region $F \subset X$ is said to be a **fundamental region** for a group

G if the following conditions hold:

1. $\bigcup_{T \in G} (F) = X$

2. $F^\circ \cap T(F^\circ) = \emptyset$

where F is the closure of a non-empty open set, F° called the **interior** of F.

Example 4.1.2. Let G = $\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$, let $F = \{ z \in \mathbb{H}^2 \mid 0 \leq Re\,(z) \leq 1\}$ and then

$F^\circ = \{z \in \mathbb{H}^2 \mid 0 < Re(z) < 1\}$. This is an example of a fundamental region because the union of all images of with F is indeed all of $\mathbb{H}^2$. The intersection of all interiors F° are empty. Elements of G translate left and right by integer increments as shown in figures 4.1 and 4.2.

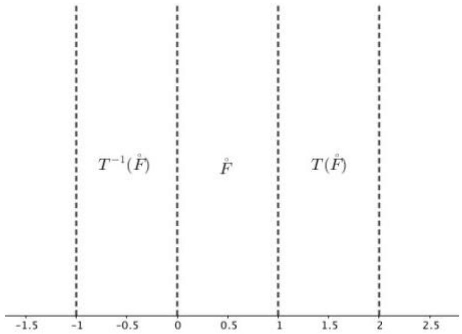

Figure 4.1: Intersection of mutually disjoint regions will be empty

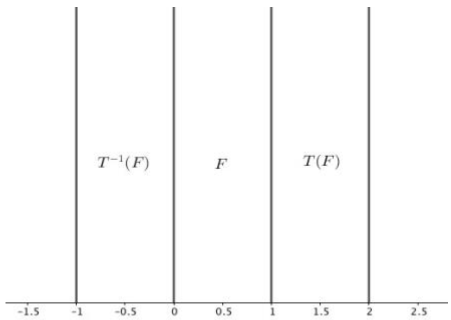

Figure 4.2: Union of regions for T ∈ G give $\mathbb{H}^2$

Example 4.1.3. Let G = $\langle\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\rangle$, let $F = \{\, z \in \mathbb{H}^2 \mid 0 \le Re\ (z) \le \frac{1}{2} \,\}$ and let $F^\circ = \{\, z \in \mathbb{H}^2 \mid 0 < Re\ (z) < \frac{1}{2} \,\}$. This is not a fundamental region because it fails condition 1. The union of all images with F does not give all of $\mathbb{H}^2$.

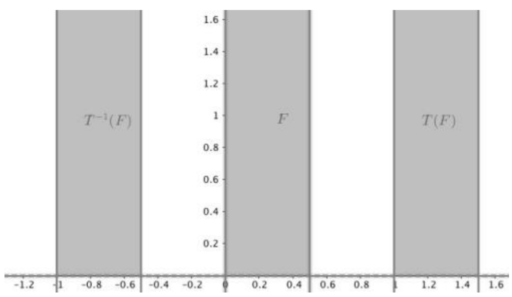

Figure 4.3: Union of regions for T ∈ G give $\mathbb{H}^2$

Example 4.1.4. Let Let G = $\langle\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\rangle$, let F = { z $\in \mathbb{H}^2$ | $0 < Re\,(z) < \frac{3}{2}$ } and let F° = { z $\in \mathbb{H}^2$ | $0 \leq Re\,(z) \leq \frac{3}{2}$ }. This is not a fundamental region because it fails condition 2. The intersection of all interiors with F° is not empty. We can see F and $T(F)$ overlap between 1 < $Re(z)$ < 1.5.

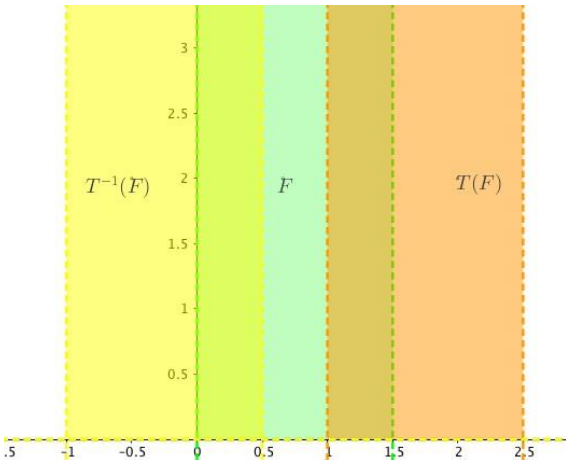

Figure 4.4: Regions are not mutually disjoint for $T \in G$

Example 4.1.5. Let G = $\langle\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\rangle$,

$$\text{let F} = \{\, z = re^{i\theta} \in \mathbb{H}^2 \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi \,\}$$

and let F° be the open set of this region. Then, F is a fundamental region since the union of all the regions for all T $\in$ G will yield $\mathbb{H}^2$. The intersection of mutually disjoint regions for $T \in G$ of F° will be empty.

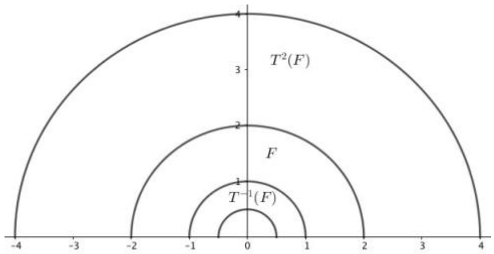

Figure 4.5: Union of regions for T ∈ G will give $\mathbb{H}$

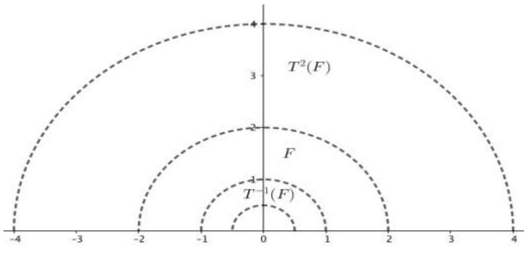

Figure 4.6: Mutually disjoint regions for T ∈ G will be empty

§ 4.2 The Dirichlet Region

Definition 4.2.1.

Let Γ be an arbitrary Fuchsian Group and let $p \in \mathbb{H}^2$ be not fixed by any element of $\Gamma - \{ Id\}$. We will define the **Dirichlet region** for Γ centered at p to be the set

$$D_p(\Gamma) = \{ z \in \mathbb{H}^2 \mid d(z,p) \leq d(z,T(p)) \,|\text{for all } T \in \Gamma\}$$

Definition 4.2.2. A **perpendicular bisector** of the geodesic segment $[z_1, z_2]$ is the unique geodesic through w, the midpoint of $[z_1, z_2]$, orthogonal to $[z_1, z_2]$.

We will denote the perpendicular bisector of the geodesic segment $[p, T(p)]$ by $Lp(T)$ and the hyperbolic half plane containing p is denoted by $Hp(T)$. Therefore, an equivalent definition for the **Dirichlet region** is given by

$$Dp(\Gamma) = \cap\, Hp(T)$$

for $T \in \Gamma$ and $T \neq Id$.

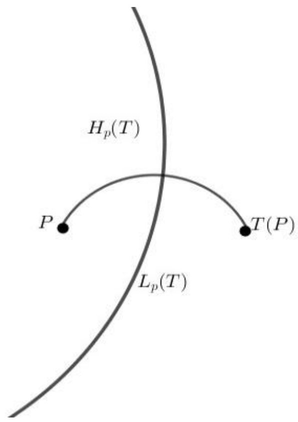

Figure 4.7: Hyperbolic half plane containing p

§ 4.3 Isometric Circles

In this subsection, we define isometric circles and examine their properties. We also will see geometrically how isometric circles of a transformation T act with the isometric circles of T^{-1}. We will then transition to examining the isometric circles of the unit disk because the model provides a convenient way to compute the Ford Region.

Let $T(z) = \frac{az+b}{cz+d} \in PSL(2, \mathbb{R})$. Since $T'(z) = (cz + d)^{-2}$, the locally Euclidean lengths are scaled by $|T'(z)| = |cz + d|^{-2}$. Thus, locally Euclidean area is scaled by $|cz + d|^{-4}$.The Euclidean areas of regions are not altered in magnitude if and only if $|cz + d| = 1$.

Proposition 4.3.1.

Let $T(z) = \frac{az+b}{cz+d}$ with $c \neq 0$, then locus of such a z is a circle

$$I(T) = \{ z \in C \mid |cz + d| = 1\}$$

where the center is $-\frac{d}{c}$ and radius $\frac{1}{|c|}$.

Proof:

We will show I(T) is a circle with center $-\frac{d}{c}$ and radius $\frac{1}{|c|}$ .Consider $|cz + d| = 1$.

Then, $|cz + d| = 1$.

$$\Rightarrow \left| z + \frac{d}{c} \right| = \frac{1}{|c|}$$

$$\Rightarrow \left| (x + iy) + \frac{d}{c} \right| = \frac{1}{|c|}$$

$$\Rightarrow \left| \left(x + \frac{d}{c}\right) + iy \right| = \frac{1}{|c|}$$

$$\Rightarrow \sqrt{\left(x + \frac{d}{c}\right)^2 + y^2} = \sqrt{\frac{1}{c^2}}$$

$$\Rightarrow \left(x + \frac{d}{c}\right)^2 + y^2 = \frac{1}{c^2}$$

This takes the form of a Euclidean circle with center $(-\frac{d}{c}, 0)$ and radius $\frac{1}{|c|}$.

Corollary 4.3.2. Let $T^{-1}(z) = \frac{-dz+b}{cz-a}$ with $c \neq 0$, then locus of such a z is a circle

$$I(T^{-1}) = \{ z \in \mathbb{C} \mid |cz - a| = 1\}$$

where the center is $\frac{a}{c}$ and radius $\frac{1}{|c|}$.

Definition 4.3.3. $I(T) = \{ z \in \mathbb{C} \mid |cz + d| = 1\}$

These $I(T)$ are called **isometric circles.**

We point out the radius of the isometric circles of a transformations T and T^{-1} are equal. Let us consider some examples and see how hyperbolic and elliptic elements differ with their respective isometric circles.

Example 4.3.4. Let $T(z) = \frac{z+1}{z+2}$ be the hyperbolic transformation. Then,

$$I(T) = \{ z \in C \mid |z + 2| = 1\}$$

where the center is -2 and the radius is 1.

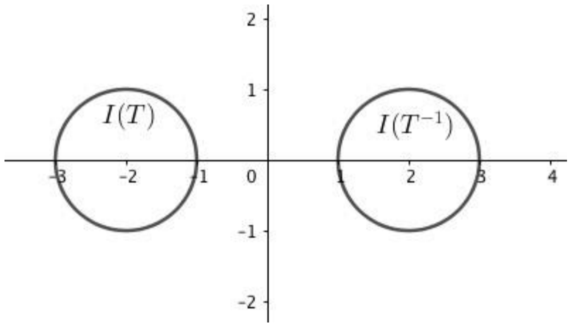

Figure 4.8: Isometric circles for hyperbolic transformation

Example 4.3.5. Let $\mathrm{T(z)} = \frac{\frac{\sqrt{2}}{2}z+\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z+\frac{\sqrt{2}}{2}}$ be the elliptic transformation. Then,

$$\mathrm{I(T)} = \left\{ z \in \mathbb{C} \mid |z-1| = \frac{1}{\left|-\frac{\sqrt{2}}{2}\right|} = \sqrt{2} \right\}$$

where the center is 1 and the radius is $\sqrt{2}$

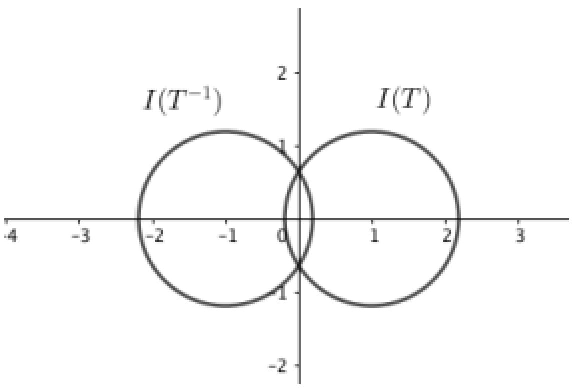

Figure 4.9: Isometric circles for elliptic transformation

Example 4.3.6. Let $T(z) = z + 1$. Observe

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z$$

We see that T is parabolic with c = 0. This means there is no unique circle with the isometric property since ∞ is a fixed point.

In these examples, we see geometrically how isometric circles of T act with the isometric circles of T^{-1}

1. When $I(T)$ and $I(T^{-1})$ intersect, T is elliptic.

2. When $I(T)$ and $I(T^{-1})$do not intersect, T is hyperbolic.

Definition 4.3.7. The **unit disk** is defined to be

$$\mathbb{B}^2 = \{ z \in \mathbb{C} \,||z| < 1\}.$$

The map

$$f(z) = \frac{zi+1}{z+i}$$

is a 1-1 map of $\mathbb{H}^2$ and provides an isometry onto $\mathbb{B}^2$.

Proposition 4.3.8.. The group of orientation preserving isometries of the unit disk given by the matrices

$$T(z) = \frac{az + \bar{c}}{cz + \bar{a}}$$

where $a, c \in \mathbb{C}$ and $a\bar{a} - c\bar{c} = 1$.

Example 4.3.9. Let $T(z) = \frac{3z+(2-2i)}{(2+2i)z+3}$ be an orientation preserving transformation of the unit disk. Then,

$$I(T) = \{ z \in \mathbb{C} \,|\, |(2+2i)z + 3| = 1\}$$

where the center is

$$\frac{3}{(2+2i\,)} = -\frac{(6-6i)}{8} = -\frac{3}{4} + \frac{3}{4}\,\mathrm{i}$$

and the radius is

$$\frac{1}{|2+2\mathrm{i}|} = \frac{1}{\sqrt{8}}$$

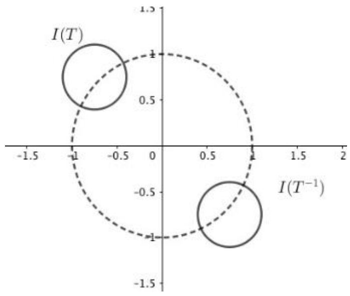

Figure 4.10: Isometric circles for orientation preserving transformation of the unit disk

Conclusion

In this project we discussed about the hyperbolic metric. We calculated hyperbolic lengths and distances. We also discussed Möbius transformations and how these transformations can be looked at algebraically by their associated matrices. The trace of a matrix determined which element in $\mathrm{PSL}(2,\mathbb{R})$ are hyperbolic, elliptic or parabolic.

We discussed how to find the fixed points of these elements. We defined what Fuchsian groups were and gave examples of them geometrically and algebraically. We discussed what it meant to be properly discontinuous and showed examples of cyclic generated groups and their orbits and stabilizers. We saw some algebraically properties of Fuchsian groups and it was shown that elements of $\mathrm{PSL}(2,\mathbb{R})$commute if and only if they share the same fixed point set. We finally looked at some examples of fundamental regions and Dirichlet regions.

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GAME THEORY

Project Report submitted to

St. MARY'S COLLEGE (Autonomous), THOOTHUKUDI.

Affiliated to

MANOMANIAM SUNNDARANAR UNIVERSITY, TIRUNELVELI

In partial fulfillment of the requirement for the award of degree of

Bachelor of Science in Mathematics

Submitted by

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(2022 – 2023)

CERTIFICATE

We hereby declare that the project report entitled **"GAME THEORY"** being submitted to **St. Mary's College (Autonomous), Thoothukudi** affiliated to **Manonmaniam Sundaranar University, Tirunelveli** in partial fulfillment for the award of degree of **Bachelor of Science in Mathematics** and it is a record of work done during the year 2022 -2023 by the following students:

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DECLARATION

We hereby declare that the project entitled "GAME THEORY" submitted for the degree of Bachelor of Science is our work carried out under the guidance of Dr.A. Punitha Tharani M.Sc., M.Phil., Ph.D., Associate Professor, Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi.

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ACKNOWLEDGEMENT

First of all, we thank the Almighty for showering his blessing to undergo this project.

With immense pleasure, we register our deep sense of gratitude to our guide and Associate professor of the Department of Mathematics **Dr. A. Punitha Tharani M.Sc., M. Phil., Ph. D** for having imparted necessary guidelines throughout the period of our studies.

We thank our beloved Principal **Sr. A.S.J. Lucia Rose M.Sc., M. Phil., Ph.D., PGDCA.,** for providing us help to carry out our project work successfully.

Finally, we thank all those who extended their helping hands towards this project.

CONTENT

INTRODUCTION

Game theory was developed by Prof. John Von Neumann and Oscar Morgenstern in 1928. Game theory is a body of knowledge that deals with making decisions when two or more rational and intelligent opponents are involved under situations of conflict and competition. The approach of game theory is to determine a rival's most profitable counter-strategy to one's own best moves. It helps in determining the best course of action for a firm in view of the expected counter moves from the competitors.

Game theory is a type of decision theory which is based on reasoning in which the choice of action is determined after considering the possible alternatives available to the opponents playing the same game. The aim is to choose the best course of action, because every player has got an alternative course of action

Many practical problems require decision-making in competitive situation where there are two or more opposing parties with conflicting interests and where the action of one depends upon the one taken by the opponent. For example, candidates for an election, advertising and marketing campaigns by competing business firms, countries involved in military battles, etc. have their conflicting interests. In a competitive situation the courses of action (alternatives) for each competitor may be either finite or infinite.

CHAPTER 1

BASIC CONCEPTS

A competitive situation will be called a 'Game', if it has the following properties:

(i) There are a finite number of competitors (participants) called players.
(ii) Each player has a finite number of strategies (alternatives) available to him.
(iii) A play of the game takes place when each player employs his strategy.
(iv) Every game results in an outcome. e.g., loss or gain or a draw, usually called payoff, to some player.

1.1 BASIC TERMS

1.1.1 Player.

The competitors in the game are known as players. A player may be individual or group of individuals, or an organisation.

1.1.2 Strategy.

A strategy for a player is defined as a set if rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt. A strategy may be of two types:

(a) **Pure strategy**.
If the players select the same strategy each time, then it is referred to as pure-strategy. In this case each player knows exactly what the other player is going to do, the objective of the players is to maximize gains or to minimize losses.

(b) **Mixed strategy**.
When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus, there is a probabilistic situation and objective of the player is to maximize expected gains or to minimize expected losses.

Optimum strategy.

A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy.

1.1.3 Value of the game.

It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair, if it is no-zero.

If the game has a saddle point, then the value of the cell at the saddle point is called the value of the game; otherwise, the value of the game is computed based on expected value calculations which will be explained later.

1.1.4 Payoff matrix.

When the players select their particular strategies, the payoff (gain or losses) can be represented in the form of a matrix called the payoff matrix. Since the game is zero-sum, therefore gain of one player is equal to the loss of other and vice-versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus, it is sufficient to construct payoff only for one of the players.

Let player A have m strategies $A_1, A_2, \ldots, A_m$ and player B have n strategies B_1, $B_2, \ldots, B_n$. Here, it is assumed that each player has his choices from amongst the pure strategies. Also it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let a_{ij} be the payoff which player A gains from player B if the player A chooses strategy A_i and player B chooses strategy B_j. Then the payoff matrix to player A is:

Player B

$$\begin{array}{c} \begin{matrix} B_1 & B_2 & \cdots & B_n \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

The payoff matrix to player B is $(-a_{ij})$.

1.2 Assumptions of Game Theory

- There are finite number of competitors (players).
- The players act reasonably.
- Every player strives to maximize gains and minimize losses.
- Each player has finite number of possible courses of action.
- The choices are assumed to be made simultaneously, so that no player knows his opponent's choice until he has decided his own course of action.
- The payoff is fixed and predetermined.
- The payoffs must represent utilities.

Two-person zero-sum game :

In a game with two players, if the gain of one player is equal to the loss of another player, then that game is called two-person zero-sum game.

Non zero sum game :

In game theory, situation where one decision maker's gain (or loss) does not necessarily result in the other decision maker's loss (or gain). In other words, where the winnings and losses of all players do not add up to zero and everyone can gain: a win-win game.

CHAPTER 2

GAME WITH PURE STRATEGIES

2.1 Minimax and Maximin Principle

2.1.1 Maximin principle:

This principle maximizes the minimum guaranteed gains of Player A. The minimum gains with respect to different alternatives of A, irrespective of B's alternative are obtained first. The maximum of these minimum gains is known as the maximin value and the corresponding alternative is called as maximin strategy.

2.1.2 Minimax principle :

This principle minimizes the maximum losses. The maximum losses with respect to different alternatives of Player B, irrespective of Player A's alternatives, are obtained first. The minimum of these maximum losses is known as the minimax value and the corresponding alternative is called as minimax strategy.

2.2 Saddle point

In a game, if the maximin value is equal to the minimax value, then the game is said to have a saddle point. The intersecting cell corresponding to these values is known as the saddle point. If the game has a saddle point, then each player has a pure strategy.

As we know, if a game has a saddle point, then the game is said to have pure strategy for each players Determination of such pure strategy of each of the players is illustrated in the following example.

Example 2.1

Two computer hardware manufacturing companies, Company A and Company B competing for supplying computers to a government department. Each company has listed its strategies for selling computers.

The strategies of Company A are:

(a) Giving special price
(b) Giving 20% worth of additional hardwares
(c) Supplying computer furniture free of cost

The strategies of Company B are:

(a) Giving special price
(b) Giving 30% worth of additional hardwares
(c) Giving free training to the users of the organization which is buying hardwares.

The estimated gains (+)/ losses (-) of company A for various possible combinations of the alternatives of both companies are summarised in the table1a. In the table, the rows represent the alternatives of company A and the columns represent the alternative of company B. Each cell value of table represents the estimated gains (+)/ losses (-) of company A in lakhs of rupees for the corresponding alternative of companies A and B. A positive cell entry denotes the gain to company A (loss to company B) and a negative cell entry represents the Loss to company A (gain to company B) Thus the table is called as the payoff matrix with respect to company A.
Determine the optimal strategy / Strategies for company A and company B

TABLE 1a Payoff Matrix of Company A

		COMPANY B		
		1	2	3
COMPANY A	1	20	15	22
	2	35	45	40
	3	18	20	25

Solution

If company A select strategy one and company B select strategy two the corresponding payoff to company A is rupees 15 lakhs This means that company B would be losing Rupees 15 lakhs. In table 1a all the cell entries are positive. So, whatever maybe the selected strategy of company B company A will always have a gain. But the magnitude of the gain varies for different combinations of rows and columns. Though company B has no chance of winning the game ,it can try to minimize the gain to company A, so that in the long run, company B will have a competitive advantage. If there are some negative cell entries in the table 1a, then there is a chance of winning the game for company B, provided company A selects any of the strategies corresponding to those negative cell values.

As per the payoff matrix shown in the table 1a company A is called as maximin player and company B is called as minimax player, which means Company A is maximizing its minimum guaranteed gain and company B is minimizing its maximum loss. Table 1a is reproduced in table 1b with necessary calculations, where

Maximin value = Minimax value =35

Hence, the game has a saddle point at the cell corresponding to row 2 and column 1. The value of the game is rupees 35 lakhs.

The optimal probabilities of selection of the strategies of company A and company B are as given below

A(p_1, p_2, p_3) = A(0,1,0) and B(q_1, q_2, q) = B (1,0,0)

TABLE 1b Payoff Matrix of Player A with Maximin and Minimax Values

		COMPANY B			
		1	2	3	Row minimum
	1	20	15	22	15
COMPANY A	2	$\boxed{35}$	45	40	35 (maximin)
	3	18	20	25	18
Column maximum		35 (minimax)	45	40	

Company A should always select its second strategy with a probability of 1 and it should select all other strategies with zero probability. Company B should always select its first strategy with a probability of 1 and it should select all other strategies with zero probability.

If anyone of the two companies deviates from its optimal strategy combination(s), the other player will have a gain. This means that company A should always select its second alternative (giving 20% worth of additional hardwares). If it deviates from this alternative, Company B will gain in terms of its reduced loss, since all the entries of table 1a are positive values (loss for company B will be less than rupees 35 lakhs)

As per the optimal solution, company B should always select it's alternative one (giving special price). If it deviates from this strategy, Company A will have an additional gain over and above the value of the game. Under this situation, the gain for company A will be more than rupees 35 lakhs.

CHAPTER 3

MIXED STRATEGIES I

If a game has no saddle point, then the game is it to have mixed strategies.

3.1 Arithmetic Method

The arithmetic method (also known as short – cut method) provides an easy method for finding optimal strategies for each player in a payoff matrix of size 2×2 , without saddle point.

Table 3a A Game with Mixed strategies

		Player B 1	Player B 2
Player A	1	a	b
	2	c	d

Algorithm to determine mixed strategies

Step 1: Find the absolute value of a - b (i.e. $|a-b|$) and write it against row 2
Step 2: Find the absolute value of c - d (i.e. $|c-d|$) and write it against row 1
Step 3: Find the absolute value of a – c (i.e. $|a-c|$) and write it against column 2
Step 4: Find the absolute value of b – d (i.e. $|b-d|$) and write it against column 1
The results of above steps as summarized in the below table. The absolute values are called as oddments.

Table : Payoff Matrix with Oddments

		Player B 1	Player B 2	Oddments
Player A	1	a	b	$\lvert c-d\rvert$
	2	c	d	$\lvert a-b\rvert$
Oddments		$\lvert b-d\rvert$	$\lvert a-c\rvert$	

Step 5: Compute the probabilities of selection of the alternatives of Player A (p_1 and p_2) and that of Player B (q_1 and q_2).

$$p_1 = \frac{|c-d|}{|a-b|+|c-d|}$$

$$p_2 = \frac{|a-b|}{|a-b|+|c-d|}$$

$$q_1 = \frac{|b-d|}{|a-c|+|b-d|}$$

$$q_2 = \frac{|a-c|}{|a-c|+|b-d|}$$

The value of the game can be computed using any one of the following formulae:

$$V = \frac{a|c-d|+c|a-b|}{|a-b|+|c-d|}$$

$$= \frac{b|c-d|+d|a-b|}{|a-b|+|c-d|}$$

$$= \frac{a|b-d|+b|a-c|}{|a-c|+|b-d|}$$

$$= \frac{c|b-d|+d|a-c|}{|a-c|+|b-d|}$$

Example 3.1

Consider payoff matrix with respect to Player A and solve it optimally:

Table 3b

		Player B 1	Player B 2
Player A	1	6	9
	2	8	4

Solution

The maximin and minimax value of the given problem are shown in table 3c

Table 3c Maximin and Minimax Values

		Player B 1	Player B 2	
Player A	1	6	9	6 (maximin)
	2	8	4	4
(minimax)		8	9	

In this problem, the maximin value (6) is not equal to the minimax (8). Hence, the game has no saddle point. Under this situation, the formulae given in this section are to be used to find the mixed strategies of the players and also the value of the game. The computations of oddments of the game are summarized in the table 3d.

Table 3d Payoff Matrix with oddments

		Player B 1	Player B 2	Oddments
Player A	1	6	9	4
	2	8	4	3
Oddments		5	2	

Let p_1 and p_2 be the probabilities of selection of Alternative 1 and of Alternative 2, respectively of Player A. Also q_1 and q_2 be the probabilities of selection of Alternative 1 and of Alternative 2, respectively.

Then we have

$$p_1 = \frac{|c-d|}{|a-b|+|c-d|} = \frac{4}{3+4} = \frac{4}{7}$$

$$p_2 = \frac{|a-b|}{|a-b|+|c-d|} = \frac{3}{3+4} = \frac{3}{7}$$

$$q_1 = \frac{|b-d|}{|a-c|+|b-d|} = \frac{5}{2+5} = \frac{5}{7}$$

$$q_2 = \frac{|a-c|}{|a-c|+|b-d|} = \frac{2}{2+5} = \frac{2}{7}$$

Where, the value of the game is

$$V = \frac{a|c-d|+c|a-b|}{|a-b|+|c-d|}$$

$$= \frac{6(4)+8(3)}{4+3}$$

$$= \frac{48}{7}$$

Hence, the strategies of Player A is: A (4/7,3/7) and of

Player B: B (5/7, 2/7).

The value of the game = $\frac{48}{7} = 6\,\frac{6}{7}$.

Dominance property

In some games, it is possible to reduce the size of the payoff matrix by eliminating the redundant rows (or columns). If a game has such a redundant rows (or columns), those rows (or columns) are dominated by some other rows (or columns) respectively.
Such property is known as dominance property.

Dominance property for rows

(a) In the payoff matrix of player A, if all the entries in a row (X) are greater than or equal to the corresponding entries of another row (Y), then row Y is dominated by row X, Under such situation, row Y of the payoff matrix can be deleted.
(b) In the payoff matrix of player A, if each of the sum of the entries of any two rows (sum of the entries of the row X and row Y) is greater than or equal to the corresponding entry of a third row (Z), then row Z is dominated by row X and row Y. Under such situation, row Z of the pay of matrix can be deleted.

Dominance properties for columns

(a) In the pay of matrix of player A, if all the entries in a column (X) are lesser than or equal to the corresponding entries of another column (Y), then column Y is dominated by column X. Under such situation, the column Y of the payoff of matrix can be deleted.
(b) In the payoff matrix of player A, if each of the sum of the entries of any two columns (sum of the entries of column X and column Y) is lesser than or equal to the corresponding of the third column (Z), then column Z is dominated by columns X and Y. Under such situation, column Z of the payoff matrix can be deleted.

3.2 Algebraic Method

This method is used to determine the probability of using different strategies by players A and B. This method becomes quite lengthy when a number of strategies for both the players are more than two.

Consider a game where the payoff matrix is: (a_{ij}) m x n. Let $(p_1,p_2....,p_m)$ and $(q_1,q_2,...., q_n)$ be the probabilities with which players A and B select their strategies $(A_1,A_2,...,A_m)$ and $(B_1,B_2,..., B_n)$ respectively .If V is the value of game, then the expected gain to player A, when player B selects strategies B_1, B_2 ,. ...B_n, one by one, is given by left-hand side of the following simultaneous-equations, respectively .Since player A is the gainer player and expects at least V, therefore, we must have

Player B

Player A	B_1	B_2	...	B_n	Probability
A_1	a_{11}	a_{12}	...	a_{1n}	P_1
A_2	a_{21}	a_{22}	...	a_{2n}	P_2
$\vdots$	$\vdots$				$\vdots$
A_m	a_{m1}	a_{m2}	...	a_{mn}	P_m
Probability	q_1	q_2	...	q_n	

$$a_{11}p_1 + a_{21}p_2 + \ldots + a_{m1}p_m \geq V$$

$$a_{12}p_1 + a_{22}p_2 + \ldots + a_{m2}p_m \geq V$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m \geq V$$

$$p_1 + p_2 + \ldots + p_m = 1, \text{ and } p_i \geq 0 \quad for\ all\ i$$

Similarly, the expected loss to player B, when Player A selects strategies A_1, A 2,…,A_m one by one, can also be determined. Since player B is the loser player, therefore, he must have :

$$a_{11}q_1 + a_{12}q_2 + \ldots + a_{1n}q_n \leq V$$

$$a_{21}q_1 + a_{22}q_2 + \ldots + a_{2n}q_n \leq V$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}q_1 + a_{m2}q_2 + \ldots + a_{mn}q_n \leq V$$

$$Where\ \ q_1 + q_2 + \ldots + q_n = 1, \text{ and } q_j \geq 0 \quad for\ all\ j$$

To get the values of p_i's and p_j's, the above inequalities are considered as equations and are then solved for given unknowns. However, if the system of equations, so obtained, is inconsistent, then at least one of the inequalities must hold as a strict inequality. The solution can now be obtained only by applying the trial and error method.

Example 3.2

A company is currently involved in negotiations with its union on the upcoming wage contract. Positive sign in table represent wage increase while negative sign represents wage reduction. What are the optimal strategies for the company as well as the union? What is the game value?

Conditional costs to the company (Rs. in lakhs)

Union Strategies

		U_1	U_2	U_3	U_4
Company Strategies	C_1	0.25	0.27	0.35	-0.02
	C_2	0.20	0.16	0.08	0.08
	C_3	0.14	0.12	0.15	0.13
	C_4	0.30	0.14	0.19	0.00

Solution:

Suppose, Company is the gainer player and Union is the looser player. Transposing payoff matrix because company's interest is to minimize the wage increase while union's interest is to get the maximum wage increase.

Company Strategies

		C_1	C_2	C_3	C_4
Union Strategies	U_1	0.25	0.20	0.14	0.30
	U_2	0.27	0.16	0.12	0.14
	U_3	0.35	0.08	0.15	0.19
	U_4	-0.02	0.08	0.13	0.00

In this payoff matrix strategy U_4 is dominated by strategy U_1 as well as U_3. After deleting this strategy, we get

Company Strategies

		C_1	C_2	C_3	C_4
Union Strategies	U_1	0.25	0.20	0.14	0.30
	U_2	0.27	0.16	0.12	0.14
	U_3	0.35	0.08	0.15	0.19

Company's point of view, strategy C_1 is dominated by C_2 as well as C_3, while C_4 is dominated C_3,Deleting Strategies C_1 and C_4 we get

Company Strategies

		C_2	C_3
Union Strategies	U_1	0.20	0.14
	U_2	0.16	0.12
	U_3	0.08	0.15

Again strategy U_2 is dominated by U_1, and is, therefore, deleted to give

Company Strategies

		C_2	C_3	Probability
Union Strategies	U_1	0.20	0.14	$0.07/0.13 = 0.538$
	U_3	0.08	0.15	$0.06/0.13 = 0.461$
		$0.01/0.13 = 0.076$	$0.12/0.13 = 0.923$	

Optimal strategy for the company : (0, 0.076, 0.923, 0)

Optimal strategy for the union : (0.538, 0, 0.461, 0)

Value of the game,V: 0.538x0.20 + 0.461x 0.08 = Rs 14360

Example 3.3

Players A and B play a game in which each player has three coins (20 p, 25 p and 50 p). Each of them selects a coin without the knowledge of the other person.
If the sum of the values of the coin is an even number, A wins B's coin.
If that sum is an odd number, B wins A's coin.

(a) Develop a payoff matrix with respect to player A.
(b) Find the optimal Strategies for the players.

Solution :

The pay of matrix with respect to player A is shown in the table 3e The maximin and minimax values are also indicated in the same table.
In the table 3e the maximum value (-20) is not equal to the minimax value (20). Hence the game has no saddle point. As a result, the game has mixed strategies.

Table 3e Payoff Matrix

		Player B			
		I	II	III	
		20p	25p	50p	Row minimum
Player A	I 20P	20	-20	50	-20 (maximin)
	II 25P	- 25	25	-25	-25
	III 50P	20	-50	50	-50
Column maximum		20 (minimax)	25	50	

Check for Dominance property.

Row III is dominated by row I and hence row III is to be deleted. The resultant matrix after deleting row III is shown in the table 3f.

Table 3f Payoff Matrix after deleting row III

		Player B I 20p	II 25p	III 50p
Player A	I 20p	20	-20	50
	II 25p	- 25	25	-25

In the table 3f the column III is dominated by the column I and hence, column III is to be deleted. The resultant Matrix after deleting column III is shown in table 3g. The oddments of the rows and columns are presented in the same table.

Table 3g Payoff Matrix after deleting column III

		Player B I 20p	II 25p	Oddments
Player A	I 20p	20	-20	50
	II 25p	- 25	25	40
	Oddments	45	45	

Let p_i be the probability of selection of the alternative i by player A, where i = 1, 2 and q_j be the probability of selection of the alternative j by player B, where j = 1, 2. Then, we have the

$$p_1 = \frac{50}{50 + 40} = \frac{5}{9}$$

$$p_2 = \frac{40}{50+40} = \frac{4}{9}$$

$$q_1 = \frac{45}{45+45} = \frac{1}{2}$$

$$q_2 = \frac{45}{45+45} = \frac{1}{2}$$

Where, value of the game is

$$V = \frac{20(50)-25(40)}{50+40} = 0$$

Hence, the strategies of Player A and Player B, and the value of the game are :

A $\left(\frac{5}{9}, \frac{4}{9}, 0\right)$, B $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, V = 0

3.3 Matrix Method

If the game matrix is in the form of a square matrix, then the optimal strategy mix as well as value of the game may be obtained by the matrix method. The solution of a two-person zero-sum game with mixed strategies with a square payoff matrix may be obtained by using the following formulae:

$$\text{Player A's optimal strategy} = \frac{[1\ 1]\, P_{adj}}{[1\ 1]\, P_{adj} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\text{Player B's optimal strategy} = \frac{[1\ 1]\, P_{cof}}{[1\ 1]\, P_{adj} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Value of the game = (Player A's optimal strategies) × (payoff matrix p_{ij}) × (Player B's optimal strategies) where P_{adj} = adjoint matrix, P_{cof} = cofactor matrix. Player A's optimal strategies are in the form of a row vector and B's optimal strategies are in the form of a column vector.

This method can be used to find a solution of a game with size of more than 2×2. However, in rare cases, the solution violates the non - negative condition of probabilities, i.e. $p_i \geq 0$, $q_j \geq 0$, although the requirement $p_1 + p_2 + \ldots + p_m = 1$

or $q_1 + q_2 + \ldots + q_n = 1$ is met.

Example 3.4

Solve the following game after reducing it to a 2×2 game

Player A	Player B		
	B_1	B_2	B_3
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

Solution

In the given game matrix, the third row is dominated by the second row and in the reduced matrix third column is dominated by the first column. So, after elimination of the third row and the third column the game matrix becomes.

Player A	Player B	
	B_1	B_2
A_1	1	7
A_2	6	2

For this reduced matrix, let us calculate P_{adj} and P_{cof} as given below:

$$P_{adj} = \begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix} \quad \text{and} \quad P_{cof} = \begin{bmatrix} 2 & -6 \\ -7 & 1 \end{bmatrix}$$

$$\text{Player A's Optimal strategies} = \frac{[1\ 1]\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}}{[1\ 1]\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{[-4\ -6]}{-10}$$

$$= \frac{[4\ 6]}{10}$$

This solution can be broken down into the optimal strategy mix for player A as $p_1 = 4/10 = 2/5$ and $p_2 = 6/10 = 3/5$, where p_1 and p_2 represent the probabilities of player A's using the his strategies A_1 and A_2, respectively.

Similarly, the optimal strategy mixture for player B is obtained as:

$$\text{Player B's optimal strategies} = \frac{[1\ 1]\begin{bmatrix} 2 & -6 \\ -7 & 1 \end{bmatrix}}{[1\ 1]\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{[-5\ -5]}{-10}$$

$$= \frac{[5\ 5]}{10}$$

This solution can be broken down into the optimal strategy mixture for player A as $q_1 = 5/10=1/2$ and $q_2 = 5/10 = 1/2$, where q_1 and q_2 represent the probabilities of player B's using the his strategies B_1 and B_2, respectively. Hence:

$$\text{Value of the game, V} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$V = 4$$

Example 3.5

Two competitors are competing for the market share of the similar product. The payoff matrix in terms of their advertising plane is shown below.

Player A	Player B		
	B_1	B_2	B_3
A_1	10	5	-2
A_2	13	12	15
A_3	16	14	10

Suggest optimal strategies for the two firms and the net outcome thereof.

Solution

Applying the rules of dominance to delete first column (dominated by second column) and then first row (dominated by second as well as third row) from the payoff matrix, we obtain the following reduced payoff matrix.

Player A	Player B	
	B_2	B_3
A_2	12	15
A_3	14	10

For this reduced matrix, let us calculate P_{adj} and P_{cof} as given below:

$$P_{adj} = \begin{bmatrix} 10 & -15 \\ -14 & 12 \end{bmatrix} \quad \text{and} \quad P_{cof} = \begin{bmatrix} 10 & -14 \\ -15 & 12 \end{bmatrix}$$

$$\text{Player A's Optimal strategies} = \frac{[1\ 1]\begin{bmatrix} 10 & -15 \\ -14 & 12 \end{bmatrix}}{[1\ 1]\begin{bmatrix} 10 & -15 \\ -14 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{[-4\ -3]}{-7}$$

$$= \frac{[4\ 3]}{7}$$

This solution can be broken down into the optimal strategy mix for player A as $p_1 = 4/7$ and $p_2 = 3/7$, where p_1 and p_2 represent the probabilities of player A's using the his strategies A_1 and A_2, respectively.

Similarly, the optimal strategy mixture for player B is obtained as:

$$\text{Player B's optimal strategies} = \frac{[1\ 1]\begin{bmatrix} 10 & -14 \\ -15 & 12 \end{bmatrix}}{[1\ 1]\begin{bmatrix} 10 & -15 \\ -14 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{[-5\ -2]}{-7}$$

$$= \frac{[5\ 2]}{7}$$

This solution can be broken down into the optimal strategy mixture for player A as $q_1 = 5/7$ and $q_2 = 2/7$, where q_1 and q_2 represent the probabilities of player B's using the his strategies B_1 and B_2, respectively. Hence:

$$\text{Value of the game, } V = \begin{bmatrix} \frac{4}{7} & \frac{3}{7} \end{bmatrix} \begin{bmatrix} 12 & 15 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix}$$

$$V = 90/7$$

CHAPTER 4

GAME WITH MIXED STRATEGIES II

GRAPHICAL METHOD FOR 2 x n or m x 2 GAMES

Some games will be of specialized nature, like 2 x n or m x 2 games. The payoff matrix of the 2 x n game will contain 2 rows and n columns, whereas the payoff matrix of the m x 2 game will contain m rows and 2 columns. If there is no saddle point for these games, One can solve them using graphical method.

4.1 GRAPHICAL METHOD FOR 2 x n

Algorithm for 2 x n game

Step 1: Reduce the size of the payoff matrix of Player A by applying the dominance property, if it exists.

Step 2: Let, x be the probability of selection of Alternative 1 by Player A and 1-x be the probability of selection of Alternative 2 by Player A. Derive the expected gain function of Player A with respect to each of the alternatives of Player B.

Step 3: For each of the gain functions which are derived in Step 2, find the value of the gain, when x is equal to 0 as well as 1.

Step 4: Plot the gain functions on a graph by assuming a suitable scale. Keep x on X-axis and the gain on Y-axis).

Step 5: Since the Player A is a maximin player, find the highest intersection point in the lower boundary of the graph. Let it be the maximin point.

Step 6: If the number of lines passing through the maximin point is only two, form a 2 x 2 payoff matrix from the original problem by retaining only the columns corresponding to those two lines and go to step 8; otherwise go to step 7.

Step 7: ldentify any two lines with opposite slopes passing through that point. Then form a 2 x 2 payoff matrix from the original problem by retaining only the columns corresponding to those two lines which are having opposite slopes.

Step 8: Solve the 2 x 2 game using oddments and find the strategies for Players A and B and also the value of the game.

Example 4.1

Consider the payoff matrix of Player A as shown in Table 4a and solve it optimally using a graphical method.

Table 4a

		Player B				
		1	2	3	4	5
Player A	1	3	0	6	-1	7
	2	-1	5	-2	2	1

Solution :

Table 4b shows the payoff matrix with maximin and minimax values. Here, the maximin value (-1) is not equal to the minimax value (2). Hence, the game has no saddle point. As a result, the players will have mixed strategies. Since the game has only two rows, it can be solved using graphical method.

Table 4b Payoff Matrix with Maximin and Minimax values

	Player B					
	1	2	3	4	5	Row minimum
	3	0	6	-1	7	-1 (maximin)
Player A	-1	5	-2	2	1	-2
Column maximum	3	5	6	2	7	
				(minimax)		

checking dominance property. In the payoff matrix as shown in Table 4b, column 5 is dominated by column 1.

Similarly, column 2 is dominated by column 4. Hence, delete column 2 and columns5.The resultant payoff matrix after deleting columns 2 and 5 is shown in Table 4c.

Table 4c Payoff Matrix after Deleting Columns 2 and 5

		Player B		
		1	3	4
Player A	1	3	6	-1
	2	-1	-2	2

Let, x be the probability of selection of Alternative 1 by Player A and 1-x be the probability of selection of Alternative 2 by Player A. Therefore, the expected payoff to Player A with respect to different alternatives of Player B is summarized in Table 4d.

Table 4d Expected Payoff functions of Player A

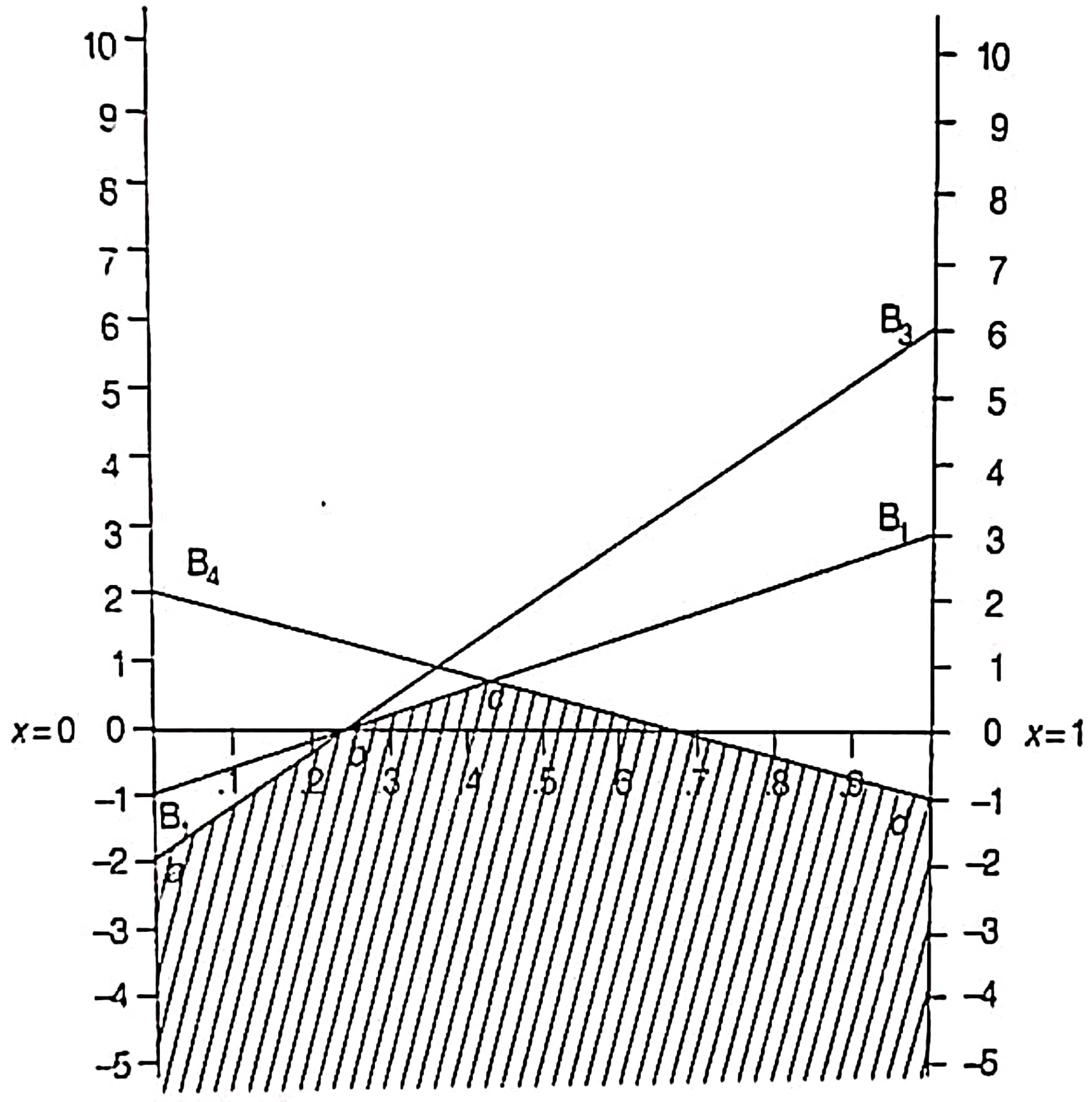
B's alternative	A's expected payoff function
1	$3x - (1 - x) = 4x - 1$
3	$6x - 2(1 - x) = 8x - 2$
4	$-x + 2(1 - x) = -3x + 2$

The computations of the expected payoff of player A with respect to each of the alternatives of Player B, when x is equal to 0 as well as 1, are summarized in table 4e.

Table 4e Expected Gain of Player A

B's alternative	A's expected payoff Function	A's expected gain	
		X= 0	X = 1
1	4x - 1	-1	3
3	8x - 2	-2	6
4	-3x + 2	2	-1

Now, the expected gain functions of Player A with respect to different alternatives of Player B are plotted in figure below.



Graph with A's payoff function.

Since, A is a maximin type player, identify the highest intersection point in the lower boundary of the graph. The lower boundary consists of the intersection points a,b, c and d . Out of these points, point c is at the highest level. Hence, the corresponding solution is the optimal solution to the given problem.

In the above figure , the line B_1 and line B_4 pass through the point c. Hence form a 2×2 payoff matrix for player A by retaining columns B_1 and B_4 as shown in table 4f. The oddments of the rows and columns are shown in the same fig

Table 4f the 2×2 pay of matrix with oddments

Player B

		1	4	Oddments
Player A	1	3	-1	3
	2	-1	2	4
Oddments		3	4	

Let p_i be the probability of selection of alternative i by player A where i = 1, 2 and q_j be the probability of selection of alternative j by Player B, where j = 1,4. Then

The strategies of player A and player B, and the value of the game are:

$$p_1 = \frac{3}{3+4} = \frac{3}{7}$$

$$p_2 = \frac{4}{3+4} = \frac{4}{7}$$

$$q_1 = \frac{3}{3+4} = \frac{3}{7}$$

$$q_4 = \frac{4}{3+4} = \frac{4}{7}$$

Where, the value of the game is

$$V = \frac{3(3) - 1(4)}{4+3} = \frac{5}{7}$$

Hence , the strategies of Player A and Player B ,and the value of the game are :

$$A\left(\frac{3}{7}, \frac{4}{7}\right) \quad , B\left(\frac{3}{7}, 0, 0, \frac{4}{7}, 0\right), \quad V = \frac{5}{7}$$

4..2 GRAPHICAL METHOD FOR m x 2

Algorithm for m x 2 game

Step 1: Reduce the size of the payoff matrix of Player A by applying the dominance property, if it exists.

Step 2: Let y be the probability of selection of Alternative 1 by Player B and 1 - y be the probability of selection of Alternative 2 by Player B. Derive the expected gain function of Player B with respect to each of alternatives of Player A.

Step 3: For each of the gain functions which are derived in step 2, find the value of the gain when y is equal to 0 as well as 1.

Step 4: Plot the gain functions on a graph by assuming a suitable scale. Keep y on X-axis and the gain on Y-axis.

Step 5: Since B is a minimax player, find the lowest intersection point in the upper boundary of the graph. Let it be the minimax point.

Step 6: If the number of lines passing through the minimax point is only two, form a 2 x 2 payoff matrix from the original problem by retaining only the rows corresponding to those two lines and go to step 8; otherwise, go to step 7.

Step 7: Identify any two lines with opposite slopes passing through that point. Then form a 2 x 2 payoff matrix from the original problem by retaining only the rows corresponding to those two lines which are having opposite slopes.

Step 8: Solve the 2 x 2 game using oddments and find the strategies for Player A and Player B and also the value of the game.

Example 4.2

Consider the payoff of Players A has shown in Table 4g and solve it optimally using graphical method.

Table 4g

		Player B	
		1	2
Player A	1	1	3
	2	3	1
	3	5	-1
	4	6	-6

Solution:

Table 4h shows the payoff matrix with maximin and minimax values. Here, the maximum value (1) is not equal to the minimax value (3). Hence, the game has no saddle point. As a result, the players will have mixed strategies. Since the game has only two columns , it can be solved using graphical method.

Table 4h Payoff Matrix with Maximin and Minimax Values

		Player B		
		1	2	
Player A	1	1	3	1 (maximin)
	2	3	1	1 (maximin)
	3	5	-1	-1
	4	6	-6	-6
		6	3	
		(minimax)		

None of the rows of the game can be deleted using the dominance property. Let y be the probability of selection of Alternative 1 by Player B and 1-y be the probability of selection of Alternative 2 by Player B. Therefore, the expected payoff functions of B with respect to different alternatives of A are summarized as in Table 4i.

Table 4i Expected Payoff Function of Player B

A's alternative	B's expected loss (+)/gain (-) function
1	y + 3(1 - y) = -2y + 3
2	3y + (1 - y) = 2y + 1
3	5y - (1 - y) = 6y - 1
4	6y - 6(1 - y) = 12y - 6

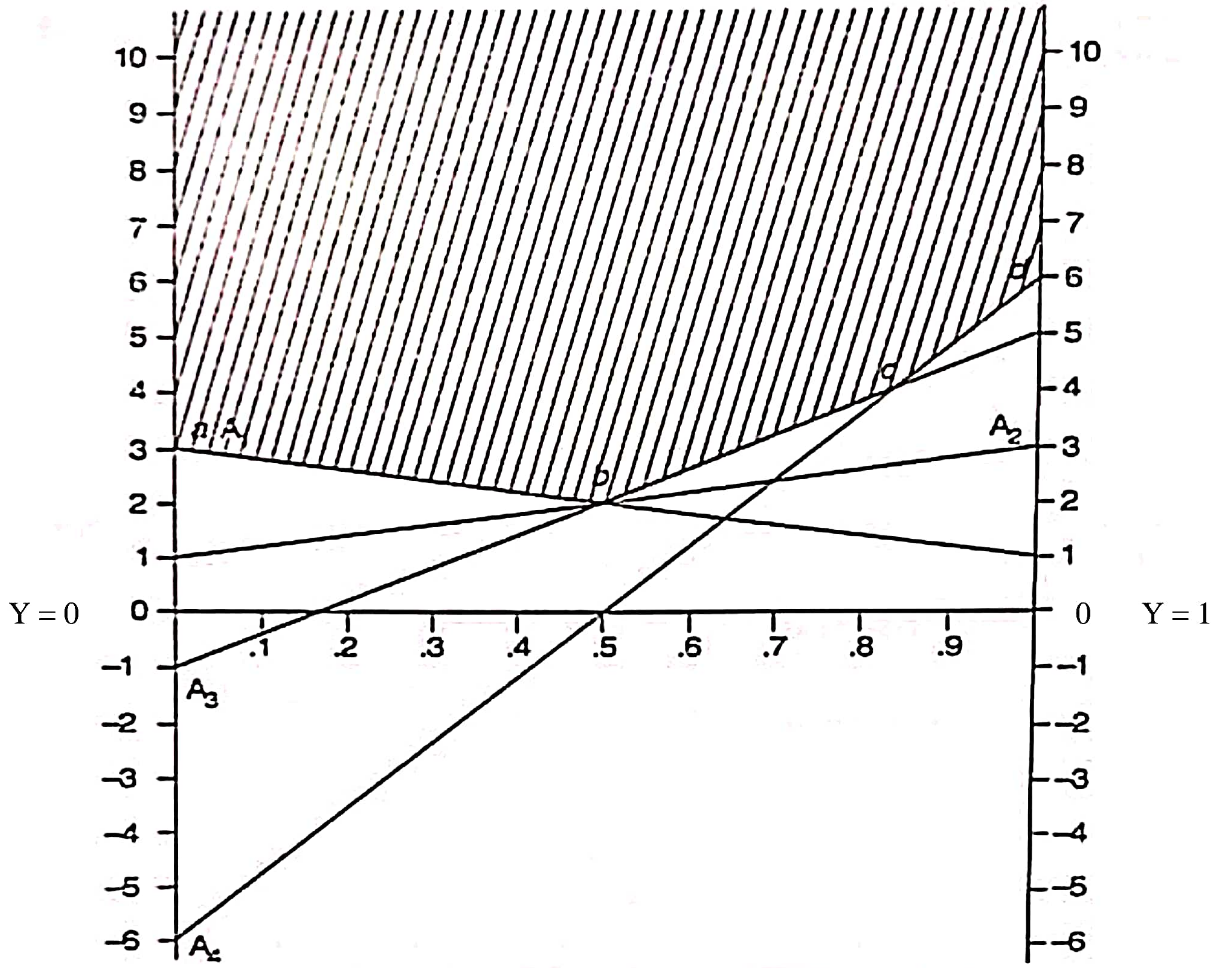
Note: A positive value of the function indicates loss to B and a negative value of the function indicates gain to B.

The computations of expected loss (+)/gain (-) of B with respect to each of the alternatives of A. when y is equal to 0 as well as 1 are summarized in table 4j

Table 4j Expected Loss (+)/Gain (-) of Player B

A's alternative	B's expected loss / gain function	B's expected function value	
		y = 0	y = 1
1	-2y + 3	3	1
2	2y + 1	1	3
3	6y - 1	-1	5
4	12y - 6	-6	6

Now, the expected functions of Player B with respect to different alternatives of Player A are plotted in Figure below.



Graph with B's payoff functions.

Since B is a minimax player, identify the lowest intersection point in the upper boundary of the graph. The upper boundary consists of the intersection points a, b, c and d. Out of these points, point b is at the lowest level. Hence, the corresponding solution is the optimal solution to the given problem. In the above figure the lines A_1, A_2 and A_3 pass through point b. Among these lines, select any two lines having opposite slopes. As per this guideline, the line A_1 and the line A_3 are selected.

Hence, form a 2 × 2 payoff matrix for Player A by retaining only the rows A_1 and A_3 as shown in Table 4k. The oddments of the rows and columns are also shown in the Table 4k.

Table 4k   2 x 2 Payoff Matrix with Oddments

		Player B		
		1	2	Oddments
Player A	1	1	3	6
	3	5	-1	2
	Oddments	4	4	

Let p_i be the probability of selection of alternative i by Player A, $i = 1, 3$ and q_j be the probability of selection of Alternative j by Player B, $j = 1, 2$. Then

$$p_1 = \frac{6}{2+6} = \frac{3}{4}, \qquad p_3 = \frac{2}{6+2} = \frac{1}{4}$$

$$q_1 = \frac{4}{4+4} = \frac{1}{2}, \qquad q_2 = \frac{4}{4+4} = \frac{1}{2}$$

$$V = \frac{1(6)+5(2)}{6+2} = 2$$

Hence, the strategies of A and B, and the value of the game are:

$$A\left(\frac{3}{4}, 0, \frac{1}{4}, 0\right), \qquad B\left(\frac{1}{2}, \frac{1}{2}\right), \qquad V = 2$$

CHAPTER 5

GAME WITH MIXED STRATEGIES – III

5.1 LINEAR PROGRAMMING METHOD

The two-person zero-sum games can also be solved by linear programming. The major advantage of using linear programming technique is that it helps to solve the mixed-strategy games of larger dimension payoff matrix.

To illustrate the transformation of a game problem to a linear programming problem, consider a payoff matrix of size m × n. Let a_{ij} be the element in the ith row and jth column of game payoff matrix, and letting p_i be the probabilities of m strategies (i = 1,2, … ,m) for player A. Then, the expected gains for player A, for each of player B's strategies will be:

$$V= \sum_{i=1}^{m} p_i a_{ij}, \qquad j= 1,2, \ldots, n$$

The aim of Player A is to select a set of strategies with probability p_i (i = 1,2, … ,m) on any play of game such that he can maximize his minimum expected gains.

Now to obtain values of probability p_i, the value of the game to player A for all strategies by player B must be at least equal to V. Thus to maximize the minimum expected gains, it is necessary that:

$$\begin{aligned} a_{11}p_1 + a_{21}p_2 + \ldots + a_{m1}p_m &\geq V \\ a_{12}p_1 + a_{22}p_2 + \ldots + a_{m2}p_m &\geq V \\ \vdots \qquad\qquad & \quad \vdots \\ a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m &\geq V \end{aligned}$$

Where $p_1 + p_2 + \ldots + p_m = 1$, and $p_i \geq 0 \quad for\ all\ i$

Dividing both sides of the m inequalities and equation by V the division is valid as long as V > 0. In case V < 0, the direction of inequality constraints must be reversed. But if V = 0, the division would be meaningless. In this case a constant can be added to all entries of the matrix, ensuring that the value of the game (V) for the revised matrix becomes more than zero. After optimal solution is obtained, the true value of the game is obtained by subtracting the same constant value. Let pi/ V = Xi, (≥0), we then have

$$a_{11}\frac{p_1}{V} + a_{21}\frac{p_2}{V} + \dots + a_{m1}\frac{p_m}{V} \geq 1$$

$$a_{12}\frac{p_1}{V} + a_{22}\frac{p_2}{V} + \dots + a_{m2}\frac{p_m}{V} \geq 1$$

$$\vdots \qquad\qquad \vdots$$

$$a_{1n}\frac{p_1}{V} + a_{2n}\frac{p_2}{V} + \dots + a_{mn}\frac{p_m}{V} \geq 1$$

$$\frac{p_1}{V} + \frac{p_2}{V} + \dots + \frac{p_m}{V} = \frac{1}{V}$$

Since the objective of player A is to maximize the value of the game, V, which is equivalent to minimizing 1/V, the resulting linear programming problem can be stated as:

$$\text{Minimize } Z_p\ (=1/V) = X_1 + X_2 + \cdots + X_m$$

Subject to the constraints

$$a_{11}X_1 + a_{21}X_2 + \dots + a_{m1}X_m \geq 1$$

$$a_{12}X_1 + a_{22}X_2 + \dots + a_{m2}X_m \geq 1$$

$$\vdots \qquad\qquad \vdots$$

$$a_{1n}X_1 + a_{2n}X_2 + \dots + a_{mn}X_m \geq 1$$

And $\quad X_1, X_2, \dots, X_m \geq 0$

Where $\quad X_i = \frac{p_i}{V} \geq 0\,; \quad i = 1,2, \dots, m$

Similarly, Player B has a similar problem with the inequalities of the constraints reversed, i.e. minimize the expected loss. Since minimizing V is equivalent to maximizing 1/V, therefore, the resulting linear programming problem can be stated as:

$$\text{Maximize } Z_q \ (=1/V) = Y_1 + Y_2 + \ \dots + Y_n$$

Subject to the constraints

$$a_{11}Y_1 + a_{12}Y_2 + \dots + a_{1n}Y_n \leq 1$$

$$a_{21}Y_1 + a_{22}Y_2 + \dots + a_{2n}Y_n \leq 1$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1}Y_1 + a_{m2}Y_2 + \dots + a_{mn}Y_n \leq 1$$

$$Y_1, Y_2, \dots, Y_n \geq 0$$

$$\text{Where } \ Y_j = \frac{q_j}{V} \geq 0 \ ; \ \ j = 1, 2, \dots, n$$

It may be noted that the LP problem for player B is the dual of LP problem for player A and vice versa. Therefore, the solution of the dual problem can be obtained from the primal simplex table. Since for both the players $Z_p = Z_q$, the expected gain to player A in the game will be exactly equal to expected loss to player B.

Example 5.1

The given table 5a represents the payoff matrix with respect to Player A. Solve it optimally using linear programming method.

Table 5a

		Player B 1	2	3
	1	1	-1	-1
Player A	2	-1	-1	3
	3	-1	2	-1

Solution

The maximin and the minimax values are computed based on Table 5b

Table 5b Payoff Matrix with Maximin and Minimax Values.

		Player B 1	2	3	Row minimum
	1	1	-1	-1	-1 (maximin)
Player A	2	-1	-1	3	-1 (maximin)
	3	-1	2	-1	-1 (maximin)
Column maximum		1 (minimax)	2	3	

Here, maximin value (-1) is not equal to the minimax value (1). Hence, the game has no saddle point and in turn, the game is said to have mixed strategies. Also, one can verify the fact that the payoff matrix cannot be reduced using dominance property. Hence, the game should be solved using the linear programming method.

Since the payoff matrix has negative values, the absolute value of the most negative value plus 1, (K = 1+1) is added to each of the entries of the payoff matrix and the corresponding revised payoff matrix is shown in table 5c.

Table 5c Payoff Matrix after adding K to each entry

		Player B 1	Player B 2	Player B 3
Player A	1	3	1	1
	2	1	1	5
	3	1	4	1

Since the linear programming formulation with respect to Player B will have only "≤" constraints, it is advisable to develop a model for Player B. The solution of the problem can be obtained in the following ways:

- Solve the model and obtain the strategies of B and the value of the game.
- Obtain the strategies of A from the optimal table of B using the concept of duality.
- The true value of the game is obtained by subtracting K from the value of the modified game.

The last step is required because K (2) is added to all the cell entries of the payoff matrix at the initial stage. Let, a_{ij} be the payoff to Player A if A selects his Alternative i and Player B selects his Alternative j. V be the value of the game (i.e. expected gain for Player A. So, A will try to maximize V and B will try to minimize V). p_i be the probability of selection of Alternative i by Player A where, i = 1, 2 and 3. q_j, be the probability of selection of Alternative j by Player B where, j = 1, 2 and 3.

Development of linear programming model with respect to Player B.

The expected loss (+)/gain (-) function to B with respect to the selection of each of the alternatives of A is presented below along with the necessary condition of $q_1 + q_2 + q_3 = 1$

$$3q_1 + q_2 + q_3 \le V$$

$$q_1 + q_2 + 5q_3 \le V$$

$$q_1 + 4q_2 + q_3 \le V$$

$$q_1 + q_2 + q_3 = 1$$

Dividing the above set of constrains by V, we get

$$3\frac{q_1}{V} + \frac{q_2}{V} + \frac{q_3}{V} \leq 1$$

$$\frac{q_1}{V} + \frac{q_2}{V} + 5\frac{q_3}{V} \leq 1$$

$$\frac{q_1}{V} + 4\frac{q_2}{V} + \frac{q_3}{V} \leq 1$$

$$\frac{q_1}{V} + \frac{q_2}{V} + \frac{q_3}{V} = \frac{1}{V}$$

Substituting $\frac{q_j}{V} = Y_j$, $j = 1, 2, 3$ in the above system of constraints,we have

$$3Y_1 + Y_2 + Y_3 \leq 1$$

$$Y_1 + Y_2 + 5Y_3 \leq 1$$

$$Y_1 + 4Y_2 + Y_3 \leq 1$$

$$Y_1 + Y_2 + Y_3 = \frac{1}{V}$$

Since the objective of player B is to minimize the value of the game, V, which is equivalent to maximizing 1/V, the resulting linear programming problem can be stated as:

$$\text{Maximize } Z_q \left(= \frac{1}{V}\right) = Y_1 + Y_2 + Y_3$$

Subject to

$$3Y_1 + Y_2 + Y_3 \leq 1$$

$$Y_1 + Y_2 + 5Y_3 \leq 1$$

$$Y_1 + 4Y_2 + Y_3 \leq 1$$

$$Y_1,\ Y_2 \text{ and } Y_3 \geq 0$$

Converting the above generalized model into a standard model yields:

$$\text{Maximize } Z_q = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$3Y_1 + Y_2 + Y_3 + S_1 = 1$$

$$Y_1 + Y_2 + 5Y_3 + S_2 = 1$$

$$Y_1 + 4Y_2 + Y_3 + S_3 = 1$$

$$Y_1,\ Y_2,\ Y_3,\ S_1,\ S_2,\ S_3 \geq 0$$

The starting initial table is shown in table 5d

Table 5d Iteration 1

CB_i	Basis	1	1	1	0	0	0	Solution	Ratio
		Y_1	Y_2	Y_3	S_1	S_2	S_3		
0	S_1	3	1	1	1	0	0	1	$\frac{1}{3} = 0.33$**
0	S_2	1	1	5	0	1	0	1	$\frac{1}{1} = 1$
0	S_3	1	4	1	0	0	1	1	$\frac{1}{1} = 1$
Z_j		0	0	0	0	0	0	0	
$C_j - Z_j$		1*	1	1	0	0	0		

*Key column ** key row

In table 5d , Y_1 is selected as the entering variable and the leaving variable is S_1 . The next iteration is shown in table 5e

Table 5e Iteration 2

CB_i	Basis	1	1	1	0	0	0	Solution	Ratio
		Y_1	Y_2	Y_3	S_1	S_2	S_3		
1	Y_1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1
0	S_2	0	$\frac{2}{3}$	$\frac{14}{3}$	$\frac{-1}{3}$	1	0	$\frac{2}{3}$	1
0	S_3	0	$\frac{11}{3}$	$\frac{2}{3}$	$\frac{-1}{3}$	0	1	$\frac{2}{3}$	$\frac{2}{11}$
	Z_j	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	
	C_j-Z_j	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{-1}{3}$	0	0		

Here Y_2 selected as the entering variable and leaving variable is S_3 . The next iteration is shown in table 5f.

Table 5f Iteration 3

CB_i	Basis	1	1	1	0	0	0	Solution	Ratio
		Y_1	Y_2	Y_3	S_1	S_2	S_3		
1	Y_1	1	0	$\frac{3}{11}$	$\frac{4}{11}$	0	$\frac{-1}{11}$	$\frac{3}{11}$	1
0	S_2	0	0	$\frac{50}{11}$	$\frac{-3}{11}$	1	$\frac{-2}{11}$	$\frac{6}{11}$	$\frac{3}{25}$
1	Y_2	0	1	$\frac{2}{11}$	$\frac{-1}{11}$	0	$\frac{3}{11}$	$\frac{2}{11}$	1
	Z_j	1	1	$\frac{5}{11}$	$\frac{3}{11}$	0	$\frac{2}{11}$	$\frac{5}{11}$	
	C_j- Z_j	0	0	$\frac{6}{11}$	$\frac{-3}{11}$	0	$\frac{-2}{11}$		

In table 5f, Y_3 selected as the entering variable and leaving variable is S_2 . The next iteration is shown in table 5g.

Table 5g Iteration 4

CB_i	Basis	1	1	1	0	0	0	Solution
		Y_1	Y_2	Y_3	S_1	S_2	S_3	
1	Y_1	1	0	0	$\frac{19}{50}$	$\frac{-3}{50}$	$\frac{-2}{25}$	$\frac{6}{25}$
1	Y_3	0	0	1	$\frac{-3}{50}$	$\frac{11}{50}$	$\frac{-1}{25}$	$\frac{3}{25}$
1	Y_2	0	1	0	$\frac{-2}{25}$	$\frac{-1}{25}$	$\frac{7}{25}$	$\frac{4}{25}$
	Z_j	1	1	1	$\frac{6}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{13}{25}$
	C_j-Z_j	0	0	0	$\frac{-6}{25}$	$\frac{-3}{25}$	$\frac{-4}{25}$	

In table 5g ,since all the $C_j - Z_j$ values are less than or equal to zero, the optimality is reached and the solution of the model is:

$$Y_1 = \frac{6}{25}, \quad Y_2 = \frac{4}{25}, \quad Y_3 = \frac{3}{25} \text{ and } Z_2 = \frac{13}{25}$$

compute the value of V and q_1 , q_2 and q_3 using the following formulae.

$$V = \frac{1}{Z_2} \text{ and } q_j = V\,Y_j, \qquad j = 1, 2, 3$$

Therefore,

$$V = \frac{1}{Z_2} = \frac{1}{(13/25)} = \frac{25}{13}$$

Where, value of the original g ame $= \frac{25}{13} - K = \frac{25}{13} - 2 = -\frac{1}{13}$

and

$$q_1 = \frac{25}{13}\frac{6}{25} = \frac{6}{13}$$

$$q_2 = \frac{25}{13}\frac{4}{25} = \frac{4}{13}$$

$$q_3 = \frac{25}{13}\frac{3}{25} = \frac{3}{13}$$

From the optimal table 5g based on the concept of duality, the values of X_1, X_2, and X_3 are obtained as shown in table 5h

Table 5h Solution of player A

Basic variable in the initial table	S_1	S_2	S_3
Corresponding dual variable	X_1	X_2	X_3
$-(C_j - Z_j)$ from table 5g	6/25	3/25	4/25

Now, the solutions of player A are :

$$X_1 = \frac{6}{25}, \; X_2 = \frac{3}{25}, \; X_3 = \frac{4}{25} \text{ and } Z_1 = \frac{13}{25}$$

compute value of V and P_1, P_2 and P_3 using the following formulae

$$V = \frac{1}{Z_1} \text{ and } \quad p_i = V\,X_i\,, \qquad\qquad i=1,2,3$$

Therefore,

$$V = \frac{1}{Z_1} = \frac{1}{(13/25)} = \frac{25}{13}$$

$$\text{The value of the original game} = \frac{25}{13} - K = \frac{25}{13} - 2 = -\frac{1}{13}$$

$$p_1 = \frac{25}{13}\frac{6}{25} = \frac{6}{13}$$

$$p_2 = \frac{25}{13}\frac{3}{25} = \frac{3}{13}$$

$$p_3 = \frac{25}{13}\frac{4}{25} = \frac{4}{13}$$

The strategies of player A and B are summarized as :

$$A\left(\frac{6}{13}, \frac{3}{13}, \frac{4}{13}\right) \text{ and } B\ \left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13}\right)$$

Where value of the original game is $-\frac{1}{13}$.

6. Limitations of game theory

- The assumption that the players have the knowledge about their own payoffs and payoffs of others is rather unrealistic. He can only make a guess of his own and his rivals' strategies.
- As the number of players increases in the game. The analysis of the gaming strategies becomes increasingly complex and difficult. In practice, there are many firms in an oligopoly situation and game theory cannot be very helpful in such situation.
- The assumptions of maximin and minimax show that the players are risk-averse and have complete knowledge the strategies. These do not seem practical.
- Rather than each player in an oligopoly situation working under uncertain conditions, the players will allow each other to share the secrets of business in order to work out a collusion. Thus, the mixed strategies are also not very useful.

7. Applications of Game Theory

The game theory is widely applied to study human as well as animal behaviours. It is utilized in economics to understand the economic behaviours, such as behaviours of consumers, markets and firms. Game theory has been commonly used in social sciences as well. It is applied in the study of sociological, political and psychological behaviours. The use of analysis based on game theory is seen in biology too. In addition to behavioural prediction, game theory utilized in the development of theories of normative or ethical behaviour.

Some are listed below

- **Economists**

Innovated antitrust policy

Auctions of radio spectrum licenses for cell phone

Program that matches medical residents to hospitals.

- **Computer scientists**

New software algorithms and routing protocols

Game Al

- **Military strategists**

Nuclear policy and notions of strategic deterrence.

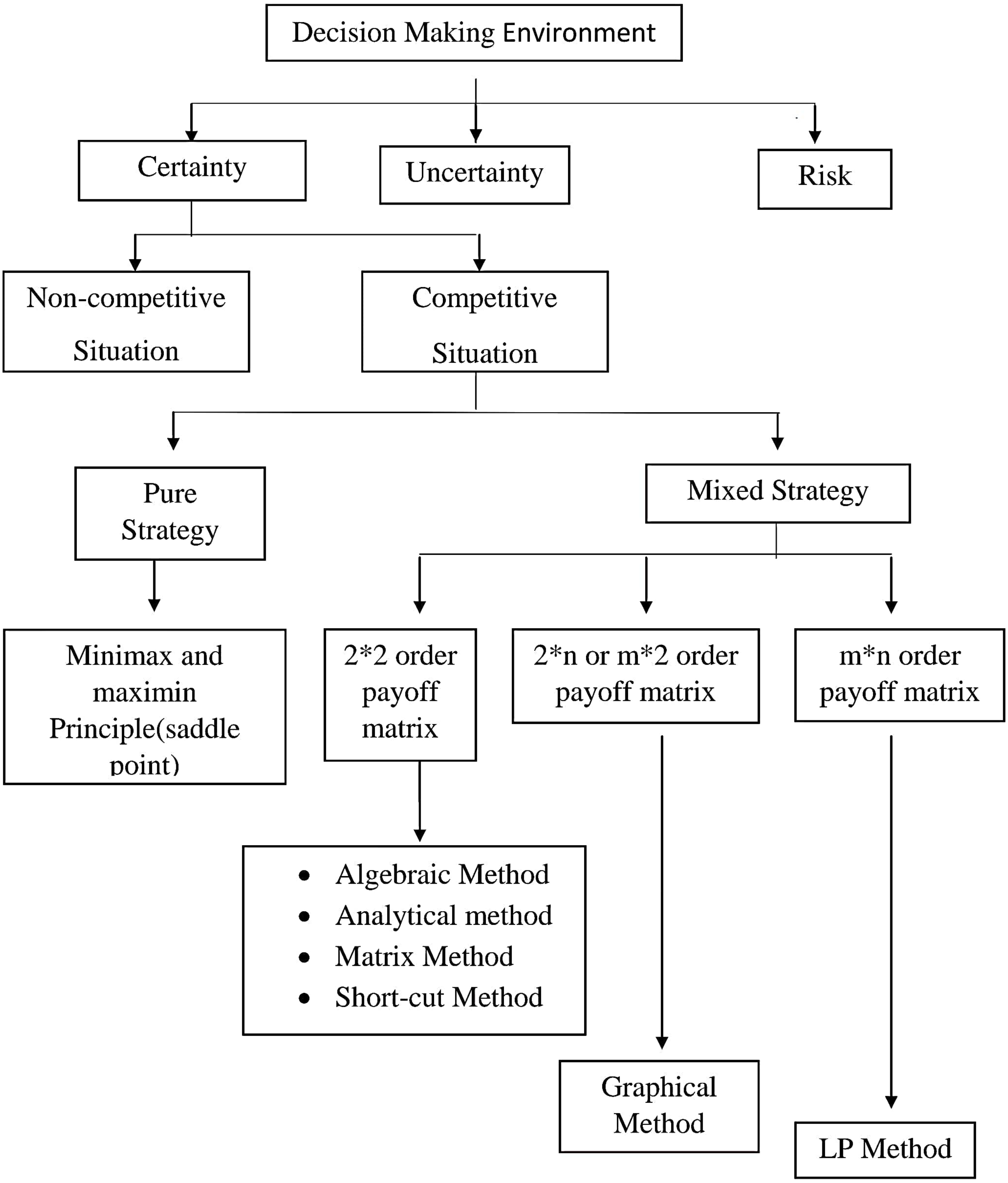
- **Sports coaching staffs**

Run versus pass or pitch fast balls versus sliders.

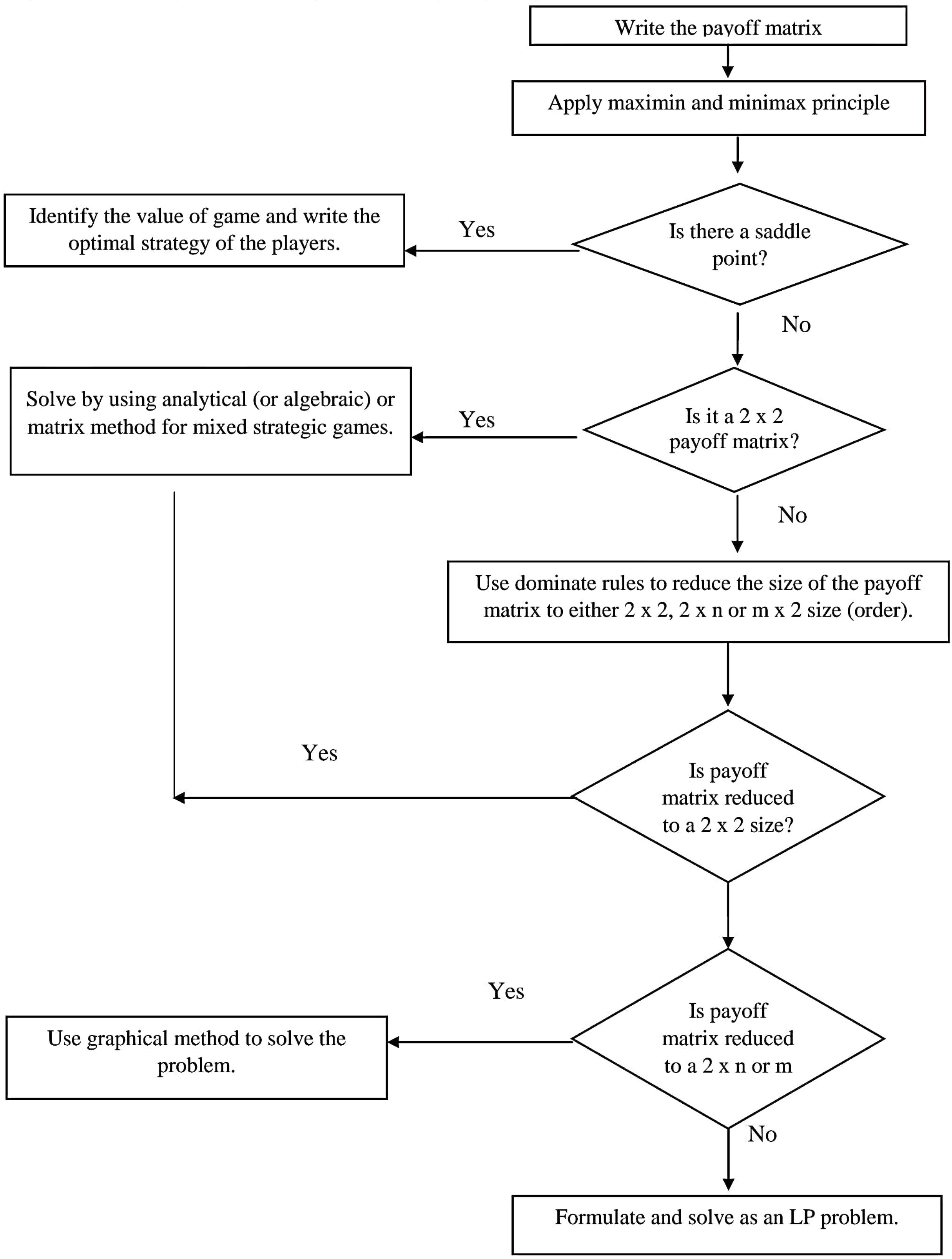
- **Biologists**

To identify the species that have the greatest likelihood of extinction.

8.Concept map



The above said methods are summarized as a flow chart



9. Conclusion

Game theory is a kind of decision theory which is based on the choice of action, and the choice of action is determined after considering the possible alternatives available to the opponent. It involves the player's decision i.e. decision makers who have different goals and objectives. The game theory determines the rules of rational behavior of these players in which the outcomes are dependent on the actions of the interdependent players. In a game theory there are number of possible outcomes, with different values to the decision.

The game theory is extremely effective in determining from advertising budgets and expenditures to the best strategy for conducting business within a market. The mathematical transitivity into practical applications makes this theory very important to businesses and mathematicians when prompted with these situations.

10. References

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